Excem Research Papers in Electronics and Electromagnetics

Number 7 July 2023 ISBN 978-2-909056-27-2

Some Results on Power in Passive Linear Time-Invariant Multiports, Part 3

FRÉDÉRIC BROYDÉ[1], and EVELYNE CLAVELIER[2]

[1]Eurexcem, 12 chemin des Hauts de Clairefontaine, 78580 Maule, France
[2]Excem, 12 chemin des Hauts de Clairefontaine, 78580 Maule, France

Corresponding author: Frédéric Broydé (e-mail: fredbroyde@eurexcem.com).

ABSTRACT We investigate a reciprocal and passive linear time-invariant multiport, having a port set coupled to a generator and a port set coupled to a load, in the harmonic steady state. Two configurations are considered, in which the port set at which the generator is connected and the port set at which the load is connected are exchanged. We improve earlier reciprocal theorems, and establish new results about the power available at output ports, and the bounds of the sets of the values of power transfer ratios, operating power gains, available power gains and unnamed power gains for all relevant excitations. The new results include five reciprocal theorems. One of them is used to generalize the Friis transmission formula.

INDEX TERMS Operating power gain, transducer power gain, available power gain, power transfer ratio, unnamed power gain, insertion power gain, passive circuits, linear circuits, reciprocity, circuit theory.

I. INTRODUCTION

This article is a sequel of [1] and [2]. In what follows, [1] is referred to as "Part 1", and [2] as "Part 2". The numbering of lemmas, theorems, etc, is a continuation of the one used in Part 1 and Part 2, but no prior knowledge of Part 1 or Part 2 is assumed. Appendix A lists some corrections to Part 2.

As in Part 1, we consider two linear time-invariant (LTI) circuits, referred to as "configurations", operating in the harmonic steady state, at a given frequency. Both comprise a device under study (DUS), which is a passive LTI multiport having 2 sets of ports, referred to as port set 1 and port set 2. Port set 1 consists of m ports numbered from 1 to m, and port set 2 consists of n ports numbered from 1 to n, where m and n are integers greater than or equal to 1. When we say that port set 1 is connected to an m-port device, we assume that the ports of the m-port device are numbered from 1 to m, and that, for any integer $p \in \{1, \ldots, m\}$, its port p is connected to port p of port set 1 (positive terminal to positive terminal and negative terminal to negative terminal). Likewise, when we say that port set 2 is connected to an n-port device, we assume that the ports of the n-port device are numbered from 1 to n, and that, for any integer $q \in \{1, \ldots, n\}$, its port q is connected to port q of port set 2 (positive terminal to positive terminal and negative terminal to negative terminal).

The two configurations are shown in Fig. 1. In configuration A (CA), port set 1 is connected to an LTI m-port generator of internal impedance matrix $\mathbf{Z}_{S1}$, and port set 2 is connected to an LTI n-port load of impedance matrix $\mathbf{Z}_{S2}$. In configuration B (CB), port set 1 is connected to an LTI m-port load of impedance matrix $\mathbf{Z}_{S1}$, and port set 2 is connected to an LTI n-port generator of internal impedance matrix $\mathbf{Z}_{S2}$. As in Part 1, we assume that the hermitian parts of $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are positive definite.

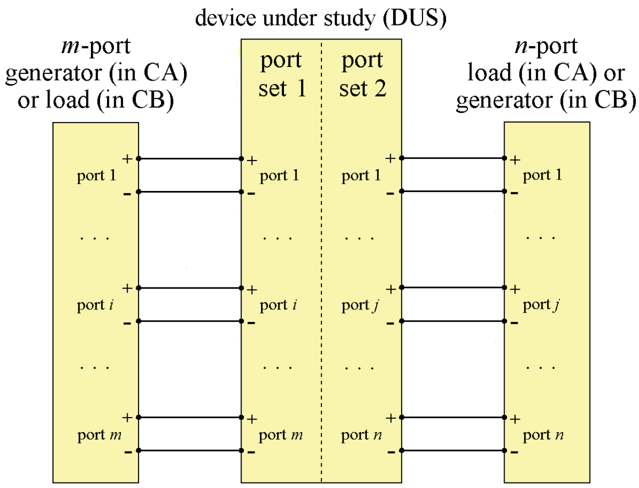

FIGURE 1. The two configurations, CA and CB, considered in the article.

The average power available from one or more ports, also referred to as "available power", is the greatest average power that can be drawn from these one or more ports by an arbitrary LTI and passive load [3, Sec. 3-8], [4]. Ignoring noise power contributions, we consider 8 average powers:

- P_{AAVG1} is the average power available from the generator connected to port set 1, in CA;
- P_{ARP1} means the average power received by port set 1, in CA;
- P_{AAVP2} means the average power available from port set 2, in CA;
- P_{ADP2} means the average power delivered by port set 2, in CA;
- P_{BAVG2} is the average power available from the generator connected to port set 2, in CB;
- P_{BRP2} means the average power received by port set 2, in CB;
- P_{BAVP1} means the average power available from port set 1, in CB; and
- P_{BDP1} means the average power delivered by port set 1, in CB.

In [5], it was shown that, in the case $m = n = 1$, these average powers satisfy

$$P_{ADP2}P_{BAVG2} = P_{AAVP2}P_{BRP2}\,, \tag{1}$$

$$P_{BDP1}P_{AAVG1} = P_{BAVP1}P_{ARP1}\,, \tag{2}$$

and, if the DUS is a reciprocal device,

$$\begin{aligned} P_{ADP2}P_{BAVG2} &= P_{AAVP2}P_{BRP2} \\ &= P_{BDP1}P_{AAVG1} = P_{BAVP1}P_{ARP1}\,. \end{aligned} \tag{3}$$

In [5], it was also shown that (1)–(3) can be used to obtain 6 reciprocal relations between 6 power ratios related to CA and 6 power ratios related to CB, these power ratios including 2 transducer power gains, 4 power transfer ratios, 2 operating power gains, 2 available power gains, and 2 unnamed power gains. Broadly speaking, the purpose of the present article is the extension of these 6 reciprocal relations to the general case $m \geqslant 1$ and $n \geqslant 1$. Appendix A lists corrections to [5].

The present work rests on results about generalized Rayleigh ratios originally presented in [6], an article on antenna theory. To avoid repeated references to [6], they are stated in a slightly revised form and proven in Section II. Section III is about our assumptions and simple or known results. In Section IV, we improve the reciprocal theorems about the transducer power gains and insertion power gains in CA and CB previously disclosed in Part 1.

Section V provides new results on operating power gains. Section VI discloses a new computation of P_{AAVP2} and P_{BAVP1}. Section VII is about power transfer ratios, and presents two new reciprocal theorems about them. Section VIII is about available power gains, and presents two new reciprocal theorems involving operating power gains and available power gains. Section IX is about unnamed power gains, and presents a new reciprocal theorem on them.

Section X treats some inequalitites involving power ratios. In Section XI, we derive relations applicable to a lossless DUS, among which several new results. Examples are provided in Section XII. Unnamed power gains are used in Section XIII to generalize the Friis transmission formula [7].

II. GENERALIZED RAYLEIGH RATIO

A. WHAT IS A GENERALIZED RAYLEIGH RATIO?

Let ν be a positive integer. The vector space of the complex column vectors of size ν is denoted by $\mathbb{C}^\nu$. For any $E \subset \mathbb{C}^\nu$, we use $E^\perp$ to denote the orthogonal complement of E, that is the set of all vectors in $\mathbb{C}^\nu$ that are orthogonal to every vector lying in E.

We use $\mathbf{1}_\nu$ to denote the identity matrix of size ν by ν. For a positive integer μ, the null matrix of size μ by ν is denoted by $\mathbf{0}_{\mu,\nu}$ or by $\mathbf{0}$ when no confusion may arise. We use $\mathrm{diag}_\nu(a_1, \ldots, a_\nu)$ to denote the diagonal matrix of diagonal entries $a_{11} = a_1$ to $a_{\nu\nu} = a_\nu$. Let $\mathbf{M}$ be a complex matrix. We use $\ker \mathbf{M}$ to denote the nullspace of $\mathbf{M}$, $\mathrm{rank}\,\mathbf{M}$ the rank of $\mathbf{M}$, $\mathbf{M}^T$ the transpose of $\mathbf{M}$, and $\mathbf{M}^*$ the hermitian adjoint of $\mathbf{M}$. If $\mathbf{M}$ is square, $\mathrm{tr}\,\mathbf{M}$ denotes the trace of $\mathbf{M}$.

Let $\mathbf{A}$ be a positive semidefinite matrix. We know [8, Sec. 7.2.6] that there exists a unique positive semidefinite matrix $\mathbf{B}$ such that $\mathbf{B}^2 = \mathbf{A}$. The matrix $\mathbf{B}$ is referred to as the unique positive semidefinite square root of $\mathbf{A}$, and is denoted by $\mathbf{A}^{1/2}$. If $\mathbf{A}$ is positive definite, $\mathbf{A}^{-1}$ and $\mathbf{A}^{1/2}$ are positive definite, and $(\mathbf{A}^{1/2})^{-1} = (\mathbf{A}^{-1})^{1/2}$, so that we can write $\mathbf{A}^{-1/2} = (\mathbf{A}^{1/2})^{-1} = (\mathbf{A}^{-1})^{1/2}$.

Observation 6. Let $\mathbf{A}$ be a positive semidefinite matrix of size ν by ν. For any $\mathbf{x} \in \mathbb{C}^\nu$, $\mathbf{x}^*\,\mathbf{A}\,\mathbf{x} = 0$ if and only if $\mathbf{x} \in \ker \mathbf{A}$.

Proof: If $\mathbf{x} \in \ker \mathbf{A}^{1/2}$, we have $\mathbf{A}\mathbf{x} = \mathbf{A}^{1/2}\mathbf{A}^{1/2}\mathbf{x} = \mathbf{A}^{1/2}\,\mathbf{0} = \mathbf{0}$, so that $\mathbf{x} \in \ker \mathbf{A}$. Conversely, let $\mathbf{x} \in \ker \mathbf{A}$. Since by [8, Sec. 7.2.6] there is a polynomial p with real coefficients such that $\mathbf{A}^{1/2} = p(\mathbf{A})$, we have $\mathbf{A}^{1/2}\mathbf{x} = p(\mathbf{A})\mathbf{x} = \mathbf{0}$, so that $\mathbf{x} \in \ker \mathbf{A}^{1/2}$.

We have proven that $\ker \mathbf{A}^{1/2} = \ker \mathbf{A}$.

For any $\mathbf{x} \in \mathbb{C}^\nu$, we have $\mathbf{x}^*\,\mathbf{A}\,\mathbf{x} = 0$ if and only if $\mathbf{x}^*\,\mathbf{A}^{1/2}\mathbf{A}^{1/2}\,\mathbf{x} = 0$ if and only if $(\mathbf{A}^{1/2}\,\mathbf{x})^*(\mathbf{A}^{1/2}\,\mathbf{x}) = 0$ if and only if $\mathbf{A}^{1/2}\,\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} \in \ker \mathbf{A}^{1/2}$.

Thus, $\mathbf{x}^*\,\mathbf{A}\,\mathbf{x} = 0$ if and only if $\mathbf{x} \in \ker \mathbf{A}$. □

Note that there are other proofs of this well-known result [8, Sec. 7.1.6].

Let $\mathbf{A}$ be an hermitian matrix of size ν by ν. The expression $\mathbf{x}^*\mathbf{A}\mathbf{x}/\mathbf{x}^*\mathbf{x}$, where $x \in \mathbb{C}^\nu$, is known as a Rayleigh ratio, or Rayleigh-Ritz ratio, or Rayleigh quotient [8, Sec. 4.2], [9, Sec. 4.2]. In this article, this concept is extended as follows. Let $\mathbf{N}$ and $\mathbf{D}$ be hermitian matrices of size ν by ν, $\mathbf{D}$ being positive semidefinite. The generalized Rayleigh ratio of $\mathbf{N}$ to $\mathbf{D}$ is a real-valued function $r : \mathbb{C}^\nu \to \mathbb{R}$ such that, for any $\mathbf{x} \in \mathbb{C}^\nu$ satisfying $\mathbf{x}^*\mathbf{D}\mathbf{x} \neq 0$, we have

$$r(\mathbf{x}) = \frac{\mathbf{x}^*\mathbf{N}\mathbf{x}}{\mathbf{x}^*\mathbf{D}\mathbf{x}}\,. \tag{4}$$

The generalized Rayleigh ratio r may be viewed as a ratio of two hermitian quadratic forms [10, Sec. 3.2.4], [11, Sec. 10.1] (also called "hermitian forms" [12, Ch. X]) in the variable $\mathbf{x}$: the hermitian quadratic form $f_\mathbf{N} : \mathbb{C}^\nu \to \mathbb{R}$ such that $f_\mathbf{N}(\mathbf{x}) = \mathbf{x}^*\mathbf{N}\mathbf{x}$ and the positive definite hermitian quadratic form $f_\mathbf{D} : \mathbb{C}^\nu \to \mathbb{R}$ such that $f_\mathbf{D}(\mathbf{x}) = \mathbf{x}^*\mathbf{D}\mathbf{x}$.

By Observation 6, the domain of definition of r, denoted by D_r, is

$$D_r = \{\mathbf{x} \in \mathbb{C}^\nu : \mathbf{x} \notin \ker \mathbf{D}\}\,, \tag{5}$$

where the colon means “such that”. Let $d = \dim \ker \mathbf{D}$ be the nullity of $\mathbf{D}$. By Observation 6, $\mathbf{D}$ is positive definite if and only if $d = 0$, that is to say if and only if $\ker \mathbf{D} = \{\mathbf{0}\}$.
Observation 7. Let $||\mathbf{x}||_2 = \sqrt{\mathbf{x}^*\mathbf{x}}$ be the euclidian vector norm of an arbitrary complex column vector $\mathbf{x}$. We use $\mathbb{S}_\nu$ to denote the hypersphere of the unit vectors of $\mathbb{C}^\nu$. It follows from (4) that, for $\mathbf{x} \neq \mathbf{0}$ and a fixed $\mathbf{x}/||\mathbf{x}||_2$, if $r(\mathbf{x})$ is defined, it does not depend on $||\mathbf{x}||_2$. Thus, the set of the values of $r(\mathbf{x})$ such that $\mathbf{x} \in D_r$ is equal to the set of the values of $r(\mathbf{x})$ such that $\mathbf{x} \in D_r \cap \mathbb{S}_\nu$.
Observation 8. If $\mathbf{N}$ is positive semidefinite, for any $\mathbf{x} \in D_r$ we have $r(\mathbf{x}) \geqslant 0$.

B. BOUNDS OF GENERALIZED RAYLEIGH RATIOS

To investigate the bounds of generalized Rayleigh ratios, we will first cover the special case where $\mathbf{D}$ is positive definite. Afterwards, we will address the general case, which is more involved.

Theorem 12. Let $\mathbf{N}$ and $\mathbf{D}$ be hermitian matrices of size ν by ν. We assume that $\mathbf{D}$ is positive definite. Let r be the generalized Rayleigh ratio of $\mathbf{N}$ to $\mathbf{D}$. Since $\mathbf{D}$ is positive definite, $D_r = \{\mathbf{x} \in \mathbb{C}^\nu : \mathbf{x} \neq \mathbf{0}\}$ and we can define

$$\mathbf{M} = \mathbf{D}^{-1/2}\mathbf{N}\mathbf{D}^{-1/2}\,. \tag{6}$$

$\mathbf{M}$ is of size ν by ν, and hermitian. Thus, its eigenvalues are real. Let $\lambda_{\max}$ be the largest eigenvalue of $\mathbf{M}$ and $\lambda_{\min}$ the smallest eigenvalue of $\mathbf{M}$. For any $\mathbf{x} \in \mathbb{C}^\nu$ satisfying $\mathbf{x} \neq \mathbf{0}$, we have

$$\lambda_{\min} = \min_{\mathbf{y}\neq\mathbf{0}} \frac{\mathbf{y}^*\mathbf{M}\mathbf{y}}{\mathbf{y}^*\mathbf{y}} \leqslant r(\mathbf{x}) \leqslant \lambda_{\max} = \max_{\mathbf{y}\neq\mathbf{0}} \frac{\mathbf{y}^*\mathbf{M}\mathbf{y}}{\mathbf{y}^*\mathbf{y}}\,. \tag{7}$$

Moreover,
- the equality $r(\mathbf{x}) = \lambda_{\max}$ is satisfied if and only if $\mathbf{x} = \mathbf{D}^{-1/2}\mathbf{y}$, where $\mathbf{y}$ is an eigenvector of $\mathbf{M}$ associated with $\lambda_{\max}$;
- the equality $r(\mathbf{x}) = \lambda_{\min}$ is satisfied if and only if $\mathbf{x} = \mathbf{D}^{-1/2}\mathbf{y}$, where $\mathbf{y}$ is an eigenvector of $\mathbf{M}$ associated with $\lambda_{\min}$; and
- $\mathbf{M}$ and $\mathbf{N}\mathbf{D}^{-1}$ are similar, so that the eigenvalues of $\mathbf{N}\mathbf{D}^{-1}$ are real, $\lambda_{\max}$ is the largest eigenvalue of $\mathbf{N}\mathbf{D}^{-1}$ and $\lambda_{\min}$ is the smallest eigenvalue of $\mathbf{N}\mathbf{D}^{-1}$.

Proof: For any $\mathbf{x} \in \mathbb{C}^\nu$, let $\mathbf{y} = \mathbf{D}^{1/2}\mathbf{x}$. Since $\mathbf{D}$ is positive definite, $\mathbf{x} \neq \mathbf{0}$ if and only if $\mathbf{y} \neq \mathbf{0}$, we have $\mathbf{x} = \mathbf{D}^{-1/2}\mathbf{y}$ and, for $\mathbf{y} \neq \mathbf{0}$, we get

$$r(\mathbf{x}) = \frac{(\mathbf{D}^{1/2}\mathbf{x})^*\mathbf{M}(\mathbf{D}^{1/2}\mathbf{x})}{(\mathbf{D}^{1/2}\mathbf{x})^*(\mathbf{D}^{1/2}\mathbf{x})} = \frac{\mathbf{y}^*\mathbf{M}\mathbf{y}}{\mathbf{y}^*\mathbf{y}}\,. \tag{8}$$

Using Rayleigh’s theorem [8, Sec. 4.2.2], we obtain (7). The other assertions of Theorem 12 relating to the equalities $r(\mathbf{x}) = \lambda_{\max}$ and $r(\mathbf{x}) = \lambda_{\min}$ result from Rayleigh’s theorem and the definition of $\mathbf{y}$. Moreover, we observe that

$$\mathbf{N}\mathbf{D}^{-1} = \mathbf{D}^{1/2}\mathbf{M}\mathbf{D}^{-1/2}\,, \tag{9}$$

so that $\mathbf{M}$ is similar to $\mathbf{N}\mathbf{D}^{-1}$. It follows that $\mathbf{M}$ and $\mathbf{N}\mathbf{D}^{-1}$ have the same eigenvalues, counting multiplicity, by [8, Sec. 1.3.4]. □

Observation 9. If $\mathbf{D}$ is positive definite and $\mathbf{N}$ is positive semidefinite, then $\mathbf{M}$ defined in Theorem 12 is positive semidefinite, so that $\lambda_{\min} \geqslant 0$.

Theorem 13. Let $\mathbf{N}$ and $\mathbf{D}$ be hermitian matrices of size ν by ν, $\mathbf{D}$ being positive semidefinite. Let r be the generalized Rayleigh ratio of $\mathbf{N}$ to $\mathbf{D}$, and let D_r be the domain of definition of r. Let $r(D_r)$ be the image of D_r under r. If $D_r \neq \varnothing$ and if there exists $\mathbf{x} \in \ker \mathbf{D}$ such that $\mathbf{x}^*\mathbf{N}\mathbf{x} \neq 0$, then $r(D_r)$ is not bounded.

Proof: We assume that $D_r \neq \varnothing$. It follows that there exists $\mathbf{y} \in D_r$. We have $\mathbf{y}^*\,\mathbf{D}\,\mathbf{y} \neq 0$. If there exists $\mathbf{x} \in \ker \mathbf{D}$ such that $\mathbf{x}^*\mathbf{N}\mathbf{x} \neq 0$, we observe that for any $\lambda \in \mathbb{R}$,

$$(\mathbf{x} + \lambda\mathbf{y})^*\,\mathbf{D}\,(\mathbf{x} + \lambda\mathbf{y}) = \lambda^2\,\mathbf{y}^*\,\mathbf{D}\,\mathbf{y}\,, \tag{10}$$

so that $\mathbf{x} + \lambda\mathbf{y} \in \ker \mathbf{D}$ if and only if $\lambda = 0$. It follows that we can define $g : \mathbb{R} \to \mathbb{R}$ such that for any $\lambda \neq 0$, $g(\lambda) = |r(\mathbf{x} + \lambda\mathbf{y})|$. For any nonzero $\lambda \in \mathbb{R}$, we have

$$g(\lambda) = \left|\frac{\mathbf{x}^*\mathbf{N}\mathbf{x} + \lambda(\mathbf{y}^*\mathbf{N}\mathbf{x} + \mathbf{x}^*\mathbf{N}\mathbf{y}) + \lambda^2\mathbf{y}^*\mathbf{N}\mathbf{y}}{\lambda^2\,\mathbf{y}^*\,\mathbf{D}\,\mathbf{y}}\right|\,, \tag{11}$$

which becomes arbitrarily large as λ approaches 0, because $\mathbf{x}^*\mathbf{N}\mathbf{x} \neq 0$. Thus, $r(D_r)$ is not bounded. □

Corollary 3. Let $\mathbf{N}$ and $\mathbf{D}$ be positive semidefinite matrices of size ν by ν. Let r be the generalized Rayleigh ratio of $\mathbf{N}$ to $\mathbf{D}$, and let D_r be the domain of definition of r. If $D_r \neq \varnothing$ and if $r(D_r)$ is bounded, then $\ker \mathbf{D} \subset \ker \mathbf{N}$.

Proof: We assume that $D_r \neq \varnothing$ and $r(D_r)$ is bounded. By Theorem 13, there is no $\mathbf{x} \in \ker \mathbf{D}$ such that $\mathbf{x}^*\mathbf{N}\mathbf{x} \neq 0$. Since $\mathbf{N}$ is positive semidefinite, we can use Observation 6 to conclude that there is no $\mathbf{x} \in \ker \mathbf{D}$ such that $\mathbf{x} \notin \ker \mathbf{N}$. □

Theorem 14. Let $\mathbf{N}$ and $\mathbf{D}$ be hermitian matrices of size ν by ν, $\mathbf{D}$ being positive semidefinite. Let r be the generalized Rayleigh ratio of $\mathbf{N}$ to $\mathbf{D}$, let D_r be the domain of definition of r, and let d be the nullity of $\mathbf{D}$. We assume that $D_r \neq \varnothing$ and $\ker \mathbf{D} \subset \ker \mathbf{N}$.

$\mathbf{D}$ being positive semidefinite, it has ν eigenvalues, counting multiplicity, and these values are real and nonnegative by [8, Sec. 7.2.1]. Let us label these eigenvalues according to a non-decreasing order $\mu_1, \ldots, \mu_\nu$. Since $D_r \neq \varnothing$, we have $d \leqslant \nu - 1$, so that $0 < \mu_{d+1} \leqslant \ldots \leqslant \mu_\nu$. For any positive integer i such that $i \leqslant d$, we have $\mu_i = 0$. $\mathbf{D}$ being hermitian, by [8, Sec. 2.5.6] there exists a unitary matrix $\mathbf{L}$ of size ν by ν such that

$$\mathbf{D} = \mathbf{L}\,\mathrm{diag}_\nu(\mu_1, \ldots, \mu_\nu)\,\mathbf{L}^* \tag{12}$$

For any $i \in \{1, \ldots, \nu\}$, let the i-th column vector of $\mathbf{L}$ be denoted by $\mathbf{L}^{<i>}$. Let $\mathcal{L}$ be the submatrix of $\mathbf{L}$, of size ν by

$\nu - d$, whose column vectors are $\mathbf{L}^{<d+1>}, \ldots, \mathbf{L}^{<\nu>}$, in this order. Let

$$\mathbf{P} = \mathcal{L}\,\mathrm{diag}_{\nu-d}\left(\frac{1}{\sqrt{\mu_{d+1}}}, \ldots, \frac{1}{\sqrt{\mu_\nu}}\right) \tag{13}$$

and

$$\mathbf{Q} = \mathbf{P}^*\,\mathbf{N}\,\mathbf{P}\,. \tag{14}$$

The matrix $\mathbf{P}$ is of size ν by $\nu - d$. The matrix $\mathbf{Q}$ is clearly hermitian, of size $\nu - d$ by $\nu - d$. Thus, its eigenvalues are real. Let $\kappa_{\max}$ be the largest eigenvalue of $\mathbf{Q}$ and $\kappa_{\min}$ the smallest eigenvalue of $\mathbf{Q}$. For any $\mathbf{x} \in D_r$, we have

$$\kappa_{\min} = \min_{\mathbf{u}\neq\mathbf{0}} \frac{\mathbf{u}^*\mathbf{Q}\mathbf{u}}{\mathbf{u}^*\mathbf{u}} \leqslant r(\mathbf{x}) \leqslant \kappa_{\max} = \max_{\mathbf{u}\neq\mathbf{0}} \frac{\mathbf{u}^*\mathbf{Q}\mathbf{u}}{\mathbf{u}^*\mathbf{u}}\,. \tag{15}$$

Moreover,

- we have $r(\mathbf{x}) = \kappa_{\max}$ if $\mathbf{x} = \mathbf{P}\mathbf{u}$, where $\mathbf{u}$ is an eigenvector of $\mathbf{Q}$ associated with $\kappa_{\max}$;
- we have $r(\mathbf{x}) = \kappa_{\min}$ if $\mathbf{x} = \mathbf{P}\mathbf{u}$, where $\mathbf{u}$ is an eigenvector of $\mathbf{Q}$ associated with $\kappa_{\min}$; and
- $\mathbf{Q}$ is similar to

$$\mathbf{R} = \mathcal{L}^*\mathbf{N}\mathcal{L}\mathrm{diag}_{\nu-d}\left(\frac{1}{\mu_{d+1}}, \ldots, \frac{1}{\mu_\nu}\right)\,, \tag{16}$$

so that the eigenvalues of $\mathbf{R}$ are real, $\kappa_{\max}$ is the largest eigenvalue of $\mathbf{R}$ and $\kappa_{\min}$ the smallest eigenvalue of $\mathbf{R}$.

Proof: Since $\mathbf{D}\mathbf{L} = \mathbf{L}\mathrm{diag}_\nu(\mu_1, \ldots, \mu_\nu)$, we know that, for any $i \in \{1, \ldots, \nu\}$, $\mathbf{L}^{<i>}$ is an eigenvector of $\mathbf{D}$ associated with the eigenvalue μ_i. It follows that $\mathbf{L}^{<1>}$ to $\mathbf{L}^{<d>}$ are vectors of $\ker \mathbf{D}$. $\mathbf{L}$ being unitary, $(\mathbf{L}^{<1>}, \ldots, \mathbf{L}^{<\nu>})$ is an orthonormal basis of $\mathbb{C}^\nu$. Thus, $(\mathbf{L}^{<d+1>}, \ldots, \mathbf{L}^{<\nu>})$ is an orthonormal basis of $(\ker \mathbf{D})^\perp$.

For any $\mathbf{x} \in \mathbb{C}^\nu$, there is a unique $p_1(\mathbf{x}) \in \ker \mathbf{D}$, and a unique $p_2(\mathbf{x}) \in (\ker \mathbf{D})^\perp$ such that $\mathbf{x} = p_1(\mathbf{x}) + p_2(\mathbf{x})$. We have $\mathbf{x}^*\,\mathbf{D}\,\mathbf{x} = p_2(\mathbf{x})^*\,\mathbf{D}\,p_2(\mathbf{x})$. Thus, if $\mathbf{x} \in D_r$, then $p_2(\mathbf{x}) \neq \mathbf{0}$. Since we assume that $\ker \mathbf{D} \subset \ker \mathbf{N}$, we also have $\mathbf{x}^*\,\mathbf{N}\,\mathbf{x} = p_2(\mathbf{x})^*\,\mathbf{N}\,p_2(\mathbf{x})$. Thus, we can assert that, if $\mathbf{x} \in D_r$, then

$$r(\mathbf{x}) = \frac{p_2(\mathbf{x})^*\,\mathbf{N}\,p_2(\mathbf{x})}{p_2(\mathbf{x})^*\,\mathbf{D}\,p_2(\mathbf{x})} = r(\mathbf{p_2}(\mathbf{x}))\,. \tag{17}$$

It follows that

$$r(D_r) = r((\ker \mathbf{D})^\perp)\,. \tag{18}$$

Let $\mathbf{x} \in D_r$ and $\mathbf{z} = p_2(\mathbf{x})$. Let $\zeta_{d+1}, \ldots, \zeta_\nu$ be the coordinates of $\mathbf{z}$ in the basis $(\mathbf{L}^{<d+1>}, \ldots, \mathbf{L}^{<\nu>})$ of $(\ker \mathbf{D})^\perp$. We introduce a column vector of size $\nu - d$, given by

$$\mathfrak{z} = \begin{pmatrix} \zeta_{d+1} \\ \vdots \\ \zeta_\nu \end{pmatrix}. \tag{19}$$

The product $\mathcal{L}\,\mathfrak{z}$ is a column vector of size ν. Using the rule for the multiplication of block matrices, we find

$$\mathcal{L}\,\mathfrak{z} = \sum_{i=d+1}^{\nu} \mathbf{L}^{<i>}\zeta_i = \mathbf{z}\,. \tag{20}$$

Using (17) and (20), we get

$$r(\mathbf{x}) = \frac{\mathfrak{z}^*\mathcal{L}^*\,\mathbf{N}\,\mathcal{L}\,\mathfrak{z}}{\mathfrak{z}^*\mathcal{L}^*\,\mathbf{D}\,\mathcal{L}\,\mathfrak{z}}\,, \tag{21}$$

and (12) leads us to

$$r(\mathbf{x}) = \frac{\mathfrak{z}^*\mathcal{L}^*\,\mathbf{N}\,\mathcal{L}\,\mathfrak{z}}{\mathfrak{z}^*\mathcal{L}^*\,\mathbf{L}\,\mathrm{diag}_\nu(\mu_1, \ldots, \mu_\nu)\,\mathbf{L}^*\,\mathcal{L}\,\mathfrak{z}}\,. \tag{22}$$

$\mathbf{L}^*\mathcal{L}$ is of size ν by $\nu - d$. Since $\mathbf{L}$ is unitary, we find that $\mathbf{L}^*\mathcal{L}$ is given by

$$\mathbf{L}^*\mathcal{L} = \begin{pmatrix} \mathbf{0}_{d,\,\nu-d} \\ \mathbf{1}_{\nu-d} \end{pmatrix}. \tag{23}$$

Using (22) and (23), we obtain

$$r(\mathbf{z}) = \frac{\mathfrak{z}^*\mathcal{L}^*\,\mathbf{N}\,\mathcal{L}\,\mathfrak{z}}{\mathfrak{z}^*\,\mathrm{diag}_{\nu-d}(\mu_{d+1}, \ldots, \mu_\nu)\,\mathfrak{z}} = \frac{\mathbf{u}^*\mathbf{Q}\mathbf{u}}{\mathbf{u}^*\mathbf{u}}\,, \tag{24}$$

where $\mathbf{u} = \mathrm{diag}_{\nu-d}(\mu_{d+1}, \ldots, \mu_\nu)^{1/2}\mathfrak{z}$, so that we have $\mathcal{L}\,\mathfrak{z} = \mathbf{P}\mathbf{u}$. Since $\mathfrak{z}$ is the column vector of the coordinates of $\mathbf{z}$ in the basis $(\mathbf{L}^{<d+1>}, \ldots, \mathbf{L}^{<\nu>})$ of $(\ker \mathbf{D})^\perp$, it follows from (18) that $r(D_r)$ is the set of all $r(\mathbf{z})$ given by (24) when $\mathfrak{z}$ takes on any nonzero value in $\mathbb{C}^{\nu-d}$. Thus, using Theorem 12, we obtain (15), and

- we have $r(\mathbf{x}) = \kappa_{\max}$ if we have $\mathbf{x} = \mathcal{L}\,\mathfrak{z}'$ in which $\mathfrak{z}' = \mathrm{diag}_{\nu-d}(\mu_{d+1}, \ldots, \mu_\nu)^{-1/2}\mathbf{u}$, where $\mathbf{u}$ is an eigenvector of $\mathbf{Q}$ associated with $\kappa_{\max}$;
- we have $r(\mathbf{x}) = \kappa_{\min}$ if we have $\mathbf{x} = \mathcal{L}\,\mathfrak{z}'$ in which $\mathfrak{z}' = \mathrm{diag}_{\nu-d}(\mu_{d+1}, \ldots, \mu_\nu)^{-1/2}\mathbf{u}$, where $\mathbf{u}$ is an eigenvector of $\mathbf{Q}$ associated with $\kappa_{\min}$; and
- $\mathbf{Q}$ is similar to $\mathbf{R}$ given by (16), so that the eigenvalues of $\mathbf{R}$ are real, $\kappa_{\max}$ is the largest eigenvalue of $\mathbf{R}$ and $\kappa_{\min}$ the smallest eigenvalue of $\mathbf{R}$.

This leads to the final results of Theorem 14. □

In the case $d = 0$, we can use Theorem 12 and Theorem 14, the latter giving the same results as the former.

Corollary 4. Let $\mathbf{N}$ and $\mathbf{D}$ be positive semidefinite matrices of size ν by ν. Let r be the generalized Rayleigh ratio of $\mathbf{N}$ to $\mathbf{D}$, and let D_r be the domain of definition of r. We assume that $D_r \neq \varnothing$. Then $r(D_r)$ is bounded if and only if $\ker \mathbf{D} \subset \ker \mathbf{N}$.

Proof: This is a direct consequence of Corollary 3 and Theorem 14. □

C. RELATED RESULTS THAT DO NOT USE A RATIO

Corollary 5. Let $\mathbf{N}$ and $\mathbf{D}$ be hermitian matrices of size ν by ν. We assume that $\mathbf{D}$ is positive definite, so that we can define $\mathbf{M} = \mathbf{D}^{-1/2}\mathbf{N}\mathbf{D}^{-1/2}$. The matrix $\mathbf{M}$ is of size ν by ν, and hermitian. Thus, its eigenvalues are real. Let $\lambda_{\max}$ be the largest eigenvalue of $\mathbf{M}$ and $\lambda_{\min}$ the smallest eigenvalue of $\mathbf{M}$. For any $\mathbf{x} \in \mathbb{C}^\nu$, we have

$$\lambda_{\min}\,\mathbf{x}^*\mathbf{D}\mathbf{x} \leqslant \mathbf{x}^*\mathbf{N}\mathbf{x} \leqslant \lambda_{\max}\,\mathbf{x}^*\mathbf{D}\mathbf{x}\,. \tag{25}$$

Moreover,

- we have $\mathbf{x}^*\mathbf{N}\mathbf{x} = \lambda_{\max}\,\mathbf{x}^*\mathbf{D}\mathbf{x}$ if $\mathbf{x} = \mathbf{D}^{-1/2}\mathbf{y}$, where $\mathbf{y}$ is an eigenvector of $\mathbf{M}$ associated with $\lambda_{\max}$;
- we have $\mathbf{x}^*\mathbf{N}\mathbf{x} = \lambda_{\min}\,\mathbf{x}^*\mathbf{D}\mathbf{x}$ if $\mathbf{x} = \mathbf{D}^{-1/2}\mathbf{y}$, where $\mathbf{y}$ is an eigenvector of $\mathbf{M}$ associated with $\lambda_{\min}$; and
- $\mathbf{M}$ and $\mathbf{N}\mathbf{D}^{-1}$ are similar, so that the eigenvalues of $\mathbf{N}\mathbf{D}^{-1}$ are real, $\lambda_{\max}$ is the largest eigenvalue of $\mathbf{N}\mathbf{D}^{-1}$ and $\lambda_{\min}$ the smallest eigenvalue of $\mathbf{N}\mathbf{D}^{-1}$.

Proof: This is a direct consequence of Theorem 12. □

Corollary 6. Let $\mathbf{N}$ and $\mathbf{D}$ be hermitian matrices of size ν by ν, $\mathbf{D}$ being positive semidefinite. Let d be the nullity of $\mathbf{D}$. We assume that $\mathbf{D} \neq \mathbf{0}$ and $\ker \mathbf{D} \subset \ker \mathbf{N}$.

$\mathbf{D}$ being positive semidefinite, it has ν eigenvalues, counting multiplicity, and these values are real and nonnegative. Let us label these eigenvalues according to a non-decreasing order $\mu_1, \ldots, \mu_\nu$. Since $\mathbf{D} \neq \mathbf{0}$, we have $d \leqslant \nu - 1$, so that $0 < \mu_{d+1} \leqslant \ldots \leqslant \mu_\nu$. For any positive integer i such that $i \leqslant d$, we have $\mu_i = 0$. $\mathbf{D}$ being hermitian, there exists a unitary matrix $\mathbf{L}$ of size ν by ν such that $\mathbf{D} = \mathbf{L}\mathrm{diag}_\nu(\mu_1, \ldots, \mu_\nu)\mathbf{L}^*$.

For any $i \in \{1, \ldots, \nu\}$, let the i-th column vector of $\mathbf{L}$ be denoted by $\mathbf{L}^{<i>}$. Let $\mathcal{L}$ be the submatrix of $\mathbf{L}$, of size ν by $\nu - d$, whose column vectors are $\mathbf{L}^{<d+1>}, \ldots, \mathbf{L}^{<\nu>}$, in this order. Let $\mathbf{P} = \mathcal{L}\,\mathrm{diag}_{\nu-d}(\mu_{d+1}^{-1/2}, \ldots, \mu_\nu^{-1/2})$ and $\mathbf{Q} = \mathbf{P}^*\,\mathbf{N}\,\mathbf{P}$. The matrix $\mathbf{Q}$ is hermitian, of size $\nu - d$ by $\nu - d$. Thus, its eigenvalues are real. Let $\kappa_{\max}$ be the largest eigenvalue of $\mathbf{Q}$ and $\kappa_{\min}$ the smallest eigenvalue of $\mathbf{Q}$. For any $\mathbf{x} \in \mathbb{C}^\nu$, we have

$$\kappa_{\min}\,\mathbf{x}^*\mathbf{D}\mathbf{x} \leqslant \mathbf{x}^*\mathbf{N}\mathbf{x} \leqslant \kappa_{\max}\,\mathbf{x}^*\mathbf{D}\mathbf{x}\,. \tag{26}$$

Moreover,

- we have $\mathbf{x}^*\mathbf{N}\mathbf{x} = \kappa_{\max}\,\mathbf{x}^*\mathbf{D}\mathbf{x}$ if $\mathbf{x} = \mathbf{P}\mathbf{u}$, where $\mathbf{u}$ is an eigenvector of $\mathbf{Q}$ associated with $\kappa_{\max}$;
- we have $\mathbf{x}^*\mathbf{N}\mathbf{x} = \kappa_{\min}\,\mathbf{x}^*\mathbf{D}\mathbf{x}$ if $\mathbf{x} = \mathbf{P}\mathbf{u}$, where $\mathbf{u}$ is an eigenvector of $\mathbf{Q}$ associated with $\kappa_{\min}$; and
- $\mathbf{Q}$ is similar to $\mathbf{R} = \mathcal{L}^*\mathbf{N}\mathcal{L}\,\mathrm{diag}_{\nu-d}(\mu_{d+1}^{-1}, \ldots, \mu_\nu^{-1})$, so that the eigenvalues of $\mathbf{R}$ are real, $\kappa_{\max}$ is the largest eigenvalue of $\mathbf{R}$ and $\kappa_{\min}$ the smallest eigenvalue of $\mathbf{R}$.

Proof: This is a direct consequence of Theorem 14. □

D. LAST REMARKS

Theorem 12 is a consequence of the results on pencils of quadratic forms and pencils of hermitian forms presented in sections 7 and 9 of [12, Ch. X]). Special cases of Corollary 5 were obtained in Theorem 3 and Theorem 5 of Part 1, and in Theorem 7 of Part 2. It seems that results similar to Theorem 13, Theorem 14, Corollary 3, Corollary 4 and Corollary 6 were first stated and proven in [6, Sec. II].

Examples of generalized Rayleigh ratios, together with different methods of computing the least upper bound and greatest lower bound of $r(\mathbf{x})$ for $\mathbf{x} \in D_r$, were provided in [6, Sec. III] and are not repeated here.

III. ASSUMPTIONS, MISCELLANEOUS RESULTS AND SIMPLE FORMULAE ON AVERAGE POWERS

A. NOTATIONS, ASSUMPTIONS AND BASIC RESULTS

In the special case where $m = n$, in addition to the powers defined in Section I, we can consider two additional average powers:

- P_{AW} is the average power which would be received by the load connected at port set 2 in CA, if the DUS was not present and this load was directly connected to the generator connected at port set 1 in CA; and
- P_{BW} is the average power which would be received by the load connected at port set 1 in CB, if the DUS was not present and this load was directly connected to the generator connected at port set 2 in CB.

Let $\mathbf{M}$ be a square complex matrix. We use $H(\mathbf{M})$ to denote the hermitian part of $\mathbf{M}$. As said above, we assume that the DUS is LTI and passive, that the generators and the loads are LTI, and that $H(\mathbf{Z}_{S1})$ and $H(\mathbf{Z}_{S2})$ are positive definite. As explained in Section IV of Part 1, this ensures that the loads are passive and that P_{AAVG1} and P_{BAVG2} are defined. The DUS being a passive $(m+n)$-port, it follows that:

$$0 \leqslant P_{ADP2} \leqslant P_{ARP1} \leqslant P_{AAVG1}\,; \tag{27}$$

P_{AAVP2} is defined and satisfies

$$0 \leqslant P_{ADP2} \leqslant P_{AAVP2} \leqslant P_{AAVG1}\,; \tag{28}$$

$$0 \leqslant P_{AW} \leqslant P_{AAVG1}\,; \tag{29}$$

$$0 \leqslant P_{BDP1} \leqslant P_{BRP2} \leqslant P_{BAVG2}\,; \tag{30}$$

P_{BAVP1} is defined and satisfies

$$0 \leqslant P_{BDP1} \leqslant P_{BAVP1} \leqslant P_{BAVG2}\,; \tag{31}$$

and

$$0 \leqslant P_{BW} \leqslant P_{BAVG2}\,. \tag{32}$$

By Lemma 1 of Part 1, we can assert that:

- we can define $\mathbf{Y}_{S1} = \mathbf{Z}_{S1}^{-1}$ and $\mathbf{Y}_{S2} = \mathbf{Z}_{S2}^{-1}$;
- $H(\mathbf{Y}_{S1})$ and $H(\mathbf{Y}_{S2})$ are positive definite; and
- instead of assuming that $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ exist and are such that $H(\mathbf{Z}_{S1})$ and $H(\mathbf{Z}_{S2})$ are positive definite, we could equivalently have assumed that $\mathbf{Y}_{S1}$ and $\mathbf{Y}_{S2}$ exist and are such that $H(\mathbf{Y}_{S1})$ and $H(\mathbf{Y}_{S2})$ are positive definite.

We use $\mathbf{V}_{O1}$ and $\mathbf{I}_{S1} = \mathbf{Y}_{S1}\mathbf{V}_{O1}$ to denote the column vectors of the rms open-circuit voltages and of the rms short-circuit currents, respectively, of the m-port generator connected to port set 1 in CA. We use $\mathbf{V}_{O2}$ and $\mathbf{I}_{S2} = \mathbf{Y}_{S2}\mathbf{V}_{O2}$ to denote the column vectors of the rms open-circuit voltages and of the rms short-circuit currents, respectively, of the n-port generator connected to port set 2 in CB. We use $\mathbf{V}_1$ and $\mathbf{I}_1$ to denote the column vectors of the rms voltages across port set 1 and of the rms currents flowing into port set 1, respectively, in a specified configuration. We use $\mathbf{V}_2$ and $\mathbf{I}_2$ to denote the column vectors of the rms voltages across port set 2 and of the rms currents flowing into port set 2, respectively, in a specified configuration.

B. AUGMENTED MULTIPORTS

As in Section IV of Part 1, we consider the ports of the DUS in the following order: ports 1 to m of port set 1, and then ports 1 to n of port set 2.

We introduce a parallel-augmented multiport, as defined in Section II of Part 1, composed of the DUS (as original multiport), of an m-port load of admittance matrix $\mathbf{Y}_{S1}$ connected in parallel with port set 1, and of an n-port load of admittance matrix $\mathbf{Y}_{S2}$ connected in parallel with port set 2. Here, the admittance matrix of the added multiport is

$$\mathbf{Y}_{ADD} = \begin{pmatrix} \mathbf{Y}_{S1} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}_{S2} \end{pmatrix} . \quad (33)$$

$H(\mathbf{Y}_{S1})$ and $H(\mathbf{Y}_{S2})$ being positive definite, $H(\mathbf{Y}_{ADD})$ is positive definite. By Theorem 1 of Part 1, the parallel-augmented multiport has an impedance matrix $\mathbf{Z}_{PAM}$. The matrix $\mathbf{Z}_{PAM}$ is of size $(m+n)$ by $(m+n)$ and it may be partitioned into four submatrices, $\mathbf{Z}_{PAM11}$ of size m by m, $\mathbf{Z}_{PAM12}$ of size m by n, $\mathbf{Z}_{PAM21}$ of size n by m and $\mathbf{Z}_{PAM22}$ of size n by n, which are such that

$$\mathbf{Z}_{PAM} = \begin{pmatrix} \mathbf{Z}_{PAM11} & \mathbf{Z}_{PAM12} \\ \mathbf{Z}_{PAM21} & \mathbf{Z}_{PAM22} \end{pmatrix} . \quad (34)$$

By Theorem 1 of Part 1, if $\mathbf{Y}_{S1}$ and $\mathbf{Y}_{S2}$ are symmetric and the original multiport is a reciprocal device, then $\mathbf{Z}_{PAM}$ is symmetric. By Corollary 1 of Part 1, in the special case where the DUS has an admittance matrix $\mathbf{Y}$, then: $\mathbf{Z}_{PAM}$ is invertible;

$$\mathbf{Z}_{PAM}^{-1} = \mathbf{Y} + \mathbf{Y}_{ADD} ; \quad (35)$$

and, if $\mathbf{Y}_{ADD}$ is symmetric, $\mathbf{Z}_{PAM}$ is symmetric if and only if $\mathbf{Y}$ is symmetric.

We also introduce a series-augmented multiport, as defined in Section II of Part 1, composed of the DUS (as original multiport), of an m-port load of impedance matrix $\mathbf{Z}_{S1}$ connected in series with port set 1, and of an n-port load of impedance matrix $\mathbf{Z}_{S2}$ connected in series with port set 2. Here, the impedance matrix of the added multiport is

$$\mathbf{Z}_{ADD} = \begin{pmatrix} \mathbf{Z}_{S1} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_{S2} \end{pmatrix} = \mathbf{Y}_{ADD}^{-1} . \quad (36)$$

$H(\mathbf{Z}_{S1})$ and $H(\mathbf{Z}_{S2})$ being positive definite, $H(\mathbf{Z}_{ADD})$ is positive definite. By Theorem 2 of Part 1, the series-augmented multiport has an admittance matrix $\mathbf{Y}_{SAM}$. The matrix $\mathbf{Y}_{SAM}$ is of size $(m+n)$ by $(m+n)$ and it may be partitioned into four submatrices, $\mathbf{Y}_{SAM11}$ of size m by m, $\mathbf{Y}_{SAM12}$ of size m by n, $\mathbf{Y}_{SAM21}$ of size n by m and $\mathbf{Y}_{SAM22}$ of size n by n, which are such that

$$\mathbf{Y}_{SAM} = \begin{pmatrix} \mathbf{Y}_{SAM11} & \mathbf{Y}_{SAM12} \\ \mathbf{Y}_{SAM21} & \mathbf{Y}_{SAM22} \end{pmatrix} . \quad (37)$$

By Theorem 2 of Part 1, if $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are symmetric and the original multiport is a reciprocal device, then $\mathbf{Y}_{SAM}$ is symmetric. By Corollary 2 of Part 1, in the special case where the DUS has an impedance matrix $\mathbf{Z}$, then: $\mathbf{Y}_{SAM}$ is invertible;

$$\mathbf{Y}_{SAM}^{-1} = \mathbf{Z} + \mathbf{Z}_{ADD} ; \quad (38)$$

and, if $\mathbf{Z}_{ADD}$ is symmetric, $\mathbf{Y}_{SAM}$ is symmetric if and only if $\mathbf{Z}$ is symmetric.

C. FORMULAS USING THE OPEN-CIRCUIT VOLTAGES

We want to compute some of the above-defined average powers, using the open-circuit voltages of the generators to define the excitations, and $\mathbf{Y}_{SAM}$ to define the DUS. Ignoring noise power contributions, and using the fact that, $H(\mathbf{Z}_{S1})$ and $H(\mathbf{Z}_{S2})$ being positive definite, they are invertible, we get [4], [13]:

$$P_{AAVG1} = \mathbf{V}_{O1}^{*} \mathbf{Y}_{AAVG1} \mathbf{V}_{O1} , \quad (39)$$

where the admittance matrix

$$\mathbf{Y}_{AAVG1} = \frac{1}{2} \left(\mathbf{Z}_{S1} + \mathbf{Z}_{S1}^{*}\right)^{-1} \quad (40)$$

is positive definite; and

$$P_{BAVG2} = \mathbf{V}_{O2}^{*} \mathbf{Y}_{BAVG2} \mathbf{V}_{O2} , \quad (41)$$

where the admittance matrix

$$\mathbf{Y}_{BAVG2} = \frac{1}{2} \left(\mathbf{Z}_{S2} + \mathbf{Z}_{S2}^{*}\right)^{-1} \quad (42)$$

is positive definite.

By inspection, ignoring noise power contributions, we find:

$$P_{ARP1} = \mathbf{V}_{O1}^{*} \mathbf{Y}_{ARP1} \mathbf{V}_{O1} , \quad (43)$$

where the admittance matrix

$$\mathbf{Y}_{ARP1} = \frac{\mathbf{Y}_{SAM11} + \mathbf{Y}_{SAM11}^{*}}{2} - \mathbf{Y}_{SAM11}^{*} \frac{\mathbf{Z}_{S1} + \mathbf{Z}_{S1}^{*}}{2} \mathbf{Y}_{SAM11} \quad (44)$$

is positive semidefinite because the DUS and the n-port load connected to port set 2 in CA are passive, and because $\mathbf{V}_{O1}$ can take on any value lying in $\mathbb{C}^m$;

$$P_{ADP2} = \mathbf{V}_{O1}^{*} \mathbf{Y}_{ADP2} \mathbf{V}_{O1} , \quad (45)$$

where the admittance matrix

$$\mathbf{Y}_{ADP2} = \mathbf{Y}_{SAM21}^{*} \frac{\mathbf{Z}_{S2} + \mathbf{Z}_{S2}^{*}}{2} \mathbf{Y}_{SAM21} \quad (46)$$

is positive semidefinite because the DUS and the n-port load connected to port set 2 in CA are passive, and because $\mathbf{V}_{O1}$ can take on any value lying in $\mathbb{C}^m$;

$$P_{AW} = \mathbf{V}_{O1}^{*} \mathbf{Y}_{AW} \mathbf{V}_{O1} , \quad (47)$$

where the admittance matrix

$$\mathbf{Y}_{AW} = \left(\mathbf{Z}_{S1} + \mathbf{Z}_{S2}\right)^{-1*} \times \frac{\mathbf{Z}_{S2} + \mathbf{Z}_{S2}^{*}}{2} \left(\mathbf{Z}_{S1} + \mathbf{Z}_{S2}\right)^{-1} \quad (48)$$

is positive definite;

$$P_{BRP2} = \mathbf{V}_{O2}^{*} \mathbf{Y}_{BRP2} \mathbf{V}_{O2} , \quad (49)$$

where the admittance matrix

$$\mathbf{Y}_{BRP2} = \frac{\mathbf{Y}_{SAM22} + \mathbf{Y}_{SAM22}^{*}}{2} - \mathbf{Y}_{SAM22}^{*} \frac{\mathbf{Z}_{S2} + \mathbf{Z}_{S2}^{*}}{2} \mathbf{Y}_{SAM22} \quad (50)$$

is positive semidefinite because the DUS and the m-port load connected to port set 1 in CB are passive, and because $\mathbf{V}_{O2}$ can take on any value lying in $\mathbb{C}^n$;

$$P_{BDP1} = \mathbf{V}_{O2}^* \mathbf{Y}_{BDP1} \mathbf{V}_{O2}\,, \tag{51}$$

where the admittance matrix

$$\mathbf{Y}_{BDP1} = \mathbf{Y}_{SAM12}^* \frac{\mathbf{Z}_{S1} + \mathbf{Z}_{S1}^*}{2} \mathbf{Y}_{SAM12} \tag{52}$$

is positive semidefinite because the DUS and the m-port load connected to port set 1 in CB are passive, and because $\mathbf{V}_{O2}$ can take on any value lying in $\mathbb{C}^n$; and

$$P_{BW} = \mathbf{V}_{O2}^* \mathbf{Y}_{BW} \mathbf{V}_{O2}\,, \tag{53}$$

where the admittance matrix

$$\mathbf{Y}_{BW} = (\mathbf{Z}_{S1} + \mathbf{Z}_{S2})^{-1*} \times \frac{\mathbf{Z}_{S1} + \mathbf{Z}_{S1}^*}{2} (\mathbf{Z}_{S1} + \mathbf{Z}_{S2})^{-1} \tag{54}$$

is positive definite.

D. FORMULAS USING THE SHORT-CIRCUIT CURRENTS

We now wish to compute the same average powers as in (39)–(54), using the short-circuit currents of the generators to define the excitations, and $\mathbf{Z}_{PAM}$ to define the DUS. Ignoring noise power contributions, and using the fact that $H(\mathbf{Y}_{S1})$ and $H(\mathbf{Y}_{S2})$ are invertible, we obtain:

$$P_{AAVG1} = \mathbf{I}_{S1}^* \mathbf{Z}_{AAVG1} \mathbf{I}_{S1}\,, \tag{55}$$

where the impedance matrix

$$\mathbf{Z}_{AAVG1} = \frac{1}{2} (\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*)^{-1} \tag{56}$$

is positive definite; and

$$P_{BAVG2} = \mathbf{I}_{S2}^* \mathbf{Z}_{BAVG2} \mathbf{I}_{S2}\,, \tag{57}$$

where the impedance matrix

$$\mathbf{Z}_{BAVG2} = \frac{1}{2} (\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*)^{-1} \tag{58}$$

is positive definite.

By inspection, ignoring noise power contributions, we get:

$$P_{ARP1} = \mathbf{I}_{S1}^* \mathbf{Z}_{ARP1} \mathbf{I}_{S1}\,, \tag{59}$$

where the impedance matrix

$$\mathbf{Z}_{ARP1} = \frac{\mathbf{Z}_{PAM11} + \mathbf{Z}_{PAM11}^*}{2} - \mathbf{Z}_{PAM11}^* \frac{\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*}{2} \mathbf{Z}_{PAM11} \tag{60}$$

is positive semidefinite because the DUS and the n-port load connected to port set 2 in CA are passive, and because $\mathbf{I}_{S1}$ can take on any value lying in $\mathbb{C}^m$;

$$P_{ADP2} = \mathbf{I}_{S1}^* \mathbf{Z}_{ADP2} \mathbf{I}_{S1}\,, \tag{61}$$

where the impedance matrix

$$\mathbf{Z}_{ADP2} = \mathbf{Z}_{PAM21}^* \frac{\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*}{2} \mathbf{Z}_{PAM21} \tag{62}$$

is positive semidefinite because the DUS and the n-port load connected to port set 2 in CA are passive, and because $\mathbf{I}_{S1}$ can take on any value lying in $\mathbb{C}^m$;

$$P_{AW} = \mathbf{I}_{S1}^* \mathbf{Z}_{AW} \mathbf{I}_{S1}\,, \tag{63}$$

where the impedance matrix

$$\mathbf{Z}_{AW} = (\mathbf{Y}_{S1} + \mathbf{Y}_{S2})^{-1*} \times \frac{\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*}{2} (\mathbf{Y}_{S1} + \mathbf{Y}_{S2})^{-1} \tag{64}$$

is positive definite;

$$P_{BRP2} = \mathbf{I}_{S2}^* \mathbf{Z}_{BRP2} \mathbf{I}_{S2}\,, \tag{65}$$

where the impedance matrix

$$\mathbf{Z}_{BRP2} = \frac{\mathbf{Z}_{PAM22} + \mathbf{Z}_{PAM22}^*}{2} - \mathbf{Z}_{PAM22}^* \frac{\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*}{2} \mathbf{Z}_{PAM22} \tag{66}$$

is positive semidefinite because the DUS and the m-port load connected to port set 1 in CB are passive, and because $\mathbf{I}_{S2}$ can take on any value lying in $\mathbb{C}^n$;

$$P_{BDP1} = \mathbf{I}_{S2}^* \mathbf{Z}_{BDP1} \mathbf{I}_{S2}\,, \tag{67}$$

where the impedance matrix

$$\mathbf{Z}_{BDP1} = \mathbf{Z}_{PAM12}^* \frac{\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*}{2} \mathbf{Z}_{PAM12} \tag{68}$$

is positive semidefinite because the DUS and the m-port load connected to port set 1 in CB are passive, and because $\mathbf{I}_{S2}$ can take on any value lying in $\mathbb{C}^n$; and

$$P_{BW} = \mathbf{I}_{S2}^* \mathbf{Z}_{BW} \mathbf{I}_{S2}\,, \tag{69}$$

where the impedance matrix

$$\mathbf{Z}_{BW} = (\mathbf{Y}_{S1} + \mathbf{Y}_{S2})^{-1*} \times \frac{\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*}{2} (\mathbf{Y}_{S1} + \mathbf{Y}_{S2})^{-1} \tag{70}$$

is positive definite.

E. REMARKS

Additional results can be obtained by applying (368)–(371) of Appendix C to the impedance and admittance matrices defined above in Section III.C and Section III.D. Also, these two sections do not cover the computation of P_{AAVP2} and P_{BAVP1}, which is complicated and treated in Section VI.

IV. TWO IMPROVED RECIPROCAL THEOREMS

A. THEOREM ON THE TRANSDUCER POWER GAINS

As in Part 1, we consider two transducer power gains: the transducer power gain in CA, given by

$$G_{AT} = \frac{P_{ADP2}}{P_{AAVG1}}\,, \tag{71}$$

and the transducer power gain in CB, given by

$$G_{BT} = \frac{P_{BDP1}}{P_{BAVG2}}\,. \tag{72}$$

It follows from (27) and (30) that we have $0 \leqslant G_{AT} \leqslant 1$ and $0 \leqslant G_{BT} \leqslant 1$.

To define the excitation in CA, let $\mathbf{X}_A$ denote one of the variables $\mathbf{V}_{O1}$ or $\mathbf{I}_{S1}$. Based on the results of Section III.C and Section III.D, we find that G_{AT} is given by

$$G_{AT} = \frac{\mathbf{X}_A^* \, \mathbf{N}_{AT} \, \mathbf{X}_A}{\mathbf{X}_A^* \, \mathbf{D}_{AT} \, \mathbf{X}_A} , \tag{73}$$

where $\mathbf{N}_{AT}$ and $\mathbf{D}_{AT}$ are hermitian matrices of size m by m, and given in Table 1. We note that $\mathbf{N}_{AT}$ is positive semidefinite, and $\mathbf{D}_{AT}$ positive definite.

TABLE 1. Variable $\mathbf{X}_A$ and associated $\mathbf{N}_{AT}$ and $\mathbf{D}_{AT}$.

Variable $\mathbf{X}_A$	$\mathbf{N}_{AT}$	$\mathbf{D}_{AT}$
$\mathbf{V}_{O1}$	$\mathbf{Y}_{ADP2}$	$\mathbf{Y}_{AAVG1}$
$\mathbf{I}_{S1}$	$\mathbf{Z}_{ADP2}$	$\mathbf{Z}_{AAVG1}$

G_{AT} is given by (73) in the form of a generalized Rayleigh ratio of $\mathbf{N}_{AT}$ to $\mathbf{D}_{AT}$, in the variable $\mathbf{X}_A$. Thus, G_{AT} depends on the excitation. Since $\mathbf{D}_{AT}$ is positive definite, G_{AT} is defined for any nonzero $\mathbf{X}_A \in \mathbb{C}^m$.

To define the excitation in CB, let $\mathbf{X}_B$ denote one of the variables $\mathbf{V}_{O2}$ or $\mathbf{I}_{S2}$. Based on the results of Section III.C and Section III.D, we find that G_{BT} is given by

$$G_{BT} = \frac{\mathbf{X}_B^* \, \mathbf{N}_{BT} \, \mathbf{X}_B}{\mathbf{X}_B^* \, \mathbf{D}_{BT} \, \mathbf{X}_B} , \tag{74}$$

where $\mathbf{N}_{BT}$ and $\mathbf{D}_{BT}$ are hermitian matrices of size n by n, and given in Table 2. We note that $\mathbf{N}_{BT}$ is positive semidefinite, and $\mathbf{D}_{BT}$ positive definite.

TABLE 2. Variable $\mathbf{X}_B$ and associated $\mathbf{N}_{BT}$ and $\mathbf{D}_{BT}$.

Variable $\mathbf{X}_B$	$\mathbf{N}_{BT}$	$\mathbf{D}_{BT}$
$\mathbf{V}_{O2}$	$\mathbf{Y}_{BDP1}$	$\mathbf{Y}_{BAVG2}$
$\mathbf{I}_{S2}$	$\mathbf{Z}_{BDP1}$	$\mathbf{Z}_{BAVG2}$

G_{BT} is given by (74) in the form of a generalized Rayleigh ratio of $\mathbf{N}_{BT}$ to $\mathbf{D}_{BT}$, in the variable $\mathbf{X}_B$. Thus, G_{BT} depends on the excitation. Since $\mathbf{D}_{BT}$ is positive definite, G_{BT} is defined for any nonzero $\mathbf{X}_B \in \mathbb{C}^n$.

By Observation 7, the set of the values of G_{AT} obtained for all $\mathbf{X}_A \in \mathbb{C}^m$ such that $\mathbf{X}_A \neq \mathbf{0}$ is equal to the set of the values of G_{AT} obtained for all $\mathbf{X}_A \in \mathbb{S}_m$; and the set of the values of G_{BT} obtained for all $\mathbf{X}_B \in \mathbb{C}^n$ such that $\mathbf{X}_B \neq \mathbf{0}$ is equal to the set of the values of G_{BT} obtained for all $\mathbf{X}_B \in \mathbb{S}_n$.

Since $\mathbf{I}_{S1} = \mathbf{Y}_{S1}\mathbf{V}_{O1}$, where $\mathbf{Y}_{S1}$ is invertible, we can assert that the set of the values of G_{AT} obtained for all nonzero $\mathbf{X}_A \in \mathbb{C}^m$, or for all $\mathbf{X}_A \in \mathbb{S}_m$, does not depend on the choice $\mathbf{X}_A = \mathbf{V}_{O1}$ or $\mathbf{X}_A = \mathbf{I}_{S1}$.

Likewise, since $\mathbf{I}_{S2} = \mathbf{Y}_{S2}\mathbf{V}_{O2}$, where $\mathbf{Y}_{S2}$ is invertible, we can assert that the set of the values of G_{BT} obtained for all nonzero $\mathbf{X}_B \in \mathbb{C}^n$, or for all $\mathbf{X}_B \in \mathbb{S}_n$, does not depend on the choice $\mathbf{X}_B = \mathbf{V}_{O2}$ or $\mathbf{X}_B = \mathbf{I}_{S2}$.

We can now state and prove a reciprocal theorem on the bounds of the sets of the values of the transducer power gains in CA and CB, which is an improved version of Theorem 4 of Part 1.

Theorem 15. Ignoring noise power contributions, we can assert that:

(a) the set of the values of the transducer power gain in CA, obtained for all nonzero $\mathbf{V}_{O1} \in \mathbb{C}^m$, or equivalently for all nonzero $\mathbf{I}_{S1} \in \mathbb{C}^m$, has a least element referred to as "minimum value" and denoted by $G_{AT\,MIN}$, and a greatest element referred to as "maximum value" and denoted by $G_{AT\,MAX}$;

(b) if $\lambda_{\min}$ and $\lambda_{\max}$ are given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{AT}$ to $\mathbf{D}_{AT}$, in the variable $\mathbf{X}_A$ according to (73) and Table 1, we have $G_{AT\,MIN} = \lambda_{\min}$ and $G_{AT\,MAX} = \lambda_{\max}$;

(c) if $\mathbf{M}$ is given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{AT}$ to $\mathbf{D}_{AT}$, in the variable $\mathbf{X}_A$, an average value of G_{AT} over a number $\min\{m, n\}$ of nonzero excitations is

$$G_{AT\,AVR} = \frac{\operatorname{tr}\mathbf{M}}{\min\{m,n\}} = \frac{\operatorname{tr}\left(\mathbf{N}_{AT}\mathbf{D}_{AT}^{-1}\right)}{\min\{m,n\}} ; \tag{75}$$

(d) $G_{AT\,AVR}$ doesn't depend on the choice of $\mathbf{X}_A$, and

$$0 \leqslant G_{AT\,MIN} \leqslant G_{AT\,AVR} \leqslant G_{AT\,MAX} \leqslant 1 ; \tag{76}$$

(e) the set of the values of the transducer power gain in CB, obtained for all nonzero $\mathbf{V}_{O2} \in \mathbb{C}^n$, or equivalently for all nonzero $\mathbf{I}_{S2} \in \mathbb{C}^n$, has a least element referred to as "minimum value" and denoted by $G_{BT\,MIN}$, and a greatest element referred to as "maximum value" and denoted by $G_{BT\,MAX}$;

(f) if $\lambda_{\min}$ and $\lambda_{\max}$ are given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{BT}$ to $\mathbf{D}_{BT}$, in the variable $\mathbf{X}_B$ according to (74) and Table 2, we have $G_{BT\,MIN} = \lambda_{\min}$ and $G_{BT\,MAX} = \lambda_{\max}$;

(g) if $\mathbf{M}$ is given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{BT}$ to $\mathbf{D}_{BT}$, in the variable $\mathbf{X}_B$, an average value of G_{BT} over a number $\min\{m, n\}$ of nonzero excitations is

$$G_{BT\,AVR} = \frac{\operatorname{tr}\mathbf{M}}{\min\{m,n\}} = \frac{\operatorname{tr}\left(\mathbf{N}_{BT}\mathbf{D}_{BT}^{-1}\right)}{\min\{m,n\}} ; \tag{77}$$

(h) $G_{BT\,AVR}$ doesn't depend on the choice of $\mathbf{X}_B$, and

$$0 \leqslant G_{BT\,MIN} \leqslant G_{BT\,AVR} \leqslant G_{BT\,MAX} \leqslant 1 ; \tag{78}$$

(i) if the DUS and both loads are reciprocal devices, then

$$G_{AT\,MAX} = G_{BT\,MAX} \tag{79}$$

and

$$G_{AT\,AVR} = G_{BT\,AVR} ; \tag{80}$$

(j) if the DUS and both loads are reciprocal devices, then

$$(m = n) \Longrightarrow (G_{AT\,MIN} = G_{BT\,MIN}) , \tag{81}$$

$$(m > n) \Longrightarrow (G_{AT\,MIN} = 0) \tag{82}$$

and

$$(m < n) \Longrightarrow (G_{BT\,MIN} = 0) . \tag{83}$$

Proof: Since $\mathbf{D}_{AT}$ and $\mathbf{D}_{BT}$ are positive definite, assertions (a), (b), (e) and (f) directly follow from Theorem 12.

In (c), by Theorem 12 and [8, Sec. 1.3.3], we obtain $\operatorname{tr}\mathbf{M} = \operatorname{tr}\left(\mathbf{N}_{AT}\mathbf{D}_{AT}^{-1}\right)$, and then the second equality of (75). By (8), each eigenvector $\mathbf{y}$ of $\mathbf{M}$ corresponds to a nonzero excitation $\mathbf{X}_A = \mathbf{D}_{AT}^{-1/2}\mathbf{y}$, and to an eigenvalue that is equal to G_{AT} for this $\mathbf{X}_A$. Using Table 1, (46) and (62), we get

$$\operatorname{rank}\left(\mathbf{N}_{AT}\mathbf{D}_{AT}^{-1}\right) = \operatorname{rank}\mathbf{N}_{AT} \leqslant \min\{m, n\}\,, \tag{84}$$

so that the number of nonzero eigenvalues of $\mathbf{M}$, counting multiplicity, is less than or equal to $\min\{m, n\}$. Since $\operatorname{tr}\mathbf{M}$ is the sum of the eigenvalues of $\mathbf{M}$, counting multiplicity, it follows that $G_{AT\,AVR}$ given by (75) is an average of G_{AT} over a number $\min\{m, n\}$ of nonzero excitations. This proves (c). Assertion (d) follows from (c), (372) of Appendix C, and the fact that, as said above, we have $0 \leqslant G_{AT} \leqslant 1$.

In (g), by Theorem 12, we have $\operatorname{tr}\mathbf{M} = \operatorname{tr}\left(\mathbf{N}_{BT}\mathbf{D}_{BT}^{-1}\right)$, which allows us to write the second equality of (77). Using Table 2, (52) and (68), we get

$$\operatorname{rank}\left(\mathbf{N}_{BT}\mathbf{D}_{BT}^{-1}\right) = \operatorname{rank}\mathbf{N}_{BT} \leqslant \min\{m, n\}\,, \tag{85}$$

which can be used to prove (g) as we used (84) to prove (c). Assertion (h) follows from (g), (373) of Appendix C, and the fact that $0 \leqslant G_{BT} \leqslant 1$.

To prove (i) and (j), we can assume $\mathbf{X}_A = \mathbf{I}_{S1}$ and $\mathbf{X}_B = \mathbf{I}_{S2}$. By Theorem 12, we only need to compare the eigenvalues of $\mathbf{A} = \mathbf{N}_{AT}\mathbf{D}_{AT}^{-1}$ with the eigenvalues of $\mathbf{B} = \mathbf{N}_{BT}\mathbf{D}_{BT}^{-1}$. It follows from Table 1, (56) and (62) that

$$\mathbf{A} = \mathbf{Z}_{PAM21}^{*}(\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^{*})\mathbf{Z}_{PAM21}(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^{*})\,, \tag{86}$$

which is of size m by m. It follows from Table 2, (58) and (68) that

$$\mathbf{B} = \mathbf{Z}_{PAM12}^{*}(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^{*})\mathbf{Z}_{PAM12}(\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^{*})\,, \tag{87}$$

which is of size n by n. If the DUS and both loads are reciprocal devices, $\mathbf{Z}_{PAM}$, $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are symmetric. Thus, $\mathbf{Y}_{S1}$ and $\mathbf{Y}_{S2}$ are symmetric and the transpose of $\mathbf{Z}_{PAM12}$ is $\mathbf{Z}_{PAM21}$, so that

$$\begin{aligned}\mathbf{B}^{T} &= (\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^{*})\\ &\quad\times \mathbf{Z}_{PAM21}(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^{*})\mathbf{Z}_{PAM21}^{*}\,.\end{aligned} \tag{88}$$

By [8, Sec. 1.4.1], the eigenvalues of $\mathbf{B}^T$ are the same as those of $\mathbf{B}$, counting multiplicity. We note that, if we write $\mathbf{C} = (\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^{*})\mathbf{Z}_{PAM21}(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^{*})$, the right hand sides of (86) and (88) are $\mathbf{Z}_{PAM21}^{*}\mathbf{C}$ and $\mathbf{C}\,\mathbf{Z}_{PAM21}^{*}$, respectively. Thus, using [8, Sec. 1.3.22] and the fact that $\mathbf{Z}_{PAM21}^{*}$ is of size m by n, we find that:

- if $m = n$, then $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues, counting multiplicity;
- if $m > n$, then $\mathbf{A}$ has the same eigenvalues as $\mathbf{B}$, counting multiplicity, together with $m - n$ additional eigenvalues equal to zero; and
- if $m < n$, then $\mathbf{B}$ has the same eigenvalues as $\mathbf{A}$, counting multiplicity, together with $n - m$ additional eigenvalues equal to zero.

This leads to the final assertions of Theorem 15. □

B. THEOREM ON THE INSERTION POWER GAINS

As in Part 1, we consider two insertion power gains in the special case where $n = m$: the insertion power gain in CA, given by

$$G_{AI} = \frac{P_{ADP2}}{P_{AW}}\,, \tag{89}$$

and the insertion power gain in CB, given by

$$G_{BI} = \frac{P_{BDP1}}{P_{BW}}\,. \tag{90}$$

G_{AI} and G_{BI} are nonnegative, but they need not be less than or equal to one.

To define the excitation in CA, let $\mathbf{X}_A$ denote one of the variables $\mathbf{V}_{O1}$ or $\mathbf{I}_{S1}$. Based on the results of Section III.C and Section III.D, we find that G_{AI} is given by

$$G_{AI} = \frac{\mathbf{X}_A^{*}\,\mathbf{N}_{AI}\,\mathbf{X}_A}{\mathbf{X}_A^{*}\,\mathbf{D}_{AI}\,\mathbf{X}_A}\,, \tag{91}$$

where $\mathbf{N}_{AI}$ and $\mathbf{D}_{AI}$ are hermitian matrices of size m by m, and given in Table 3. We note that $\mathbf{N}_{AI}$ is positive semidefinite, and $\mathbf{D}_{AI}$ positive definite.

TABLE 3. Variable $\mathbf{X}_A$ and associated $\mathbf{N}_{AI}$ and $\mathbf{D}_{AI}$.

Variable $\mathbf{X}_A$	$\mathbf{N}_{AI}$	$\mathbf{D}_{AI}$
$\mathbf{V}_{O1}$	$\mathbf{Y}_{ADP2}$	$\mathbf{Y}_{AW}$
$\mathbf{I}_{S1}$	$\mathbf{Z}_{ADP2}$	$\mathbf{Z}_{AW}$

G_{AI} is given by (91) in the form of a generalized Rayleigh ratio of $\mathbf{N}_{AI}$ to $\mathbf{D}_{AI}$, in the variable $\mathbf{X}_A$. Thus, G_{AI} depends on the excitation. Since $\mathbf{D}_{AI}$ is positive definite, G_{AI} is defined for any nonzero $\mathbf{X}_A \in \mathbb{C}^m$.

To define the excitation in CB, let $\mathbf{X}_B$ denote one of the variables $\mathbf{V}_{O2}$ or $\mathbf{I}_{S2}$. Based on the results of Section III.C and Section III.D, we find that G_{BI} is given by

$$G_{BI} = \frac{\mathbf{X}_B^{*}\,\mathbf{N}_{BI}\,\mathbf{X}_B}{\mathbf{X}_B^{*}\,\mathbf{D}_{BI}\,\mathbf{X}_B}\,, \tag{92}$$

where $\mathbf{N}_{BI}$ and $\mathbf{D}_{BI}$ are hermitian matrices of size m by m, and given in Table 4. We note that $\mathbf{N}_{BI}$ is positive semidefinite, and $\mathbf{D}_{BI}$ positive definite.

TABLE 4. Variable $\mathbf{X}_B$ and associated $\mathbf{N}_{BI}$ and $\mathbf{D}_{BI}$.

Variable $\mathbf{X}_B$	$\mathbf{N}_{BI}$	$\mathbf{D}_{BI}$
$\mathbf{V}_{O2}$	$\mathbf{Y}_{BDP1}$	$\mathbf{Y}_{BW}$
$\mathbf{I}_{S2}$	$\mathbf{Z}_{BDP1}$	$\mathbf{Z}_{BW}$

G_{BI} is given by (92) in the form of a generalized Rayleigh ratio of $\mathbf{N}_{BI}$ to $\mathbf{D}_{BI}$, in the variable $\mathbf{X}_B$. Thus, G_{BI} depends on the excitation. Since $\mathbf{D}_{BI}$ is positive definite, G_{BI} is defined for any nonzero $\mathbf{X}_B \in \mathbb{C}^m$.

By Observation 7, the set of the values of G_{AI} obtained for all $\mathbf{X}_A \in \mathbb{C}^m$ such that $\mathbf{X}_A \neq \mathbf{0}$ is equal to the set of the values of G_{AI} obtained for all $\mathbf{X}_A \in \mathbb{S}_m$; and the set of the values of G_{BI} obtained for all $\mathbf{X}_B \in \mathbb{C}^m$ such that $\mathbf{X}_B \neq \mathbf{0}$ is equal to the set of the values of G_{BI} obtained for all $\mathbf{X}_B \in \mathbb{S}_m$.

Since $\mathbf{I}_{S1} = \mathbf{Y}_{S1}\mathbf{V}_{O1}$, where $\mathbf{Y}_{S1}$ is invertible, we can assert that the set of the values of G_{AI} obtained for all nonzero $\mathbf{X}_A \in \mathbb{C}^m$, or for all $\mathbf{X}_A \in \mathbb{S}_m$, does not depend on the choice $\mathbf{X}_A = \mathbf{V}_{O1}$ or $\mathbf{X}_A = \mathbf{I}_{S1}$.

Likewise, since $\mathbf{I}_{S2} = \mathbf{Y}_{S2}\mathbf{V}_{O2}$, where $\mathbf{Y}_{S2}$ is invertible, we can assert that the set of the values of G_{BI} obtained for all nonzero $\mathbf{X}_B \in \mathbb{C}^m$, or for all $\mathbf{X}_B \in \mathbb{S}_m$, does not depend on the choice $\mathbf{X}_B = \mathbf{V}_{O2}$ or $\mathbf{X}_B = \mathbf{I}_{S2}$.

We can now state and prove a reciprocal theorem on the bounds of the sets of the values of G_{AI} and G_{BI}, which is a better version of Theorem 6 of Part 1.

Theorem 16. If $m = n$, ignoring noise power contributions, we can assert that:

(a) the set of the values of the insertion power gain in CA, obtained for all nonzero $\mathbf{V}_{O1} \in \mathbb{C}^m$, or equivalently for all nonzero $\mathbf{I}_{S1} \in \mathbb{C}^m$, has a least element referred to as "minimum value" and denoted by $G_{AI\,MIN}$, and a greatest element referred to as "maximum value" and denoted by $G_{AI\,MAX}$;

(b) if $\lambda_{\min}$ and $\lambda_{\max}$ are given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{AI}$ to $\mathbf{D}_{AI}$, in the variable $\mathbf{X}_A$ according to (91) and Table 3, we have $G_{AI\,MIN} = \lambda_{\min}$ and $G_{AI\,MAX} = \lambda_{\max}$;

(c) if $\mathbf{M}$ is given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{AI}$ to $\mathbf{D}_{AI}$, in the variable $\mathbf{X}_A$, an average value of G_{AI} over m nonzero excitations is

$$G_{AI\,AVR} = \frac{\operatorname{tr}\mathbf{M}}{m} = \frac{\operatorname{tr}\left(\mathbf{N}_{AI}\mathbf{D}_{AI}^{-1}\right)}{m}\,; \tag{93}$$

(d) $G_{AI\,AVR}$ doesn't depend on the choice of $\mathbf{X}_A$, and

$$0 \leqslant G_{AI\,MIN} \leqslant G_{AI\,AVR} \leqslant G_{AI\,MAX}\,; \tag{94}$$

(e) the set of the values of the transducer power gain in CB, obtained for all nonzero $\mathbf{V}_{O2} \in \mathbb{C}^m$, or equivalently for all nonzero $\mathbf{I}_{S2} \in \mathbb{C}^m$, has a least element referred to as "minimum value" and denoted by $G_{BI\,MIN}$, and a greatest element referred to as "maximum value" and denoted by $G_{BI\,MAX}$;

(f) if $\lambda_{\min}$ and $\lambda_{\max}$ are given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{BI}$ to $\mathbf{D}_{BI}$, in the variable $\mathbf{X}_B$ according to (92) and Table 4, we have $G_{BI\,MIN} = \lambda_{\min}$ and $G_{BI\,MAX} = \lambda_{\max}$;

(g) if $\mathbf{M}$ is given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{BI}$ to $\mathbf{D}_{BI}$, in the variable $\mathbf{X}_B$, an average value of G_{BI} over m nonzero excitations is

$$G_{BI\,AVR} = \frac{\operatorname{tr}\mathbf{M}}{m} = \frac{\operatorname{tr}\left(\mathbf{N}_{BI}\mathbf{D}_{BI}^{-1}\right)}{m}\,; \tag{95}$$

(h) $G_{BI\,AVR}$ doesn't depend on the choice of $\mathbf{X}_B$, and

$$0 \leqslant G_{BI\,MIN} \leqslant G_{BI\,AVR} \leqslant G_{BI\,MAX}\,; \tag{96}$$

(i) assuming that the DUS and both loads are reciprocal devices, if there exist two complex numbers Z_{S1} and Z_{S2} such that $\mathbf{Z}_{S1} = Z_{S1}\mathbf{1}_m$ and $\mathbf{Z}_{S2} = Z_{S2}\mathbf{1}_m$, or if $\mathbf{Z}_{PAM21}$, $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are circulant, then

$$G_{AI\,MAX} = G_{BI\,MAX}\,, \tag{97}$$

$$G_{AI\,AVR} = G_{BI\,AVR} \tag{98}$$

and

$$G_{AI\,MIN} = G_{BI\,MIN}\,. \tag{99}$$

Proof: Since $\mathbf{D}_{AI}$ and $\mathbf{D}_{BI}$ are positive definite, assertions (a), (b), (e) and (f) directly follow from Theorem 12.

In (c), by Theorem 12 and [8, Sec. 1.3.3], we obtain $\operatorname{tr}\mathbf{M} = \operatorname{tr}\left(\mathbf{N}_{AI}\mathbf{D}_{AI}^{-1}\right)$, and then the second equality of (93). By (8), each eigenvector $\mathbf{y}$ of $\mathbf{M}$ corresponds to a nonzero excitation $\mathbf{X}_A = \mathbf{D}_{AI}^{-1/2}\mathbf{y}$, and to an eigenvalue that is equal to G_{AI} for this $\mathbf{X}_A$. Using Table 3, (46) and (62), we get

$$\operatorname{rank}\left(\mathbf{N}_{AI}\mathbf{D}_{AI}^{-1}\right) = \operatorname{rank}\mathbf{N}_{AI} \leqslant m\,, \tag{100}$$

so that the number of nonzero eigenvalues of $\mathbf{M}$, counting multiplicity, is less than or equal to m. Since $\operatorname{tr}\mathbf{M}$ is the sum of the eigenvalues of $\mathbf{M}$, counting multiplicity, it follows that $G_{AI\,AVR}$ given by (93) is an average of G_{AI} over m nonzero excitations. This proves (c). Assertion (d) follows from (c), (372) of Appendix C, and $G_{AI} \geqslant 0$.

In (g), by Theorem 12, we have $\operatorname{tr}\mathbf{M} = \operatorname{tr}\left(\mathbf{N}_{BI}\mathbf{D}_{BI}^{-1}\right)$, which allows us to write the second equality of (95). Using Table 4, (52) and (68), we get

$$\operatorname{rank}\left(\mathbf{N}_{BI}\mathbf{D}_{BI}^{-1}\right) = \operatorname{rank}\mathbf{N}_{BI} \leqslant m\,, \tag{101}$$

which can be used to prove (g) as we used (100) to prove (c). This, (373) of Appendix C, and $G_{BI} \geqslant 0$ lead us to (h).

To prove (i), we can assume $\mathbf{X}_A = \mathbf{I}_{S1}$ and $\mathbf{X}_B = \mathbf{I}_{S2}$. By Theorem 12, we only need to compare the eigenvalues of $\mathbf{A} = \mathbf{N}_{AI}\mathbf{D}_{AI}^{-1}$ with the eigenvalues of $\mathbf{B} = \mathbf{N}_{BI}\mathbf{D}_{BI}^{-1}$. It follows from Table 3, (62) and (64) that

$$\begin{aligned}\mathbf{A} &= \mathbf{Z}_{PAM21}^*(\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*)\mathbf{Z}_{PAM21}\\ &\times(\mathbf{Y}_{S1} + \mathbf{Y}_{S2})(\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*)^{-1}(\mathbf{Y}_{S1} + \mathbf{Y}_{S2})^*\,,\end{aligned} \tag{102}$$

which is of size m by m. It follows from Table 4, (68) and (70) that

$$\begin{aligned}\mathbf{B} &= \mathbf{Z}_{PAM12}^*(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*)\mathbf{Z}_{PAM12}\\ &\times(\mathbf{Y}_{S1} + \mathbf{Y}_{S2})(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*)^{-1}(\mathbf{Y}_{S1} + \mathbf{Y}_{S2})^*\,,\end{aligned} \tag{103}$$

which is of size m by m. If the DUS and both loads are reciprocal devices, $\mathbf{Z}_{PAM}$, $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are symmetric. Thus, $\mathbf{Y}_{S1}$ and $\mathbf{Y}_{S2}$ are symmetric and the transpose of $\mathbf{Z}_{PAM12}$ is $\mathbf{Z}_{PAM21}$, so that

$$\begin{aligned}\mathbf{B}^T &= (\mathbf{Y}_{S1} + \mathbf{Y}_{S2})^*(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*)^{-1}\\ &\times(\mathbf{Y}_{S1} + \mathbf{Y}_{S2})\mathbf{Z}_{PAM21}(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*)\mathbf{Z}_{PAM21}^*\,.\end{aligned} \tag{104}$$

We need an additional assumption, suitable to allow us to remove: $(\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*)$ and $(\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*)^{-1}$ from (102); and $(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*)$ and $(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*)^{-1}$ from (104). A first possibility is that we assume that there exist two complex numbers Z_{S1} and Z_{S2} such that $\mathbf{Z}_{S1} = Z_{S1}\mathbf{1}_m$ and $\mathbf{Z}_{S2} = Z_{S2}\mathbf{1}_m$. A second possibility is that we assume that $\mathbf{Z}_{PAM21}$, $\mathbf{Z}_{S1}$

and $\mathbf{Z}_{S2}$ are circulant, because: circulant matrices commute; linear combinations of circulant matrices are circulant; and the inverse of an invertible circulant matrix is circulant [8, Sec. 0.9.6]. Using either assumption, we obtain

$$\mathbf{A} = \mathbf{Z}^*_{PAM21}\mathbf{Z}_{PAM21}(\mathbf{Y}_{S1}+\mathbf{Y}_{S2})(\mathbf{Y}_{S1}+\mathbf{Y}_{S2})^* \,, \tag{105}$$

and

$$\mathbf{B}^T = \mathbf{Z}_{PAM21}(\mathbf{Y}_{S1}+\mathbf{Y}_{S2})(\mathbf{Y}_{S1}+\mathbf{Y}_{S2})^*\mathbf{Z}^*_{PAM21} \,. \tag{106}$$

By [8, Sec. 1.4.1], the eigenvalues of $\mathbf{B}^T$ are the same as those of $\mathbf{B}$, counting multiplicity. We then observe that, if we write $\mathbf{C} = \mathbf{Z}_{PAM21}(\mathbf{Y}_{S1}+\mathbf{Y}_{S2})(\mathbf{Y}_{S1}+\mathbf{Y}_{S2})^*$, the right hand sides of (105) and (106) are $\mathbf{Z}^*_{PAM21}\mathbf{C}$ and $\mathbf{C}\,\mathbf{Z}^*_{PAM21}$, respectively. Thus, using [8, Sec. 1.3.22], we find that $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues, counting multiplicity, which leads to the final assertions of Theorem 16. □

V. OPERATING POWER GAINS

An operating power gain is sometimes called “power gain” [14, Sec. 3.2]. It could also be called “efficiency” since we are considering a passive DUS. We consider two operating power gains: the operating power gain in CA, given by

$$G_{AO} = \frac{P_{ADP2}}{P_{ARP1}} \,, \tag{107}$$

and the operating power gain in CB, given by

$$G_{BO} = \frac{P_{BDP1}}{P_{BRP2}} \,. \tag{108}$$

It follows from (27) and (30) that we have $0 \leqslant G_{AO} \leqslant 1$ and $0 \leqslant G_{BO} \leqslant 1$.

To define the excitation in CA, let $\mathbf{X}_A$ denote one of the variables $\mathbf{V}_{O1}$ or $\mathbf{I}_{S1}$. Based on the results of Section III.C and Section III.D, we find that G_{AO} is given by

$$G_{AO} = \frac{\mathbf{X}_A^*\,\mathbf{N}_{AO}\,\mathbf{X}_A}{\mathbf{X}_A^*\,\mathbf{D}_{AO}\,\mathbf{X}_A} \,, \tag{109}$$

where $\mathbf{N}_{AO}$ and $\mathbf{D}_{AO}$ are hermitian matrices of size m by m, and given in Table 5. We note that $\mathbf{N}_{AO}$ and $\mathbf{D}_{AO}$ are positive semidefinite.

TABLE 5. Variable $\mathbf{X}_A$ and associated $\mathbf{N}_{AO}$ and $\mathbf{D}_{AO}$.

Variable $\mathbf{X}_A$	$\mathbf{N}_{AO}$	$\mathbf{D}_{AO}$
$\mathbf{V}_{O1}$	$\mathbf{Y}_{ADP2}$	$\mathbf{Y}_{ARP1}$
$\mathbf{I}_{S1}$	$\mathbf{Z}_{ADP2}$	$\mathbf{Z}_{ARP1}$

G_{AO} is given by (109) in the form of a generalized Rayleigh ratio of $\mathbf{N}_{AO}$ to $\mathbf{D}_{AO}$, in the variable $\mathbf{X}_A$. It follows that G_{AO} depends on the excitation, and that, according to the explanations provided in Section II.A, G_{AO} is defined for $\mathbf{X}_A \in D(\mathbf{D}_{AO})$, where

$$D(\mathbf{D}_{AO}) = \{\mathbf{X}_A \in \mathbb{C}^m : \mathbf{X}_A \notin \ker \mathbf{D}_{AO}\} \,. \tag{110}$$

To define the excitation in CB, let $\mathbf{X}_B$ denote one of the variables $\mathbf{V}_{O2}$ or $\mathbf{I}_{S2}$. Based on the results of Section III.C and Section III.D, we find that G_{BO} is given by

$$G_{BO} = \frac{\mathbf{X}_B^*\,\mathbf{N}_{BO}\,\mathbf{X}_B}{\mathbf{X}_B^*\,\mathbf{D}_{BO}\,\mathbf{X}_B} \,, \tag{111}$$

where $\mathbf{N}_{BO}$ and $\mathbf{D}_{BO}$ are hermitian matrices of size n by n, and given in Table 6. We note that $\mathbf{N}_{BO}$ and $\mathbf{D}_{BO}$ are positive semidefinite.

TABLE 6. Variable $\mathbf{X}_B$ and associated $\mathbf{N}_{BO}$ and $\mathbf{D}_{BO}$.

Variable $\mathbf{X}_B$	$\mathbf{N}_{BO}$	$\mathbf{D}_{BO}$
$\mathbf{V}_{O2}$	$\mathbf{Y}_{BDP1}$	$\mathbf{Y}_{BRP2}$
$\mathbf{I}_{S2}$	$\mathbf{Z}_{BDP1}$	$\mathbf{Z}_{BRP2}$

G_{BO} is given by (111) in the form of a generalized Rayleigh ratio of $\mathbf{N}_{BO}$ to $\mathbf{D}_{BO}$, in the variable $\mathbf{X}_B$. It follows that G_{BO} depends on the excitation, and is defined for $\mathbf{X}_B \in D(\mathbf{D}_{BO})$, where

$$D(\mathbf{D}_{BO}) = \{\mathbf{X}_B \in \mathbb{C}^n : \mathbf{X}_B \notin \ker \mathbf{D}_{BO}\} \,. \tag{112}$$

By Observation 7, the set of the values of G_{AO} obtained for all $\mathbf{X}_A \in D(\mathbf{D}_{AO})$ is equal to the set of the values of G_{AO} obtained for all $\mathbf{X}_A \in D(\mathbf{D}_{AO}) \cap \mathbb{S}_m$. Likewise, we can assert that the set of the values of G_{BO} obtained for all $\mathbf{X}_B \in D(\mathbf{D}_{BO})$ is equal to the set of the values of G_{BO} obtained for all $\mathbf{X}_B \in D(\mathbf{D}_{BO}) \cap \mathbb{S}_n$.

Since $\mathbf{I}_{S1} = \mathbf{Y}_{S1}\mathbf{V}_{O1}$, where $\mathbf{Y}_{S1}$ is invertible, it is possible to show that the set of the values of G_{AO} obtained for all $\mathbf{X}_A \in D(\mathbf{D}_{AO})$, or for all $\mathbf{X}_A \in D(\mathbf{D}_{AO}) \cap \mathbb{S}_m$, does not depend on the choice $\mathbf{X}_A = \mathbf{V}_{O1}$ or $\mathbf{X}_A = \mathbf{I}_{S1}$.

Likewise, since $\mathbf{I}_{S2} = \mathbf{Y}_{S2}\mathbf{V}_{O2}$, where $\mathbf{Y}_{S2}$ is invertible, we can show that the set of the values of G_{BO} obtained for all $\mathbf{X}_B \in D(\mathbf{D}_{BO})$, or for all $\mathbf{X}_B \in D(\mathbf{D}_{BO}) \cap \mathbb{S}_n$, does not depend on the choice $\mathbf{X}_B = \mathbf{V}_{O2}$ or $\mathbf{X}_B = \mathbf{I}_{S2}$.

We can now state and prove two new theorems on the operating power gains.

Theorem 17. We assume that we have $D(\mathbf{D}_{AO}) \neq \varnothing$, since otherwise studying G_{AO} is not interesting. Ignoring noise power contributions, we can assert that:

(a) we have $\ker \mathbf{D}_{AO} \subset \ker \mathbf{N}_{AO}$ so that Theorem 14 can be applied to the generalized Rayleigh ratio of $\mathbf{N}_{AO}$ to $\mathbf{D}_{AO}$;
(b) the set of the values of the operating power gain in CA, obtained for all $\mathbf{X}_A \in D(\mathbf{D}_{AO})$, has a least element referred to as “minimum value” and denoted by $G_{AO\,MIN}$, and a greatest element referred to as “maximum value” and denoted by $G_{AO\,MAX}$;
(c) if $\kappa_{\min}$ and $\kappa_{\max}$ are given by Theorem 14 applied to the generalized Rayleigh ratio of $\mathbf{N}_{AO}$ to $\mathbf{D}_{AO}$, in the variable $\mathbf{X}_A$ according to (109) and Table 5, we have $G_{AO\,MIN} = \kappa_{\min}$ and $G_{AO\,MAX} = \kappa_{\max}$;
(d) if d is the nullity of $\mathbf{D}_{AO}$, and if $\mathbf{Q}$ and $\mathbf{R}$ are given by Theorem 14 applied to the generalized Rayleigh ratio of $\mathbf{N}_{AO}$ to $\mathbf{D}_{AO}$, in the variable $\mathbf{X}_A$, an average value of G_{AO} over a number $N = \min\{m-d, n\}$ of nonzero excitations $\mathbf{X}_A \in D(\mathbf{D}_{AO})$ is

$$G_{AO\,AVR} = \frac{\operatorname{tr}\mathbf{Q}}{N} = \frac{\operatorname{tr}\mathbf{R}}{N} \,; \tag{113}$$

(e) we have

$$0 \leqslant G_{AO\,MIN} \leqslant G_{AO\,AVR} \leqslant G_{AO\,MAX} \leqslant 1\,; \quad (114)$$

(f) if $\mathbf{D}_{AO}$ is positive definite and if $\lambda_{\min}$ and $\lambda_{\max}$ are given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{AO}$ to $\mathbf{D}_{AO}$, in the variable $\mathbf{X}_A$ according to (109) and Table 5, we have $G_{AO\,MIN} = \lambda_{\min}$ and $G_{AO\,MAX} = \lambda_{\max}$;

(g) if $\mathbf{D}_{AO}$ is positive definite and if $\mathbf{M}$ is given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{AO}$ to $\mathbf{D}_{AO}$, then $G_{AO\,AVR}$ doesn't depend on the choice of the variable $\mathbf{X}_A$, and we have

$$G_{AO\,AVR} = \frac{\operatorname{tr}\mathbf{M}}{\min\{m,n\}} = \frac{\operatorname{tr}\left(\mathbf{N}_{AO}\mathbf{D}_{AO}^{-1}\right)}{\min\{m,n\}}\,. \quad (115)$$

Proof: We have already observed that power conservation entails $G_{AO} \leqslant 1$. Since $\mathbf{D}_{AO}$ is positive semidefinite, we can apply Corollary 3 to the generalized Rayleigh ratio of $\mathbf{N}_{AO}$ to $\mathbf{D}_{AO}$. It follows that $\ker \mathbf{D}_{AO} \subset \ker \mathbf{N}_{AO}$. Thus, the assumptions of Theorem 14 applied to the generalized Rayleigh ratio of $\mathbf{N}_{AO}$ to $\mathbf{D}_{AO}$ are satisfied. This proves (a), and also (b) and (c), which directly follow from Theorem 14.

In (d), by Theorem 14 and [8, Sec. 1.3.3], we obtain $\operatorname{tr}\mathbf{Q} = \operatorname{tr}\mathbf{R}$, which allows us to write the second equality of (113). Let $\mathbf{L}$, $\mathcal{L}$ and $\mathbf{P}$ be given by Theorem 14 applied to the generalized Rayleigh ratio of $\mathbf{N}_{AO}$ to $\mathbf{D}_{AO}$. By (24), each eigenvector $\mathbf{u}$ of $\mathbf{Q}$ corresponds to a nonzero excitation $\mathbf{X}_A = \mathbf{P}\mathbf{u}$, and to an eigenvalue that is equal to G_{AO} for this $\mathbf{X}_A$. We have $\operatorname{rank}\mathbf{Q} \leqslant m - d$, and $\operatorname{rank}\mathbf{Q} \leqslant \operatorname{rank}\mathbf{N}_{AO}$. Using Table 5, (46) and (62), we get

$$\operatorname{rank}\mathbf{Q} \leqslant \min\{m-d, n\}\,, \quad (116)$$

so that the number of nonzero eigenvalues of $\mathbf{Q}$, counting multiplicity, is less than or equal to $N = \min\{m-d, n\}$. Since $\operatorname{tr}\mathbf{Q}$ is the sum of the eigenvalues of $\mathbf{Q}$, counting multiplicity, it follows that $G_{AO\,AVR}$ given by (113) is an average of G_{AO} over a number N of nonzero excitations $\mathbf{X}_A \in D(\mathbf{D}_{AO})$. This leads us to (d) and (e).

If $\mathbf{D}_{AO}$ is positive definite, we have $d = 0$ and $\mathcal{L} = \mathbf{L}$, so that, according to (13), we have $\mathbf{P}\mathbf{L}^* = \mathbf{P}\mathbf{L}^{-1} = \mathbf{D}_{AO}^{-1/2}$. Consequently, it follows from (6) and (14) that

$$\mathbf{L}\mathbf{Q}\mathbf{L}^{-1} = \mathbf{D}_{AO}^{-1/2}\mathbf{N}_{AO}\mathbf{D}_{AO}^{-1/2} = \mathbf{M}\,. \quad (117)$$

Thus, if $\mathbf{D}_{AO}$ is positive definite, $\mathbf{M}$ is similar to $\mathbf{Q}$. It follows that $\mathbf{M}$ and $\mathbf{Q}$ have the same eigenvalues, counting multiplicity, by [8, Sec. 1.3.4]. This, Theorem 12, Theorem 14 and (372) of Appendix C lead us to (f) and (g). □

Theorem 18. We assume that we have $D(\mathbf{D}_{BO}) \neq \varnothing$, since otherwise studying G_{BO} is not interesting. Ignoring noise power contributions, we can assert that:

(a) we have $\ker \mathbf{D}_{BO} \subset \ker \mathbf{N}_{BO}$ so that Theorem 14 can be applied to the generalized Rayleigh ratio of $\mathbf{N}_{BO}$ to $\mathbf{D}_{BO}$;

(b) the set of the values of the operating power gain in CB, obtained for all $\mathbf{X}_B \in D(\mathbf{D}_{BO})$, has a least element referred to as "minimum value" and denoted by $G_{BO\,MIN}$, and a greatest element referred to as "maximum value" and denoted by $G_{BO\,MAX}$;

(c) if $\kappa_{\min}$ and $\kappa_{\max}$ are given by Theorem 14 applied to the generalized Rayleigh ratio of $\mathbf{N}_{BO}$ to $\mathbf{D}_{BO}$, in the variable $\mathbf{X}_B$ according to (111) and Table 6, we have $G_{BO\,MIN} = \kappa_{\min}$ and $G_{BO\,MAX} = \kappa_{\max}$;

(d) if d is the nullity of $\mathbf{D}_{BO}$, and if $\mathbf{Q}$ and $\mathbf{R}$ are given by Theorem 14 applied to the generalized Rayleigh ratio of $\mathbf{N}_{BO}$ to $\mathbf{D}_{BO}$, in the variable $\mathbf{X}_B$, an average value of G_{BO} over a number $N = \min\{m, n-d\}$ of nonzero excitations $\mathbf{X}_B \in D(\mathbf{D}_{BO})$ is

$$G_{BO\,AVR} = \frac{\operatorname{tr}\mathbf{Q}}{N} = \frac{\operatorname{tr}\mathbf{R}}{N}\,; \quad (118)$$

(e) we have

$$0 \leqslant G_{BO\,MIN} \leqslant G_{BO\,AVR} \leqslant G_{BO\,MAX} \leqslant 1\,; \quad (119)$$

(f) if $\mathbf{D}_{BO}$ is positive definite and if $\lambda_{\min}$ and $\lambda_{\max}$ are given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{BO}$ to $\mathbf{D}_{BO}$, in the variable $\mathbf{X}_B$ according to (111) and Table 6, we have $G_{BO\,MIN} = \lambda_{\min}$ and $G_{BO\,MAX} = \lambda_{\max}$;

(g) if $\mathbf{D}_{BO}$ is positive definite and if $\mathbf{M}$ is given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{BO}$ to $\mathbf{D}_{BO}$, then $G_{BO\,AVR}$ doesn't depend on the choice of the variable $\mathbf{X}_B$, and we have

$$G_{BO\,AVR} = \frac{\operatorname{tr}\mathbf{M}}{\min\{m,n\}} = \frac{\operatorname{tr}\left(\mathbf{N}_{BO}\mathbf{D}_{BO}^{-1}\right)}{\min\{m,n\}}\,. \quad (120)$$

Proof: Theorem 18 is Theorem 17 with a different labeling of the ports. □

Neither Theorem 17 nor Theorem 18 qualifies as a reciprocal theorem about the bounds of the sets of the values of the operating power gains in CA and CB. This comment also applies to the following new theorem.

Theorem 19. We assume $D(\mathbf{D}_{AO}) \neq \varnothing$ and $D(\mathbf{D}_{BO}) \neq \varnothing$. Ignoring noise power contributions, we can assert that:

(a) if $\mathbf{Z}_{PAM11}$ is invertible, for a specified DUS and a specified $\mathbf{Y}_{S2}$, $G_{AO\,MIN}$ and $G_{AO\,MAX}$ do not depend on $\mathbf{Y}_{S1}$;

(b) if $\mathbf{Y}_{SAM11}$ is invertible, for a specified DUS and a specified $\mathbf{Z}_{S2}$, $G_{AO\,MIN}$ and $G_{AO\,MAX}$ do not depend on $\mathbf{Z}_{S1}$;

(c) if $\mathbf{Z}_{PAM22}$ is invertible, for a specified DUS and a specified $\mathbf{Y}_{S1}$, $G_{BO\,MIN}$ and $G_{BO\,MAX}$ do not depend on $\mathbf{Y}_{S2}$; and

(d) if $\mathbf{Y}_{SAM11}$ is invertible, for a specified DUS and a specified $\mathbf{Z}_{S1}$, $G_{BO\,MIN}$ and $G_{BO\,MAX}$ do not depend on $\mathbf{Z}_{S2}$.

Proof: In CA, if $\mathbf{Z}_{PAM11}$ is invertible, then port set 1 of the DUS has an admittance matrix

$$\mathbf{Y}_{APP1} = \mathbf{Z}_{PAM11}^{-1} - \mathbf{Y}_{S1}\,. \tag{121}$$

Accordingly, the vector $\mathbf{V}_1$ may lie anywhere in $\mathbb{C}^m$, we have $\mathbf{I}_1 = \mathbf{Y}_{APP1}\mathbf{V}_1$ for any $\mathbf{V}_1 \in \mathbb{C}^m$, and $\mathbf{Y}_{APP1}$ is positive semidefinite because $P_{ARP1} \geqslant 0$ for any $\mathbf{V}_1 \in \mathbb{C}^m$. Of course, $\mathbf{Y}_{APP1}$ does not depend on $\mathbf{Y}_{S1}$.

In CA, if $\mathbf{Z}_{PAM11}$ is invertible, for a specified DUS and a specified $\mathbf{Y}_{S2}$, it follows that P_{ADP2} and P_{ARP1} are completely determined by $\mathbf{V}_1$, so that any change in $\mathbf{Y}_{S1}$ can be compensated by a change in $\mathbf{I}_{S1} = (\mathbf{Y}_{APP1} + \mathbf{Y}_{S1})\mathbf{V}_1$ to obtain the same $\mathbf{V}_1$ and the same $\mathbf{I}_1$, hence the same P_{ADP2} and the same P_{ARP1}, so that G_{AOMIN} and G_{AOMAX} do not depend on $\mathbf{Y}_{S1}$. This proves (a).

In CA, if $\mathbf{Y}_{SAM11}$ is invertible, then port set 1 of the DUS has an impedance matrix

$$\mathbf{Z}_{APP1} = \mathbf{Y}_{SAM11}^{-1} - \mathbf{Z}_{S1}\,. \tag{122}$$

Accordingly, the vector $\mathbf{I}_1$ may lie anywhere in $\mathbb{C}^m$, we have $\mathbf{V}_1 = \mathbf{Z}_{APP1}\mathbf{I}_1$ for any $\mathbf{I}_1 \in \mathbb{C}^m$, and $\mathbf{Z}_{APP1}$ is positive semidefinite because $P_{ARP1} \geqslant 0$ for any $\mathbf{I}_1 \in \mathbb{C}^m$. Of course, $\mathbf{Z}_{APP1}$ does not depend on $\mathbf{Z}_{S1}$.

In CA, if $\mathbf{Y}_{SAM11}$ is invertible, for a specified DUS and a specified $\mathbf{Z}_{S2}$, it follows that P_{ADP2} and P_{ARP1} are completely determined by $\mathbf{I}_1$, so that any change in $\mathbf{Z}_{S1}$ can be compensated by a change in $\mathbf{V}_{O1} = (\mathbf{Z}_{APP1} + \mathbf{Z}_{S1})\mathbf{I}_1$ to obtain the same $\mathbf{I}_1$ and the same $\mathbf{V}_1$, hence the same P_{ADP2} and the same P_{ARP1}, so that G_{AOMIN} and G_{AOMAX} do not depend on $\mathbf{Z}_{S1}$. This proves (b).

Regarding (c) and (d), they correspond to (a) and (b), respectively, with a different labeling of the port sets. □

VI. AVAILABLE POWERS AT OUTPUT PORTS

A. Plan

To investigate more power ratios, we need to compute the available powers at the output ports of the DUS, that is to say P_{AAVP2} in CA and P_{BAVP1} in CB.

We will study two new configurations, using short-circuit currents in Section VI-B, and using open-circuit voltages in Section VI-C. The theorems covering the computation of P_{AAVP2} and P_{BAVP1} will be obtained in Section VI-D.

B. Some Results Using Short-Circuit Currents

We consider the parallel-augmented multiport defined in Section III-B. Port set 1 of the parallel-augmented multiport corresponds to port set 1 of the DUS connected to an m-port load of admittance matrix $\mathbf{Y}_{S1}$. Port set 2 of the parallel-augmented multiport corresponds to port set 2 of the DUS connected to an m-port load of admittance matrix $\mathbf{Y}_{S2}$.

We will use the equivalent circuit of the DUS defined in Corollary 1 of Part 1, composed of: the parallel-augmented multiport defined in Section III-B; an m-port circuit of admittance matrix $-\mathbf{Y}_{S1}$ connected in parallel with port set 1 of the parallel-augmented multiport; and an n-port circuit of admittance matrix $-\mathbf{Y}_{S2}$ connected in parallel with port set 2 of the parallel-augmented multiport.

In a configuration C (CC), port set 1 of the DUS is connected to an LTI m-port generator of internal admittance matrix $\mathbf{Y}_{S1}$ and rms short-circuit current vector $\mathbf{I}_{S1}$, as in CA, and port set 2 of the DUS is connected to an LTI n-port device, which need neither be passive nor have an admittance matrix. We see that an equivalent circuit of CC comprises: the parallel-augmented multiport, of impedance matrix $\mathbf{Z}_{PAM}$; an m-port current source delivering $\mathbf{I}_{S1}$ connected in parallel with port set 1 of the parallel-augmented multiport; an n-port circuit of admittance matrix $-\mathbf{Y}_{S2}$ connected in parallel with port set 2 of the parallel-augmented multiport; and said LTI n-port device also connected in parallel with port set 2 of the parallel-augmented multiport. It follows that, in CC, we have

$$\mathbf{V}_2 = \mathbf{Z}_{PAM21}\mathbf{I}_{S1} + \mathbf{Z}_{PAM22}(\mathbf{I}_2 + \mathbf{Y}_{S2}\mathbf{V}_2)\,. \tag{123}$$

In a configuration D (CD), port set 2 of the DUS is connected to an LTI n-port generator of internal admittance matrix $\mathbf{Y}_{S2}$ and rms short-circuit current vector $\mathbf{I}_{S2}$, as in CB, and port set 1 of the DUS is connected to an LTI m-port device, which need neither be passive nor have an admittance matrix. We find that, in CD, we have

$$\mathbf{V}_1 = \mathbf{Z}_{PAM12}\mathbf{I}_{S2} + \mathbf{Z}_{PAM11}(\mathbf{I}_1 + \mathbf{Y}_{S1}\mathbf{V}_1)\,. \tag{124}$$

Let $\mathbf{I}_{C2}$ be the column vector of size n given by

$$\mathbf{I}_{C2} = \mathbf{I}_2 + \mathbf{Y}_{S2}\mathbf{V}_2\,, \tag{125}$$

and $\mathbf{I}_{D1}$ be the column vector of size m given by

$$\mathbf{I}_{D1} = \mathbf{I}_1 + \mathbf{Y}_{S1}\mathbf{V}_1\,. \tag{126}$$

Lemma 4. In CC, the LTI n-port device connected to port set 2 of the DUS produces a relationship between $\mathbf{V}_2$ and $\mathbf{I}_2$, but if we leave this relationship undetermined, that is to say if this LTI n-port device is not specified, then the vector $\mathbf{I}_{C2} = \mathbf{I}_2 + \mathbf{Y}_{S2}\mathbf{V}_2$ may lie anywhere in $\mathbb{C}^n$.

Likewise, in CD, the LTI m-port device connected to port set 1 of the DUS produces a relationship between $\mathbf{V}_1$ and $\mathbf{I}_1$, but if we leave this relationship undetermined, that is to say if this LTI m-port device is not specified, then the vector $\mathbf{I}_{D1} = \mathbf{I}_1 + \mathbf{Y}_{S1}\mathbf{V}_1$ may lie anywhere in $\mathbb{C}^m$.

Proof: In CC, $\mathbf{I}_{S1}$ is the vector of the rms currents flowing in port set 1 of the parallel-augmented multiport, and $\mathbf{I}_{C2} = \mathbf{I}_2 + \mathbf{Y}_{S2}\mathbf{V}_2$ is the vector of the rms currents flowing in port set 2 of the parallel-augmented multiport. The fact that $\mathbf{Z}_{PAM}$ exists entails that the parallel-augmented multiport creates no constraint on $\mathbf{I}_{S1}$ and $\mathbf{I}_{C2}$, so that $\mathbf{Z}_{PAM}$ can be measured by injecting arbitrary currents in the ports of the parallel-augmented multiport. Thus, in CC, if the n-port device connected to port set 2 is not specified, $\mathbf{I}_{C2}$ may lie anywhere in $\mathbb{C}^n$. In practice, we can decide that the LTI n-port device is an n-port generator of internal admittance matrix $\mathbf{Y}_{S2}$ and rms short-circuit current $\mathbf{I}_{S2}$, as in CB. In this case, $\mathbf{I}_{S2} = \mathbf{I}_{C2}$, which may lie anywhere in $\mathbb{C}^n$.

The argument is similar for CD. □

Observation 10. In contrast, since the DUS need not have an impedance matrix, $\mathbf{I}_2$ may be constrained to lie in a proper subspace of $\mathbb{C}^n$ and $\mathbf{I}_1$ may be constrained to lie in a proper subspace of $\mathbb{C}^m$.

Lemma 5. In CC, let P_{CDP2} be the average power delivered by port set 2. Ignoring noise power contributions, we find

$$2P_{CDP2} = \mathbf{I}_{S1}^* \mathbf{Z}_{PAM21}^* (\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*) \mathbf{Z}_{PAM21} \mathbf{I}_{S1} - \mathbf{I}_{C2}^* \mathbf{Z}_{E2} \mathbf{I}_{C2} + 2\mathrm{Re}\left(\mathbf{I}_{S1}^* \mathbf{Z}_{PAM21}^* \mathbf{K}_{E2} \mathbf{I}_{C2} \right) , \quad (127)$$

where $\mathrm{Re}(z)$ denotes the real part of $z \in \mathbb{C}$, where the impedance matrix $\mathbf{Z}_{E2}$ is of size n by n and given by

$$\mathbf{Z}_{E2} = \mathbf{Z}_{PAM22} + \mathbf{Z}_{PAM22}^* - \mathbf{Z}_{PAM22}^* (\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*) \mathbf{Z}_{PAM22} , \quad (128)$$

and where the dimensionless matrix $\mathbf{K}_{E2}$ is of size n by n and given by

$$\mathbf{K}_{E2} = (\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*) \mathbf{Z}_{PAM22} - \mathbf{1}_n \, . \quad (129)$$

In CD, let P_{DDP1} be the average power delivered by port set 1. Ignoring noise power contributions, we find

$$2P_{DDP1} = \mathbf{I}_{S2}^* \mathbf{Z}_{PAM12}^* (\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*) \mathbf{Z}_{PAM12} \mathbf{I}_{S2} - \mathbf{I}_{D1}^* \mathbf{Z}_{E1} \mathbf{I}_{D1} + 2\mathrm{Re}\left(\mathbf{I}_{S2}^* \mathbf{Z}_{PAM12}^* \mathbf{K}_{E1} \mathbf{I}_{D1} \right) , \quad (130)$$

where the impedance matrix $\mathbf{Z}_{E1}$ is of size m by m and given by

$$\mathbf{Z}_{E1} = \mathbf{Z}_{PAM11} + \mathbf{Z}_{PAM11}^* - \mathbf{Z}_{PAM11}^* (\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*) \mathbf{Z}_{PAM11} , \quad (131)$$

and where the dimensionless matrix $\mathbf{K}_{E1}$ is of size m by m and given by

$$\mathbf{K}_{E1} = (\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*) \mathbf{Z}_{PAM11} - \mathbf{1}_m \, . \quad (132)$$

Proof: In CC, P_{CDP2} is given by

$$P_{CDP2} = -\frac{1}{2} (\mathbf{V}_2^* \mathbf{I}_2 + \mathbf{I}_2^* \mathbf{V}_2) \, . \quad (133)$$

so that, using (123) and (125), we get (134) shown at the bottom of this page. We then get (135) shown at the bottom of this page, which leads us to (127)–(129).

The proof for CD corresponds to the proof for CC, with a different labeling of the port sets. □

Lemma 6. $\mathbf{Z}_{E2}$ and $\mathbf{Z}_{E1}$ are positive semidefinite.

Proof: By (128), $\mathbf{Z}_{E2}$ is hermitian. Let $\lambda_{\min}$ be the smallest eigenvalue of $\mathbf{Z}_{E2}$. Since, by Lemma 4, $\mathbf{I}_{C2}$ can be any complex column vector of size n, we can assume that $\mathbf{I}_{C2} = \mu \mathbf{J}$, where $\mathbf{J}$ is an eigenvector of $\mathbf{Z}_{E2}$ associated with the eigenvalue $\lambda_{\min}$, and where μ is an arbitrary complex number. In this case, we have:

$$\mathbf{I}_{C2}^* \mathbf{Z}_{E2} \mathbf{I}_{C2} = \lambda_{\min} |\mu|^2 \mathbf{J}^* \mathbf{J} \, . \quad (136)$$

Since $\mathbf{J}^* \mathbf{J} > 0$ and $|\mu|$ can be arbitrarily large, it follows from (127) and (136) that P_{CDP2} could be arbitrarily large if $\lambda_{\min}$ was negative. But this is impossible because, the DUS being passive, P_{CDP2} must be less than P_{AAVG1}. We may conclude that $\lambda_{\min}$ is nonnegative, so that $\mathbf{Z}_{E2}$ is positive semidefinite by [8, Sec. 7.2.1].

The proof for $\mathbf{Z}_{E1}$ corresponds to the proof for $\mathbf{Z}_{E2}$, with a different labeling of the port sets. □

Lemma 7. We have

$$\ker \mathbf{Z}_{E2} \subset \ker \left(\mathbf{Z}_{PAM21}^* \mathbf{K}_{E2} \right) \quad (137)$$

and

$$\ker \mathbf{Z}_{E1} \subset \ker \left(\mathbf{Z}_{PAM12}^* \mathbf{K}_{E1} \right) \, . \quad (138)$$

Proof: Let $\mathrm{Im}(z)$ denote the imaginary part of $z \in \mathbb{C}$. For any $\mathbf{J} \in \ker \mathbf{Z}_{E2}$, since, by Lemma 4, $\mathbf{I}_{C2}$ can be any complex column vector of size n, we can posit $\mathbf{I}_{C2} = \mu \mathbf{J}$, where μ is an arbitrary complex number. We get:

$$2P_{CDP2} = \mathbf{I}_{S1}^* \mathbf{Z}_{PAM21}^* (\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*) \mathbf{Z}_{PAM21} \mathbf{I}_{S1} + 2\mathrm{Re}(\mu) \mathrm{Re}\left(\mathbf{I}_{S1}^* \mathbf{Z}_{PAM21}^* \mathbf{K}_{E2} \mathbf{J} \right) - 2\mathrm{Im}(\mu) \mathrm{Im}\left(\mathbf{I}_{S1}^* \mathbf{Z}_{PAM21}^* \mathbf{K}_{E2} \mathbf{J} \right) . \quad (139)$$

Since P_{CDP2} must be less than P_{AAVG1} for any value of μ in $\mathbb{C}$, it follows that

$$\mathbf{I}_{S1}^* \mathbf{Z}_{PAM21}^* \mathbf{K}_{E2} \mathbf{J} = 0 \, . \quad (140)$$

Since $\mathbf{Z}_{PAM21}$ and $\mathbf{K}_{E2}$ are independent of $\mathbf{I}_{S1}$, since $\mathbf{Z}_{E2}$ is independent of $\mathbf{I}_{S1}$ so that $\mathbf{J}$ is independent of $\mathbf{I}_{S1}$, and since (140) is applicable to any $\mathbf{I}_{S1} \in \mathbb{C}^n$, it follows that $\mathbf{J} \in \ker(\mathbf{Z}_{PAM21}^* \mathbf{K}_{E2})$. We have proven (137).

The proof for (138) corresponds to the proof for (137), with a different labeling of the port sets. □

$$2P_{CDP2} = -(\mathbf{Z}_{PAM21} \mathbf{I}_{S1} + \mathbf{Z}_{PAM22} \mathbf{I}_{C2})^* \left[\mathbf{I}_{C2} - \mathbf{Y}_{S2} (\mathbf{Z}_{PAM21} \mathbf{I}_{S1} + \mathbf{Z}_{PAM22} \mathbf{I}_{C2}) \right] - \left[\mathbf{I}_{C2} - \mathbf{Y}_{S2} (\mathbf{Z}_{PAM21} \mathbf{I}_{S1} + \mathbf{Z}_{PAM22} \mathbf{I}_{C2}) \right]^* (\mathbf{Z}_{PAM21} \mathbf{I}_{S1} + \mathbf{Z}_{PAM22} \mathbf{I}_{C2}) \, . \quad (134)$$

$$2P_{CDP2} = \mathbf{I}_{S1}^* \mathbf{Z}_{PAM21}^* (\mathbf{Y}_{S2} + \mathbf{Y}_{S2}) \mathbf{Z}_{PAM21} \mathbf{I}_{S1} + \mathbf{I}_{C2}^* \left[\mathbf{Z}_{PAM22}^* (\mathbf{Y}_{S2} \mathbf{Z}_{PAM22} - \mathbf{1}_n) + (\mathbf{Z}_{PAM22}^* \mathbf{Y}_{S2}^* - \mathbf{1}_n) \mathbf{Z}_{PAM22} \right] \mathbf{I}_{C2} + \mathbf{I}_{S1}^* \mathbf{Z}_{PAM21}^* \left[(\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*) \mathbf{Z}_{PAM22} - \mathbf{1}_n \right] \mathbf{I}_{C2} + \mathbf{I}_{C2}^* \left[\mathbf{Z}_{PAM22}^* (\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*) - \mathbf{1}_n \right] \mathbf{Z}_{PAM21} \mathbf{I}_{S1} \, . \quad (135)$$

Lemma 8. We have

$$\mathrm{range}\,(\mathbf{K}_{E2}^{*}\mathbf{Z}_{PAM21}) \subset \mathrm{range}\,\mathbf{Z}_{E2} \tag{141}$$

and

$$\mathrm{range}\,(\mathbf{K}_{E1}^{*}\mathbf{Z}_{PAM12}) \subset \mathrm{range}\,\mathbf{Z}_{E1} \tag{142}$$

Proof: By Lemma 7, for any $\mathbf{J} \in \mathbb{C}^n$, we have

$$(\mathbf{Z}_{E2}\mathbf{J} = \mathbf{0}) \Longrightarrow (\mathbf{Z}_{PAM21}^{*}\mathbf{K}_{E2}\mathbf{J} = \mathbf{0}) \tag{143}$$

so that, since $\mathbf{Z}_{E2}$ is hermitian,

$$(\mathbf{J}^{*}\mathbf{Z}_{E2} = \mathbf{0}) \Longrightarrow (\mathbf{J}^{*}\mathbf{K}_{E2}^{*}\mathbf{Z}_{PAM21} = \mathbf{0})\ . \tag{144}$$

We use the standard scalar product of $\mathbb{C}^n$ to define orthogonality [8, Sec. 0.6]. In (144), $\mathbf{J}^{*}\mathbf{Z}_{E2} = \mathbf{0}$ means that $\mathbf{J}$ is orthogonal to each column vector of $\mathbf{Z}_{E2}$, or equivalently that $\mathbf{J}$ is orthogonal to $\mathrm{range}\,\mathbf{Z}_{E2}$. In (144), $\mathbf{J}^{*}\mathbf{K}_{E2}^{*}\mathbf{Z}_{PAM21} = \mathbf{0}$ means that $\mathbf{J}$ is orthogonal to $\mathrm{range}(\mathbf{K}_{E2}^{*}\mathbf{Z}_{PAM21})$.

If $\mathbf{Z}_{E2}$ is invertible, there is nothing to prove to obtain (141). In the opposite case, using q to denote $\mathrm{rank}\,\mathbf{Z}_{E2}$, where $q < n$, and using a Gram-Schmidt orthonormalization process, we can build an orthonormal basis $\mathbf{J}_1, \ldots, \mathbf{J}_n$ of $\mathbb{C}^n$, such that $\mathbf{J}_1, \ldots, \mathbf{J}_q$ is an orthonormal basis of $\mathrm{range}\,\mathbf{Z}_{E2}$. Here, for any $k \in \{q+1, \ldots, n\}$, $\mathbf{J}_k$ is orthogonal to $\mathrm{range}\,\mathbf{Z}_{E2}$, so that, by (144), $\mathbf{J}_k$ is orthogonal to $\mathrm{range}(\mathbf{K}_{E2}^{*}\mathbf{Z}_{PAM21})$.

Let $\mathbf{V}$ be an arbitrary element of $\mathbb{C}^n$, of coordinates $v_1, \ldots, v_n$ in the basis $\mathbf{J}_1, \ldots, \mathbf{J}_n$. For any $k \in \{1, \ldots, n\}$, we have $v_k = \mathbf{J}_k^{*}\mathbf{V}$. Thus, if $\mathbf{V} \in \mathrm{range}(\mathbf{K}_{E2}^{*}\mathbf{Z}_{PAM21})$, then $v_{q+1} = \ldots = v_n = 0$, so that $\mathbf{V} \in \mathrm{range}\,\mathbf{Z}_{E2}$. We have proven (141).

The proof for (142) corresponds to the proof for (141), with a different labeling of the port sets. □

Lemma 9. Let $\mathbf{A}$ be an arbitrary complex matrix. We use $\mathbf{A}^{\dagger}$, to denote the Moore-Penrose generalized inverse of $\mathbf{A}$. We assert that: $\mathbf{Z}_{E2}^{\dagger}$ and $\mathbf{Z}_{E1}^{\dagger}$ are positive semidefinite;

$$\mathbf{X} = \mathbf{Z}_{E2}^{\dagger}\mathbf{K}_{E2}^{*}\mathbf{Z}_{PAM21}\mathbf{I}_{S1} \tag{145}$$

is a solution of the equation

$$\mathbf{Z}_{E2}\mathbf{X} = \mathbf{K}_{E2}^{*}\mathbf{Z}_{PAM21}\mathbf{I}_{S1} \tag{146}$$

in the variable $\mathbf{X} \in \mathbb{C}^n$; and

$$\mathbf{X} = \mathbf{Z}_{E1}^{\dagger}\mathbf{K}_{E1}^{*}\mathbf{Z}_{PAM12}\mathbf{I}_{S2} \tag{147}$$

is a solution of the equation

$$\mathbf{Z}_{E1}\mathbf{X} = \mathbf{K}_{E1}^{*}\mathbf{Z}_{PAM12}\mathbf{I}_{S2} \tag{148}$$

in the variable $\mathbf{X} \in \mathbb{C}^m$.

Proof: Let p be a positive integer. If $\mathbf{A}$ is of size p by p and positive semidefinite, there exist a unitary matrix $\mathbf{U}$ and a real diagonal matrix $\Lambda = \mathrm{diag}_p(\lambda_1, \ldots, \lambda_p)$ such that we have $\lambda_1 \geqslant \ldots \geqslant \lambda_p \geqslant 0$ and $\mathbf{A} = \mathbf{U}\Lambda\mathbf{U}^{*}$. Here, $\mathbf{A} = \mathbf{U}\Lambda\mathbf{U}^{*}$ is a diagonalization of $\mathbf{A}$ and a singular value decomposition of $\mathbf{A}$. If $\mathbf{A}$ is invertible, since $\mathbf{A}^{-1} = \mathbf{A}^{\dagger}$, it follows from [8, Sec. 7.2.1] that $\mathbf{A}^{\dagger}$ is positive definite. If $\mathbf{A}$ is not invertible, we write $r = \mathrm{rank}\,\mathbf{A}$, and by [8, Sec. 7.3.P7] we get

$$\mathbf{A}^{\dagger} = \mathbf{U}\mathrm{diag}_p\left(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_r}, 0, \ldots, 0\right)\mathbf{U}^{*}\ , \tag{149}$$

which is hermitian and positive semidefinite according to [8, Sec. 7.2.1].

Thus, $\mathbf{A}^{\dagger}$ is positive semidefinite if $\mathbf{A}$ is positive semidefinite. It follows from Lemma 6 that $\mathbf{Z}_{E2}^{\dagger}$ and $\mathbf{Z}_{E1}^{\dagger}$ are positive semidefinite

By Lemma 8, (146) has at least one solution, and (148) has at least one solution. Consequently, by [15, Sec. 4.3] or [16, Sec. 5.7 to 5.8] or [8, Sec. 7.3.P9], we find that: (145) is a solution of the equation (146); and (147) is a solution of the equation (148). □

Theorem 20. In CC, for any $\mathbf{I}_{S1} \in \mathbb{C}^m$, if we study P_{CDP2} as a function of $\mathbf{I}_{C2}$, which by Lemma 4 may lie anywhere in $\mathbb{C}^n$, we find that P_{CDP2} has a maximum, denoted by $P_{CDP2\,\max}$ and given by

$$P_{CDP2\,\max} = \mathbf{I}_{S1}^{*}\mathbf{Z}_{CDP2\,\max}\mathbf{I}_{S1}\ , \tag{150}$$

where the impedance matrix

$$\mathbf{Z}_{CDP2\,\max} = \mathbf{Z}_{PAM21}^{*} \times \frac{\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^{*} + \mathbf{K}_{E2}\mathbf{Z}_{E2}^{\dagger}\mathbf{K}_{E2}^{*}}{2}\,\mathbf{Z}_{PAM21} \tag{151}$$

is positive semidefinite. Thus, $P_{CDP2\,\max}$ is nonnegative.

In CD, for any $\mathbf{I}_{S2} \in \mathbb{C}^n$, if we study P_{DDP1} as a function of $\mathbf{I}_{D1}$, which by Lemma 4 may lie anywhere in $\mathbb{C}^m$, we find that P_{DDP1} has a maximum, denoted by $P_{DDP1\,\max}$ and given by

$$P_{DDP1\,\max} = \mathbf{I}_{S2}^{*}\mathbf{Z}_{DDP1\,\max}\mathbf{I}_{S2}\ , \tag{152}$$

where the impedance matrix

$$\mathbf{Z}_{DDP1\,\max} = \mathbf{Z}_{PAM12}^{*} \times \frac{\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^{*} + \mathbf{K}_{E1}\mathbf{Z}_{E1}^{\dagger}\mathbf{K}_{E1}^{*}}{2}\,\mathbf{Z}_{PAM12} \tag{153}$$

is positive semidefinite. Thus, $P_{DDP1\,\max}$ is nonnegative.

Proof: According to Lemma 5, we have

$$\begin{aligned} 2P_{CDP2} = {} & \mathbf{I}_{S1}^{*}\mathbf{Z}_{PAM21}^{*}(\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^{*})\mathbf{Z}_{PAM21}\mathbf{I}_{S1} \\ & - \mathbf{I}_{C2}^{*}\mathbf{Z}_{E2}\mathbf{I}_{C2} \\ & + \mathbf{I}_{S1}^{*}\mathbf{Z}_{PAM21}^{*}\mathbf{K}_{E2}\mathbf{I}_{C2} + \mathbf{I}_{C2}^{*}\mathbf{K}_{E2}^{*}\mathbf{Z}_{PAM21}\mathbf{I}_{S1}\ . \end{aligned} \tag{154}$$

A variation $\Delta\mathbf{I}_{C2}$ in $\mathbf{I}_{C2}$ entails a variation ΔP_{CDP2} in P_{CDP2}, where ΔP_{CDP2} is given by

$$\begin{aligned} \Delta P_{CDP2} = \frac{1}{2}\big[& \Delta\mathbf{I}_{C2}^{*}(\mathbf{K}_{E2}^{*}\mathbf{Z}_{PAM21}\mathbf{I}_{S1} - \mathbf{Z}_{E2}\mathbf{I}_{C2}) \\ & + (\mathbf{I}_{S1}^{*}\mathbf{Z}_{PAM21}^{*}\mathbf{K}_{E2} - \mathbf{I}_{C2}^{*}\mathbf{Z}_{E2})\Delta\mathbf{I}_{C2} \\ & - \Delta\mathbf{I}_{C2}^{*}\mathbf{Z}_{E2}\Delta\mathbf{I}_{C2}\big]\ . \end{aligned} \tag{155}$$

Using Landau's little-o notation and the fact that $\mathbf{Z}_{E2}$ is hermitian, we obtain

$$\Delta P_{CDP2} = o\left(||\Delta \mathbf{I}_{C2}||_2\right) + \mathrm{Re}\left(\Delta \mathbf{I}_{C2}^{*}\left(\mathbf{K}_{E2}^{*}\mathbf{Z}_{PAM21}\mathbf{I}_{S1} - \mathbf{Z}_{E2}\mathbf{I}_{C2}\right)\right). \quad (156)$$

A stationary point of P_{CDP2} exists if and only if, for any $\Delta \mathbf{I}_{C2}$, we have $\Delta P_{CDP2} = o(||\Delta \mathbf{I}_{C2}||_2)$, that is if and only if we have

$$\mathrm{Re}(\Delta \mathbf{I}_{C2}^{*})\mathrm{Re}(\mathbf{K}_{E2}^{*}\mathbf{Z}_{PAM21}\mathbf{I}_{S1} - \mathbf{Z}_{E2}\mathbf{I}_{C2}) - \mathrm{Im}(\Delta \mathbf{I}_{C2}^{*})\mathrm{Im}(\mathbf{K}_{E2}^{*}\mathbf{Z}_{PAM21}\mathbf{I}_{S1} - \mathbf{Z}_{E2}\mathbf{I}_{C2}) = 0\,, \quad (157)$$

for any $\Delta \mathbf{I}_{C2}$. Consequently, $\mathbf{I}_{C2}$ is a stationary point of P_{CDP2} if and only if

$$\mathbf{Z}_{E2}\mathbf{I}_{C2} = \mathbf{K}_{E2}^{*}\mathbf{Z}_{PAM21}\mathbf{I}_{S1}\,. \quad (158)$$

Since, according to Lemma 9, (158) has a solution given by (145), it follows that a stationary point of P_{CDP2} exists. Using (154), (158), and the fact that $\mathbf{Z}_{E2}$ is hermitian, we find that, at any of the stationary points, the stationary value is

$$P_{CDP2} = \frac{1}{2}\left[\mathbf{I}_{S1}^{*}\mathbf{Z}_{PAM21}^{*}(\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^{*})\mathbf{Z}_{PAM21}\mathbf{I}_{S1} + \mathbf{I}_{C2}^{*}\mathbf{Z}_{E2}\mathbf{I}_{C2}\right]. \quad (159)$$

If $\mathbf{I}_{C2}$ is a stationary point of P_{CDP2}, (157) is satisfied, so that, according to (155), we have

$$\Delta P_{CDP2} = -\frac{1}{2}\Delta \mathbf{I}_{C2}^{*}\mathbf{Z}_{E2}\Delta \mathbf{I}_{C2}\,. \quad (160)$$

Thus, the single stationary value of P_{CDP2} is a maximum, since $\mathbf{Z}_{E2}$ is positive semidefinite by Lemma 6. Let us use $P_{CDP2\,\max}$ to denote this maximum, which is given by (159) where $\mathbf{I}_{C2}$ is any solution of (158). Using Lemma 9, we get

$$P_{CDP2\,\max} = \frac{1}{2}\left[\mathbf{I}_{S1}^{*}\mathbf{Z}_{PAM21}^{*}(\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^{*})\mathbf{Z}_{PAM21}\mathbf{I}_{S1} + \mathbf{I}_{S1}^{*}\mathbf{Z}_{PAM21}^{*}\mathbf{K}_{E2}\mathbf{Z}_{E2}^{\dagger}\mathbf{Z}_{E2}\mathbf{Z}_{E2}^{\dagger}\mathbf{K}_{E2}^{*}\mathbf{Z}_{PAM21}\mathbf{I}_{S1}\right]. \quad (161)$$

By [15, Sec. 4.3] or [8, Sec. 7.3.P7], $\mathbf{Z}_{E2}^{\dagger}$ satisfies

$$\mathbf{Z}_{E2}^{\dagger} = \mathbf{Z}_{E2}^{\dagger}\mathbf{Z}_{E2}\mathbf{Z}_{E2}^{\dagger}\,, \quad (162)$$

so that (161) leads us to (150)–(151). Moreover, the impedance matrix defined by (151) is positive semidefinite because $\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^{*}$ is positive definite and $\mathbf{Z}_{E2}^{\dagger}$ is positive semidefinite according to Lemma 9.

The proof for (152)–(153) corresponds to the proof for (150)–(151), with a different labeling of the port sets. □

C. SOME RESULTS USING OPEN-CIRCUIT VOLTAGES

To use open-circuit voltages, we need to define: the admittance matrix $\mathbf{Y}_{F2}$ given by

$$\mathbf{Y}_{F2} = \mathbf{Y}_{SAM22} + \mathbf{Y}_{SAM22}^{*} - \mathbf{Y}_{SAM22}^{*}(\mathbf{Z}_{S2} + \mathbf{Z}_{S2}^{*})\mathbf{Y}_{SAM22}\,, \quad (163)$$

which is of size n by n; the dimensionless matrix $\mathbf{K}_{F2}$ given by

$$\mathbf{K}_{F2} = (\mathbf{Z}_{S2} + \mathbf{Z}_{S2}^{*})\mathbf{Y}_{SAM22} - \mathbf{1}_n\,, \quad (164)$$

which is of size n by n; the admittance matrix $\mathbf{Y}_{F1}$ given by

$$\mathbf{Y}_{F1} = \mathbf{Y}_{SAM11} + \mathbf{Y}_{SAM11}^{*} - \mathbf{Y}_{SAM11}^{*}(\mathbf{Z}_{S1} + \mathbf{Z}_{S1}^{*})\mathbf{Y}_{SAM11}\,, \quad (165)$$

which is of size m by m; and the dimensionless matrix $\mathbf{K}_{F1}$ given by

$$\mathbf{K}_{F1} = (\mathbf{Z}_{S1} + \mathbf{Z}_{S1}^{*})\mathbf{Y}_{SAM11} - \mathbf{1}_m\,, \quad (166)$$

which is of size m by m.

Let $\mathbf{V}_{C2}$ be the column vector of size n given by

$$\mathbf{V}_{C2} = \mathbf{V}_2 + \mathbf{Z}_{S2}\mathbf{I}_2\,, \quad (167)$$

and $\mathbf{V}_{D1}$ be the column vector of size m given by

$$\mathbf{V}_{D1} = \mathbf{V}_1 + \mathbf{Z}_{S1}\mathbf{I}_1\,. \quad (168)$$

Theorem 21. In CC, for any $\mathbf{V}_{O1} \in \mathbb{C}^m$, if we study P_{CDP2} as a function of $\mathbf{V}_{C2}$, which may lie anywhere in $\mathbb{C}^n$, we find that P_{CDP2} has a maximum, denoted by $P_{CDP2\,\max}$ and given by

$$P_{CDP2\,\max} = \mathbf{V}_{O1}^{*}\mathbf{Y}_{CDP2\,\max}\mathbf{V}_{O1}\,, \quad (169)$$

where the admittance matrix

$$\mathbf{Y}_{CDP2\,\max} = \mathbf{Y}_{SAM21}^{*} \times \frac{\mathbf{Z}_{S2} + \mathbf{Z}_{S2}^{*} + \mathbf{K}_{F2}\mathbf{Y}_{F2}^{\dagger}\mathbf{K}_{F2}^{*}}{2}\mathbf{Y}_{SAM21} \quad (170)$$

is positive semidefinite. Thus, $P_{CDP2\,\max}$ is nonnegative.

In CD, for any $\mathbf{V}_{O2} \in \mathbb{C}^n$, if we study P_{DDP1} as a function of $\mathbf{V}_{D1}$, which may lie anywhere in $\mathbb{C}^m$, we find that P_{DDP1} has a maximum, denoted by $P_{DDP1\,\max}$ and given by

$$P_{DDP1\,\max} = \mathbf{V}_{O2}^{*}\mathbf{Y}_{DDP1\,\max}\mathbf{V}_{O2}\,, \quad (171)$$

where the admittance matrix

$$\mathbf{Y}_{DDP1\,\max} = \mathbf{Y}_{SAM12}^{*} \times \frac{\mathbf{Z}_{S1} + \mathbf{Z}_{S1}^{*} + \mathbf{K}_{F1}\mathbf{Y}_{F1}^{\dagger}\mathbf{K}_{F1}^{*}}{2}\mathbf{Y}_{SAM12} \quad (172)$$

is positive semidefinite. Thus, $P_{DDP1\,\max}$ is nonnegative.

Proof: Theorem 21 follows from Theorem 20 and the properties of dual networks [17, Ch. 10]. □

Corollary 7. In CC, the value of $P_{CDP2\,\max}$ given by Theorem 20 for a given $\mathbf{I}_{S1} \in \mathbb{C}^m$ is equal to the value of $P_{CDP2\,\max}$ given by Theorem 21 for $\mathbf{V}_{O1} = \mathbf{Z}_{S1}\mathbf{I}_{S1}$.

Likewise, in CD, the value of $P_{DDP1\,\max}$ given by Theorem 20 for a given $\mathbf{I}_{S2} \in \mathbb{C}^n$ is equal to the value of $P_{CDP2\,\max}$ given by Theorem 21 for $\mathbf{V}_{O2} = \mathbf{Z}_{S2}\mathbf{I}_{S2}$.

Proof: By (125) and (167), $\mathbf{V}_{C2} = \mathbf{Z}_{S2}\mathbf{I}_{C2}$, so that, since $\mathbf{Z}_{S2}$ is invertible, a maximum of P_{CDP2} for $\mathbf{I}_{C2}$ lying anywhere in $\mathbb{C}^n$ means the same thing as a maximum of P_{CDP2} for $\mathbf{V}_{C2}$ lying anywhere in $\mathbb{C}^n$.

Likewise, by (126) and (168), $\mathbf{V}_{D1} = \mathbf{Z}_{S1}\mathbf{I}_{D1}$, so that, since $\mathbf{Z}_{S1}$ is invertible, a maximum of P_{DDP1} for $\mathbf{I}_{D1}$ lying anywhere in $\mathbb{C}^m$ means the same thing as a maximum of P_{DDP1} for $\mathbf{V}_{D1}$ lying anywhere in $\mathbb{C}^m$. □

D. COMPUTATION OF THE AVAILABLE POWERS AT THE OUTPUT PORTS

Recall that an available power is defined in Section I as the greatest average power that can be drawn from one or more ports by an arbitrary LTI and passive load.

Observation 11. Neither Theorem 20 nor Theorem 21 prove that $P_{CDP2\,\max}$ can be reached using an LTI n-port device connected to port set 2 in CC, this n-port device being passive. Thus, at this stage, $P_{CDP2\,\max}$ need not be the available power at port set 2. Likewise, neither Theorem 20 nor Theorem 21 prove that $P_{DDP1\,\max}$ can be reached using an LTI m-port device connected to port set 1 in CD, this m-port device being passive. Thus, at this stage, $P_{DDP1\,\max}$ need not be the available power at port set 1.

Lemma 10. We assert that:

(a) a passive LTI n-port device, having an admittance matrix $\mathbf{Y}_{L2}$, is such that $P_{CDP2} = P_{CDP2\,\max}$ when it is connected to port set 2 in CC, if and only if there exists $\mathbf{I}_{C2\,\max} \in \mathbb{C}^n$ such that

$$\mathbf{Z}_{PAM21}\mathbf{I}_{S1} + \mathbf{Z}_{PAM22}\mathbf{I}_{C2\,\max} \in \ker\left(\mathbf{Z}^*_{PAM22}[\mathbf{Y}_{L2} + \mathbf{Y}^*_{S2}] - \mathbf{1}_n\right) \quad (173)$$

and

$$\mathbf{Z}_{E2}\mathbf{I}_{C2\,\max} = \mathbf{K}^*_{E2}\mathbf{Z}_{PAM21}\mathbf{I}_{S1}\,; \quad (174)$$

(b) a passive LTI n-port device, having an impedance matrix $\mathbf{Z}_{L2}$, is such that $P_{CDP2} = P_{CDP2\,\max}$ when it is connected to port set 2 in CC, if and only if there exists $\mathbf{V}_{C2\,\max} \in \mathbb{C}^n$ such that

$$\mathbf{Y}_{SAM21}\mathbf{V}_{O1} + \mathbf{Y}_{SAM22}\mathbf{V}_{C2\,\max} \in \ker\left(\mathbf{Y}^*_{SAM22}[\mathbf{Z}_{L2} + \mathbf{Z}^*_{S2}] - \mathbf{1}_n\right) \quad (175)$$

and

$$\mathbf{Y}_{F2}\mathbf{V}_{C2\,\max} = \mathbf{K}^*_{F2}\mathbf{Y}_{SAM21}\mathbf{V}_{O1}\,; \quad (176)$$

(c) a passive LTI m-port device, having an admittance matrix $\mathbf{Y}_{L1}$, is such that $P_{DDP1} = P_{DDP1\,\max}$ when it is connected to port set 1 in CD, if and only if there exists $\mathbf{I}_{D1\,\max} \in \mathbb{C}^m$ such that

$$\mathbf{Z}_{PAM12}\mathbf{I}_{S2} + \mathbf{Z}_{PAM11}\mathbf{I}_{D1\,\max} \in \ker\left(\mathbf{Z}^*_{PAM11}[\mathbf{Y}_{L1} + \mathbf{Y}^*_{S1}] - \mathbf{1}_m\right) \quad (177)$$

and

$$\mathbf{Z}_{E1}\mathbf{I}_{D1\,\max} = \mathbf{K}^*_{E1}\mathbf{Z}_{PAM12}\mathbf{I}_{S2}\,; \quad (178)$$

(d) a passive LTI m-port device, having an impedance matrix $\mathbf{Z}_{L1}$, is such that $P_{DDP1} = P_{DDP1\,\max}$ when it is connected to port set 1 in CD, if and only if there exists $\mathbf{V}_{D1\,\max} \in \mathbb{C}^m$ such that

$$\mathbf{Y}_{SAM12}\mathbf{V}_{O2} + \mathbf{Y}_{SAM11}\mathbf{V}_{D1\,\max} \in \ker\left(\mathbf{Y}^*_{SAM11}[\mathbf{Z}_{L1} + \mathbf{Z}^*_{S1}] - \mathbf{1}_m\right) \quad (179)$$

and

$$\mathbf{Y}_{F1}\mathbf{V}_{D1\,\max} = \mathbf{K}^*_{F1}\mathbf{Y}_{SAM12}\mathbf{V}_{O2}\,. \quad (180)$$

Proof: It follows from (123) and (125) that we have

$$\mathbf{V}_2 = \mathbf{Z}_{PAM21}\mathbf{I}_{S1} + \mathbf{Z}_{PAM22}\mathbf{I}_{C2}\,. \quad (181)$$

$\mathbf{I}_{C2}$ is a stationary point of P_{CDP2} if and only if the condition (158) is satisfied. Using (181) in (158), we obtain

$$\mathbf{Z}_{E2}\mathbf{I}_{C2} = \mathbf{K}^*_{E2}\left(\mathbf{V}_2 - \mathbf{Z}_{PAM22}\mathbf{I}_{C2}\right)\,. \quad (182)$$

If a passive LTI n-port device having an admittance matrix $\mathbf{Y}_{L2}$ is connected to port set 2, we have $\mathbf{I}_2 = -\mathbf{Y}_{L2}\mathbf{V}_2$. Thus, it follows from (125) that

$$\mathbf{I}_{C2} = (\mathbf{Y}_{S2} - \mathbf{Y}_{L2})\mathbf{V}_2\,. \quad (183)$$

The n-port device of admittance matrix $\mathbf{Y}_{L2}$ is such that $P_{CDP2} = P_{CDP2\,\max}$ when it is connected to port set 2 in CC, if and only if we can simultaneously satisfy: (181), which represents the characteristics of port set 2 of the DUS; (182), which expresses that $\mathbf{I}_{C2}$ is a stationary point; and (183) which represents the characteristics of the n-port device of admittance matrix $\mathbf{Y}_{L2}$.

Combining (182) and (183), we obtain

$$\left[(\mathbf{Z}_{E2} + \mathbf{K}^*_{E2}\mathbf{Z}_{PAM22})(\mathbf{Y}_{S2} - \mathbf{Y}_{L2}) - \mathbf{K}^*_{E2}\right]\mathbf{V}_2 = \mathbf{0}\,. \quad (184)$$

It follows from (128) and (129) that

$$\begin{aligned}(\mathbf{Z}_{E2} &+ \mathbf{K}^*_{E2}\mathbf{Z}_{PAM22})(\mathbf{Y}_{S2} - \mathbf{Y}_{L2}) - \mathbf{K}^*_{E2} \\ &= \mathbf{Z}^*_{PAM22}(\mathbf{Y}_{S2} - \mathbf{Y}_{L2}) - \mathbf{Z}^*_{PAM22}(\mathbf{Y}_{S2} + \mathbf{Y}^*_{S2}) + \mathbf{1}_n \\ &= -\mathbf{Z}^*_{PAM22}(\mathbf{Y}_{L2} + \mathbf{Y}^*_{S2}) + \mathbf{1}_n\,. \end{aligned} \quad (185)$$

Combining (184) and (185), we get

$$\left[\mathbf{Z}^*_{PAM22}(\mathbf{Y}_{L2} + \mathbf{Y}^*_{S2}) - \mathbf{1}_n\right]\mathbf{V}_2 = \mathbf{0}\,. \quad (186)$$

Taking into account (185), we can easily check that (181), (182) and (183) are simultaneously satisfied if and only if (158), (181) and (186) are simultaneously satisfied.

We can also eliminate $\mathbf{V}_2$ from (181) and (186) and note that the DUS enforces (181). It follows that the n-port device of admittance matrix $\mathbf{Y}_{L2}$ is such that $P_{CDP2} = P_{CDP2\,\max}$ when it is connected to port set 2 in CC, if and only if we can simultaneously satisfy (173) and (174).

We have proven assertion (a). Assertion (b) follows from assertion (a) and the properties of dual networks. Assertion (c) corresponds to assertion (a) with a different labeling of port sets. Assertion (d) corresponds to assertion (b) with a different labeling of port sets. □

Theorem 22. Ignoring noise power contributions, we have:

(a) if $\mathbf{Z}_{PAM22}$ is invertible, then $\mathbf{Y}_{L2} = \mathbf{Z}_{PAM22}^{-1*} - \mathbf{Y}_{S2}^*$ is such that any solution of (174) satisfies (173), and

$$P_{AAVP2} = \mathbf{I}_{S1}^* \mathbf{Z}_{AAVP2} \mathbf{I}_{S1}\,, \tag{187}$$

where the impedance matrix

$$\mathbf{Z}_{AAVP2} = \mathbf{Z}_{PAM21}^* \times \frac{\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^* + \mathbf{K}_{E2}\mathbf{Z}_{E2}^{\dagger}\mathbf{K}_{E2}^*}{2} \mathbf{Z}_{PAM21} \tag{188}$$

is positive semidefinite. Moreover, if $H(\mathbf{Z}_{PAM22}^{-1} - \mathbf{Y}_{S2})$ is invertible, then

$$\mathbf{Z}_{AAVP2} = \frac{1}{2}\mathbf{Z}_{PAM21}^* \mathbf{Z}_{PAM22}^{-1*} \times \left(\mathbf{Z}_{PAM22}^{-1} + \mathbf{Z}_{PAM22}^{-1*} - \mathbf{Y}_{S2} - \mathbf{Y}_{S2}^*\right)^{-1} \times \mathbf{Z}_{PAM22}^{-1} \mathbf{Z}_{PAM21}\,; \tag{189}$$

(b) if $\mathbf{Y}_{SAM22}$ is invertible, then $\mathbf{Z}_{L2} = \mathbf{Y}_{SAM22}^{-1*} - \mathbf{Z}_{S2}^*$ is such that any solution of (176) satisfies (175), and

$$P_{AAVP2} = \mathbf{V}_{O1}^* \mathbf{Y}_{AAVP2} \mathbf{V}_{O1}\,, \tag{190}$$

where the admittance matrix

$$\mathbf{Y}_{AAVP2} = \mathbf{Y}_{SAM21}^* \times \frac{\mathbf{Z}_{S2} + \mathbf{Z}_{S2}^* + \mathbf{K}_{F2}\mathbf{Y}_{F2}^{\dagger}\mathbf{K}_{F2}^*}{2} \mathbf{Y}_{SAM21} \tag{191}$$

is positive semidefinite. Moreover, if $H(\mathbf{Y}_{SAM22}^{-1} - \mathbf{Z}_{S2})$ is invertible, then

$$\mathbf{Y}_{AAVP2} = \frac{1}{2}\mathbf{Y}_{SAM21}^* \mathbf{Y}_{SAM22}^{-1*} \times \left(\mathbf{Y}_{SAM22}^{-1} + \mathbf{Y}_{SAM22}^{-1*} - \mathbf{Z}_{S2} - \mathbf{Z}_{S2}^*\right)^{-1} \times \mathbf{Y}_{SAM22}^{-1} \mathbf{Y}_{SAM21}\,; \tag{192}$$

(c) if $\mathbf{Z}_{PAM11}$ is invertible, then $\mathbf{Y}_{L1} = \mathbf{Z}_{PAM11}^{-1*} - \mathbf{Y}_{S1}^*$ is such that any solution of (178) satisfies (177), and

$$P_{BAVP1} = \mathbf{I}_{S2}^* \mathbf{Z}_{BAVP1} \mathbf{I}_{S2}\,, \tag{193}$$

where the impedance matrix

$$\mathbf{Z}_{BAVP1} = \mathbf{Z}_{PAM12}^* \times \frac{\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^* + \mathbf{K}_{E1}\mathbf{Z}_{E1}^{\dagger}\mathbf{K}_{E1}^*}{2} \mathbf{Z}_{PAM12} \tag{194}$$

is positive semidefinite. Moreover, if $H(\mathbf{Z}_{PAM11}^{-1} - \mathbf{Y}_{S1})$ is invertible, then

$$\mathbf{Z}_{BAVP1} = \frac{1}{2}\mathbf{Z}_{PAM12}^* \mathbf{Z}_{PAM11}^{-1*} \times \left(\mathbf{Z}_{PAM11}^{-1} + \mathbf{Z}_{PAM11}^{-1*} - \mathbf{Y}_{S1} - \mathbf{Y}_{S1}^*\right)^{-1} \times \mathbf{Z}_{PAM11}^{-1} \mathbf{Z}_{PAM12}\,; \tag{195}$$

(d) if $\mathbf{Y}_{SAM11}$ is invertible, then $\mathbf{Z}_{L1} = \mathbf{Y}_{SAM11}^{-1*} - \mathbf{Z}_{S1}^*$ is such that any solution of (180) satisfies (179), and

$$P_{BAVP1} = \mathbf{V}_{O2}^* \mathbf{Y}_{BAVP1} \mathbf{V}_{O2}\,, \tag{196}$$

where the admittance matrix

$$\mathbf{Y}_{BAVP1} = \mathbf{Y}_{SAM12}^* \times \frac{\mathbf{Z}_{S1} + \mathbf{Z}_{S1}^* + \mathbf{K}_{F1}\mathbf{Y}_{F1}^{\dagger}\mathbf{K}_{F1}^*}{2} \mathbf{Y}_{SAM12} \tag{197}$$

is positive semidefinite. Moreover, if $H(\mathbf{Y}_{SAM11}^{-1} - \mathbf{Z}_{S1})$ is invertible, then

$$\mathbf{Y}_{BAVP1} = \frac{1}{2}\mathbf{Y}_{SAM12}^* \mathbf{Y}_{SAM11}^{-1*} \times \left(\mathbf{Y}_{SAM11}^{-1} + \mathbf{Y}_{SAM11}^{-1*} - \mathbf{Z}_{S1} - \mathbf{Z}_{S1}^*\right)^{-1} \times \mathbf{Y}_{SAM11}^{-1} \mathbf{Y}_{SAM12}\,. \tag{198}$$

Proof: $\mathbf{Z}_{PAM22}$ being invertible, $\mathbf{Y}_{L2} = \mathbf{Z}_{PAM22}^{-1*} - \mathbf{Y}_{S2}^*$ is such that $\mathbf{Z}_{PAM22}^*(\mathbf{Y}_{L2} + \mathbf{Y}_{S2}^*) - \mathbf{1}_n = \mathbf{0}$, so that (173) is satisfied for any solution of (174), which exists by Lemma 9. Thus, (187)-(188) follow from Theorem 20 and Lemma 10.

Port set 2 may be viewed as an n-port generator of internal admittance matrix $\mathbf{Y}_{T1} = \mathbf{Z}_{PAM22}^{-1} - \mathbf{Y}_{S2}$ and rms short-circuit current vector $\mathbf{Z}_{PAM22}^{-1}\mathbf{Z}_{PAM21}\mathbf{I}_{S1}$ in CA, and as a load of admittance matrix $\mathbf{Y}_{T1}$ in CB. Thus, $H(\mathbf{Y}_{T1})$ is positive semidefinite. By [8, Sec. 7.2.1], it is positive definite if it is invertible. Thus, (189) follows from the maximum power transfer theorem for multiports [4], [13]. Appendix B shows that (189) can be alternatively derived from (188).

We have proven assertion (a). Assertion (b) follows from assertion (a) and the properties of dual networks. Assertions (c) and (d) correspond to assertions (a) and (b), respectively, with a different labeling of port sets. □

Corollary 8. Ignoring noise power contributions, we assert that:

(a) if $\mathbf{Z}_{PAM22}$ is invertible, it follows from (a) of Theorem 22 and $\mathbf{I}_{S1} = \mathbf{Y}_{S1}\mathbf{V}_{O1}$ that P_{AAVP2} is also given by (190), where the admittance matrix

$$\mathbf{Y}_{AAVP2} = \mathbf{Y}_{S1}^* \mathbf{Z}_{PAM21}^* \times \frac{\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^* + \mathbf{K}_{E2}\mathbf{Z}_{E2}^{\dagger}\mathbf{K}_{E2}^*}{2} \mathbf{Z}_{PAM21}\mathbf{Y}_{S1} \tag{199}$$

is positive semidefinite. Moreover, if $H(\mathbf{Z}_{PAM22}^{-1} - \mathbf{Y}_{S2})$ is invertible, then

$$\mathbf{Y}_{AAVP2} = \frac{1}{2}\mathbf{Y}_{S1}^* \mathbf{Z}_{PAM21}^* \mathbf{Z}_{PAM22}^{-1*} \times \left(\mathbf{Z}_{PAM22}^{-1} + \mathbf{Z}_{PAM22}^{-1*} - \mathbf{Y}_{S2} - \mathbf{Y}_{S2}^*\right)^{-1} \times \mathbf{Z}_{PAM22}^{-1} \mathbf{Z}_{PAM21}\mathbf{Y}_{S1}\,; \tag{200}$$

(b) if $\mathbf{Y}_{SAM22}$ is invertible, it follows from (b) of Theorem 22 and $\mathbf{V}_{O1} = \mathbf{Z}_{S1}\mathbf{I}_{S1}$ that P_{AAVP2} is also given by (187), where the impedance matrix

$$\mathbf{Z}_{AAVP2} = \mathbf{Z}_{S1}^* \mathbf{Y}_{SAM21}^* \times \frac{\mathbf{Z}_{S2} + \mathbf{Z}_{S2}^* + \mathbf{K}_{F2}\mathbf{Y}_{F2}^{\dagger}\mathbf{K}_{F2}^*}{2} \mathbf{Y}_{SAM21}\mathbf{Z}_{S1} \tag{201}$$

is positive semidefinite. Moreover, if $H(\mathbf{Y}_{SAM22}^{-1} - \mathbf{Z}_{S2})$ is invertible, then

$$\mathbf{Z}_{AAVP2} = \frac{1}{2}\mathbf{Z}_{S1}^{*}\mathbf{Y}_{SAM21}^{*}\mathbf{Y}_{SAM22}^{-1*} \times \left(\mathbf{Y}_{SAM22}^{-1} + \mathbf{Y}_{SAM22}^{-1*} - \mathbf{Z}_{S2} - \mathbf{Z}_{S2}^{*}\right)^{-1} \times \mathbf{Y}_{SAM22}^{-1}\mathbf{Y}_{SAM21}\mathbf{Z}_{S1}\,; \tag{202}$$

(c) if $\mathbf{Z}_{PAM11}$ is invertible, it follows from (c) of Theorem 22 and $\mathbf{I}_{S2} = \mathbf{Y}_{S2}\mathbf{V}_{O2}$ that P_{BAVP1} is also given by (196), where the admittance matrix

$$\mathbf{Y}_{BAVP1} = \mathbf{Y}_{S2}^{*}\mathbf{Z}_{PAM12}^{*} \times \frac{\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^{*} + \mathbf{K}_{E1}\mathbf{Z}_{E1}^{\dagger}\mathbf{K}_{E1}^{*}}{2}\mathbf{Z}_{PAM12}\mathbf{Y}_{S2} \tag{203}$$

is positive semidefinite. Moreover, if $H(\mathbf{Z}_{PAM11}^{-1} - \mathbf{Y}_{S1})$ is invertible, then

$$\mathbf{Y}_{BAVP1} = \frac{1}{2}\mathbf{Y}_{S2}^{*}\mathbf{Z}_{PAM12}^{*}\mathbf{Z}_{PAM11}^{-1*} \times \left(\mathbf{Z}_{PAM11}^{-1} + \mathbf{Z}_{PAM11}^{-1*} - \mathbf{Y}_{S1} - \mathbf{Y}_{S1}^{*}\right)^{-1} \times \mathbf{Z}_{PAM11}^{-1}\mathbf{Z}_{PAM12}\mathbf{Y}_{S2}\,; \tag{204}$$

(d) if $\mathbf{Y}_{SAM11}$ is invertible, it follows from (d) of Theorem 22 and $\mathbf{V}_{O2} = \mathbf{Z}_{S2}\mathbf{I}_{S2}$ that P_{BAVP1} is also given by (193), where the impedance matrix

$$\mathbf{Z}_{BAVP1} = \mathbf{Z}_{S2}^{*}\mathbf{Y}_{SAM12}^{*} \times \frac{\mathbf{Z}_{S1} + \mathbf{Z}_{S1}^{*} + \mathbf{K}_{F1}\mathbf{Y}_{F1}^{\dagger}\mathbf{K}_{F1}^{*}}{2}\mathbf{Y}_{SAM12}\mathbf{Z}_{S2} \tag{205}$$

is positive semidefinite. Moreover, if $H(\mathbf{Y}_{SAM11}^{-1} - \mathbf{Z}_{S1})$ is invertible, then

$$\mathbf{Z}_{BAVP1} = \frac{1}{2}\mathbf{Z}_{S2}^{*}\mathbf{Y}_{SAM12}^{*}\mathbf{Y}_{SAM11}^{-1*} \times \left(\mathbf{Y}_{SAM11}^{-1} + \mathbf{Y}_{SAM11}^{-1*} - \mathbf{Z}_{S1} - \mathbf{Z}_{S1}^{*}\right)^{-1} \times \mathbf{Y}_{SAM11}^{-1}\mathbf{Y}_{SAM12}\mathbf{Z}_{S2}\,. \tag{206}$$

Proof: If $\mathbf{Z}_{PAM22}$ is invertible, it follows from (a) of Theorem 22 and $\mathbf{I}_{S1} = \mathbf{Y}_{S1}\mathbf{V}_{O1}$ that P_{AAVP2} is also given by

$$P_{AAVP2} = \mathbf{V}_{O1}^{*}\mathbf{Y}_{AAVP2}'\mathbf{V}_{O1}\,, \tag{207}$$

where the admittance matrix

$$\mathbf{Y}_{AAVP2}' = \mathbf{Y}_{S1}^{*}\mathbf{Z}_{PAM21}^{*} \times \frac{\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^{*} + \mathbf{K}_{E2}\mathbf{Z}_{E2}^{\dagger}\mathbf{K}_{E2}^{*}}{2}\mathbf{Z}_{PAM21}\mathbf{Y}_{S1} \tag{208}$$

is positive semidefinite. Moreover, if $H(\mathbf{Z}_{PAM22}^{-1} - \mathbf{Y}_{S2})$ is invertible, then

$$\mathbf{Y}_{AAVP2}' = \frac{1}{2}\mathbf{Y}_{S1}^{*}\mathbf{Z}_{PAM21}^{*}\mathbf{Z}_{PAM22}^{-1*} \times \left(\mathbf{Z}_{PAM22}^{-1} + \mathbf{Z}_{PAM22}^{-1*} - \mathbf{Y}_{S2} - \mathbf{Y}_{S2}^{*}\right)^{-1} \times \mathbf{Z}_{PAM22}^{-1}\mathbf{Z}_{PAM21}\mathbf{Y}_{S1}\,. \tag{209}$$

It follows from (190) and (207) that, if $\mathbf{Z}_{PAM22}$ and $\mathbf{Y}_{SAM22}$ are invertible, then, for any $\mathbf{V}_{O1} \in \mathbb{C}^{m}$, we have

$$\mathbf{V}_{O1}^{*}\mathbf{Y}_{AAVP2}\mathbf{V}_{O1} = \mathbf{V}_{O1}^{*}\mathbf{Y}_{AAVP2}'\mathbf{V}_{O1}\,. \tag{210}$$

Thus, using (367) of Appendix C, we may conclude that $\mathbf{Y}_{AAVP2}' = \mathbf{Y}_{AAVP2}$. This is what allows us to obtain (199)–(200) from (208)–(209). Similar reasonings can be used to obtain (201)–(206). □

We now define two convenient propositions:

- proposition $\mathcal{P}_1$ is true if and only if $\mathbf{Z}_{PAM11}$ is invertible or $\mathbf{Y}_{SAM11}$ is invertible, or both;
- proposition $\mathcal{P}_2$ is true if and only if $\mathbf{Z}_{PAM22}$ is invertible or $\mathbf{Y}_{SAM22}$ is invertible, or both.

Corollary 9. Ignoring noise power contributions, we assert that:

(a) it follows from (a) of Theorem 22 and (b) of Corollary 8 that, if proposition $\mathcal{P}_2$ is true, P_{AAVP2} can be computed as a function of the variable $\mathbf{I}_{S1}$ using (187);
(b) it follows from (b) of Theorem 22 and (a) of Corollary 8 that, if proposition $\mathcal{P}_2$ is true, P_{AAVP2} can be computed as a function of the variable $\mathbf{V}_{O1}$ using (190);
(c) it follows from (c) of Theorem 22 and (d) of Corollary 8 that, if proposition $\mathcal{P}_1$ is true, P_{BAVP1} can be computed as a function of the variable $\mathbf{I}_{S2}$ using (193);
(d) it follows from (d) of Theorem 22 and (c) of Corollary 8 that, if proposition $\mathcal{P}_1$ is true, P_{BAVP1} can be computed as a function of the variable $\mathbf{V}_{O2}$ using (196).

There exist connections between some conditions used in Theorem 22 and Corollary 8, which are presented in the following Lemma.

Lemma 11. We assert that:

(a) $\mathbf{Y}_{SAM11}$ and $H(\mathbf{Y}_{SAM11}^{-1} - \mathbf{Z}_{S1})$ are invertible if and only if $\mathbf{Z}_{PAM11}$ and $H(\mathbf{Z}_{PAM11}^{-1} - \mathbf{Y}_{S1})$ are invertible;
(b) $\mathbf{Y}_{SAM22}$ and $H(\mathbf{Y}_{SAM22}^{-1} - \mathbf{Z}_{S2})$ are invertible if and only if $\mathbf{Z}_{PAM22}$ and $H(\mathbf{Z}_{PAM22}^{-1} - \mathbf{Y}_{S2})$ are invertible.

Proof: If $\mathbf{Z}_{PAM11}$ is invertible, port set 1 has an admittance matrix $\mathbf{Y}_{T2} = \mathbf{Z}_{PAM11}^{-1} - \mathbf{Y}_{S1}$ in CA, which must be such that $H(\mathbf{Y}_{T2})$ is positive semidefinite. Thus, if $\mathbf{Z}_{PAM11}$ and $H(\mathbf{Z}_{PAM11}^{-1} - \mathbf{Y}_{S1})$ are invertible, $H(\mathbf{Y}_{T2})$ is positive definite, so that, by Lemma 1 of Part 1, $\mathbf{Y}_{T2}$ is invertible, port set 1 has an impedance matrix $\mathbf{Z}_{T2} = \mathbf{Y}_{T2}^{-1}$ and $H(\mathbf{Z}_{T2})$ is positive definite. Thus, by Corollary 2 of Part 1, $\mathbf{Y}_{SAM11}$ is invertible and $\mathbf{Y}_{SAM11}^{-1} = \mathbf{Z}_{T2} + \mathbf{Z}_{S1}$, so that $H(\mathbf{Y}_{SAM11}^{-1} - \mathbf{Z}_{S1}) = H(\mathbf{Z}_{T2})$ is invertible.

We have shown that $\mathbf{Y}_{SAM11}$ and $H(\mathbf{Y}_{SAM11}^{-1} - \mathbf{Z}_{S1})$ are invertible if $\mathbf{Z}_{PAM11}$ and $H(\mathbf{Z}_{PAM11}^{-1} - \mathbf{Y}_{S1})$ are invertible. It follows from the properties of dual networks that $\mathbf{Z}_{PAM11}$ and $H(\mathbf{Z}_{PAM11}^{-1} - \mathbf{Y}_{S1})$ are invertible if $\mathbf{Y}_{SAM11}$ and $H(\mathbf{Y}_{SAM11}^{-1} - \mathbf{Z}_{S1})$ are invertible. This proves (a). Assertion (b) is assertion (a) with a different labeling of port sets. □

Lemma 11 allows us to define two propositions which will be convenient in what follows, especially when we use Theorem 22 or Corollary 8:

- proposition $\mathcal{P}_3$ is true if and only if $\mathbf{Y}_{SAM11}$ and $H(\mathbf{Y}_{SAM11}^{-1}-\mathbf{Z}_{S1})$ are invertible, or equivalently if and only if $\mathbf{Z}_{PAM11}$ and $H(\mathbf{Z}_{PAM11}^{-1}-\mathbf{Y}_{S1})$ are invertible;
- proposition $\mathcal{P}_4$ is true if and only if $\mathbf{Y}_{SAM22}$ and $H(\mathbf{Y}_{SAM22}^{-1}-\mathbf{Z}_{S2})$ are invertible, or equivalently if and only if $\mathbf{Z}_{PAM22}$ and $H(\mathbf{Z}_{PAM22}^{-1}-\mathbf{Y}_{S2})$ are invertible.

VII. POWER TRANSFER RATIOS

A. DEFINITIONS AND BASIC FORMULAE

We introduce the power transfer ratio in CA at port set 1 of the DUS, given by

$$t_{A1}=\frac{P_{ARP1}}{P_{AAVG1}}\,, \tag{211}$$

which by (27) satisfies $0\leqslant t_{A1}\leqslant 1$. We introduce the power transfer ratio in CA at port set 2 of the DUS, given by

$$t_{A2}=\frac{P_{ADP2}}{P_{AAVP2}}\,, \tag{212}$$

which by (28) satisfies $0\leqslant t_{A2}\leqslant 1$. If $n=m$, we introduce the power transfer ratio in CA without the DUS, given by

$$t_{AW}=\frac{P_{AW}}{P_{AAVG1}}\,, \tag{213}$$

which by (29) satisfies $0\leqslant t_{AW}\leqslant 1$.

We introduce the power transfer ratio in CB at port set 1 of the DUS, given by

$$t_{B1}=\frac{P_{BDP1}}{P_{BAVP1}}\,, \tag{214}$$

which by (31) satisfies $0\leqslant t_{B1}\leqslant 1$. We introduce the power transfer ratio in CB at port set 2 of the DUS, given by

$$t_{B2}=\frac{P_{BRP2}}{P_{BAVG2}}\,, \tag{215}$$

which by (30) satisfies $0\leqslant t_{B2}\leqslant 1$. If $n=m$, we introduce the power transfer ratio in CB without the DUS, given by

$$t_{BW}=\frac{P_{BW}}{P_{BAVG2}}\,, \tag{216}$$

which by (32) satisfies $0\leqslant t_{BW}\leqslant 1$.

We have currently considered 6 power gains (two of them being insertion power gains, hence valid only if $m=n$) and 6 power transfer ratios. Some equalities connect the ones that are defined for a given excitation:

$$G_{AT}=G_{AO}\,t_{A1}\ \text{ and }\ G_{BT}=G_{BO}\,t_{B2}\,; \tag{217}$$

and, in the case $m=n$,

$$G_{AT}=G_{AI}\,t_{AW}\ \text{ and }\ G_{BT}=G_{BI}\,t_{BW}\,. \tag{218}$$

To define the excitation in CA, let $\mathbf{X}_A$ denote one of the variables $\mathbf{V}_{O1}$ or $\mathbf{I}_{S1}$.

Based on the results of Section III.C and Section III.D, we find that t_{A1} is given by

$$t_{A1}=\frac{\mathbf{X}_A^*\ \mathbf{N}_{A1}\ \mathbf{X}_A}{\mathbf{X}_A^*\ \mathbf{D}_{A1}\ \mathbf{X}_A}\,, \tag{219}$$

where $\mathbf{N}_{A1}$ and $\mathbf{D}_{A1}$ are hermitian matrices of size m by m, and given in Table 7. We note that $\mathbf{N}_{A1}$ is positive semidefinite and $\mathbf{D}_{A1}$ is positive definite.

TABLE 7. Variable $\mathbf{X}_A$ and associated $\mathbf{N}_{A1}$ and $\mathbf{D}_{A1}$.

Variable $\mathbf{X}_A$	$\mathbf{N}_{A1}$	$\mathbf{D}_{A1}$
$\mathbf{V}_{O1}$	$\mathbf{Y}_{ARP1}$	$\mathbf{Y}_{AAVG1}$
$\mathbf{I}_{S1}$	$\mathbf{Z}_{ARP1}$	$\mathbf{Z}_{AAVG1}$

t_{A1} is given by (219) in the form of a generalized Rayleigh ratio of $\mathbf{N}_{A1}$ to $\mathbf{D}_{A1}$, in the variable $\mathbf{X}_A$. Thus, t_{A1} depends on the excitation. Since $\mathbf{D}_{A1}$ is positive definite, t_{A1} is defined for any nonzero $\mathbf{X}_A\in\mathbb{C}^m$.

Based on Section III.C, Section III.D and Corollary 9, we find that, if proposition $\mathcal{P}_2$ is true, t_{A2} is given by

$$t_{A2}=\frac{\mathbf{X}_A^*\ \mathbf{N}_{A2}\ \mathbf{X}_A}{\mathbf{X}_A^*\ \mathbf{D}_{A2}\ \mathbf{X}_A}\,, \tag{220}$$

where $\mathbf{N}_{A2}$ and $\mathbf{D}_{A2}$ are hermitian matrices of size m by m, and given in Table 8. We note that $\mathbf{N}_{A2}$ and $\mathbf{D}_{A2}$ are positive semidefinite.

TABLE 8. Variable $\mathbf{X}_A$ and associated $\mathbf{N}_{A2}$ and $\mathbf{D}_{A2}$.

Variable $\mathbf{X}_A$	Applicability	$\mathbf{N}_{A2}$	$\mathbf{D}_{A2}$
$\mathbf{V}_{O1}$	proposition $\mathcal{P}_2$ is true	$\mathbf{Y}_{ADP2}$	$\mathbf{Y}_{AAVP2}$
$\mathbf{I}_{S1}$	proposition $\mathcal{P}_2$ is true	$\mathbf{Z}_{ADP2}$	$\mathbf{Z}_{AAVP2}$

If proposition $\mathcal{P}_2$ is true, t_{A2} is given by (220) in the form of a generalized Rayleigh ratio of $\mathbf{N}_{A2}$ to $\mathbf{D}_{A2}$, in the variable $\mathbf{X}_A$. It follows that t_{A2} depends on the excitation, and that, according to the explanations provided in Section II.A, t_{A2} is defined for $\mathbf{X}_A\in D(\mathbf{D}_{A2})$, where

$$D(\mathbf{D}_{A2})=\{\mathbf{X}_A\in\mathbb{C}^m:\mathbf{X}_A\notin\ker\mathbf{D}_{A2}\}\,. \tag{221}$$

Based on the results of Section III.C and Section III.D, we find that, if $n=m$, then t_{AW} is given by

$$t_{AW}=\frac{\mathbf{X}_A^*\ \mathbf{N}_{AW}\ \mathbf{X}_A}{\mathbf{X}_A^*\ \mathbf{D}_{AW}\ \mathbf{X}_A}\,, \tag{222}$$

where $\mathbf{N}_{AW}$ and $\mathbf{D}_{AW}$ are hermitian matrices of size m by m, and given in Table 9. We note that $\mathbf{N}_{AW}$ and $\mathbf{D}_{AW}$ are positive definite.

TABLE 9. Variable $\mathbf{X}_A$ and associated $\mathbf{N}_{AW}$ and $\mathbf{D}_{AW}$.

Variable $\mathbf{X}_A$	$\mathbf{N}_{AW}$	$\mathbf{D}_{AW}$
$\mathbf{V}_{O1}$	$\mathbf{Y}_{AW}$	$\mathbf{Y}_{AAVG1}$
$\mathbf{I}_{S1}$	$\mathbf{Z}_{AW}$	$\mathbf{Z}_{AAVG1}$

t_{AW} is given by (222) in the form of a generalized Rayleigh ratio of $\mathbf{N}_{AW}$ to $\mathbf{D}_{AW}$, in the variable $\mathbf{X}_A$. Thus, t_{AW} depends on the excitation. Since $\mathbf{D}_{AW}$ is positive definite, t_{AW} is defined for any nonzero $\mathbf{X}_A\in\mathbb{C}^m$.

To define the excitation in CB, let $\mathbf{X}_B$ denote one of the variables $\mathbf{V}_{O2}$ or $\mathbf{I}_{S2}$.

Based on Section III.C, Section III.D and Corollary 9, we find that, if proposition $\mathcal{P}_1$ is true, t_{B1} is given by

$$t_{B1} = \frac{\mathbf{X}_B^* \, \mathbf{N}_{B1} \, \mathbf{X}_B}{\mathbf{X}_B^* \, \mathbf{D}_{B1} \, \mathbf{X}_B} \,, \tag{223}$$

where $\mathbf{N}_{B1}$ and $\mathbf{D}_{B1}$ are hermitian matrices of size n by n, and given in Table 10. We note that $\mathbf{N}_{B1}$ and $\mathbf{D}_{B1}$ are positive semidefinite.

TABLE 10. Variable $\mathbf{X}_B$ and associated $\mathbf{N}_{B1}$ and $\mathbf{D}_{B1}$.

Variable $\mathbf{X}_B$	Applicability	$\mathbf{N}_{B1}$	$\mathbf{D}_{B1}$
$\mathbf{V}_{O2}$	proposition $\mathcal{P}_1$ is true	$\mathbf{Y}_{BDP1}$	$\mathbf{Y}_{BAVP1}$
$\mathbf{I}_{S2}$	proposition $\mathcal{P}_1$ is true	$\mathbf{Z}_{BDP1}$	$\mathbf{Z}_{BAVP1}$

If proposition $\mathcal{P}_1$ is true, t_{B1} is given by (223) in the form of a generalized Rayleigh ratio of $\mathbf{N}_{B1}$ to $\mathbf{D}_{B1}$, in the variable $\mathbf{X}_B$. Thus, t_{B1} depends on the excitation and is defined for $\mathbf{X}_B \in D(\mathbf{D}_{B1})$, where

$$D(\mathbf{D}_{B1}) = \{\mathbf{X}_B \in \mathbb{C}^n : \mathbf{X}_B \notin \ker \mathbf{D}_{B1}\} \,. \tag{224}$$

Based on the results of Section III.C and Section III.D, we find that t_{B2} is given by

$$t_{B2} = \frac{\mathbf{X}_B^* \, \mathbf{N}_{B2} \, \mathbf{X}_B}{\mathbf{X}_B^* \, \mathbf{D}_{B2} \, \mathbf{X}_B} \,, \tag{225}$$

where $\mathbf{N}_{B2}$ and $\mathbf{D}_{B2}$ are hermitian matrices of size n by n, and given in Table 11. We note that $\mathbf{N}_{B2}$ is positive semidefinite and $\mathbf{D}_{B2}$ is positive definite.

TABLE 11. Variable $\mathbf{X}_B$ and associated $\mathbf{N}_{B2}$ and $\mathbf{D}_{B2}$.

Variable $\mathbf{X}_B$	$\mathbf{N}_{B2}$	$\mathbf{D}_{B2}$
$\mathbf{V}_{O2}$	$\mathbf{Y}_{BRP2}$	$\mathbf{Y}_{BAVG2}$
$\mathbf{I}_{S2}$	$\mathbf{Z}_{BRP2}$	$\mathbf{Z}_{BAVG2}$

t_{B2} is given by (225) in the form of a generalized Rayleigh ratio of $\mathbf{N}_{B2}$ to $\mathbf{D}_{B2}$, in the variable $\mathbf{X}_B$. Thus, t_{B2} depends on the excitation. Since $\mathbf{D}_{B2}$ is positive definite, t_{B2} is defined for any nonzero $\mathbf{X}_B \in \mathbb{C}^n$.

Based on the results of Section III.C and Section III.D, we find that, if $n = m$, then t_{BW} is given by

$$t_{BW} = \frac{\mathbf{X}_B^* \, \mathbf{N}_{BW} \, \mathbf{X}_B}{\mathbf{X}_B^* \, \mathbf{D}_{BW} \, \mathbf{X}_B} \,, \tag{226}$$

where $\mathbf{N}_{BW}$ and $\mathbf{D}_{BW}$ are hermitian matrices of size m by m, and given in Table 12. We note that $\mathbf{N}_{BW}$ and $\mathbf{D}_{BW}$ are positive definite.

TABLE 12. Variable $\mathbf{X}_B$ and associated $\mathbf{N}_{BW}$ and $\mathbf{D}_{BW}$.

Variable $\mathbf{X}_B$	$\mathbf{N}_{BW}$	$\mathbf{D}_{BW}$
$\mathbf{V}_{O2}$	$\mathbf{Y}_{BW}$	$\mathbf{Y}_{BAVG2}$
$\mathbf{I}_{S2}$	$\mathbf{Z}_{BW}$	$\mathbf{Z}_{BAVG2}$

t_{BW} is given by (226) in the form of a generalized Rayleigh ratio of $\mathbf{N}_{BW}$ to $\mathbf{D}_{BW}$, in the variable $\mathbf{X}_B$. Thus, t_{BW} depends on the excitation. Since $\mathbf{D}_{BW}$ is positive definite, t_{BW} is defined for any nonzero $\mathbf{X}_B \in \mathbb{C}^m$.

B. BOUNDS OF THE POWER TRANSFER RATIOS WITHOUT THE DUS

By Observation 7, for $m = n$, we can assert that: the set of the values of t_{AW} obtained for all $\mathbf{X}_A \in \mathbb{C}^m$ such that $\mathbf{X}_A \neq \mathbf{0}$ is equal to the set of the values of t_{AW} obtained for all $\mathbf{X}_A \in \mathbb{S}_m$; and the set of the values of t_{BW} obtained for all $\mathbf{X}_B \in \mathbb{C}^m$ such that $\mathbf{X}_B \neq \mathbf{0}$ is equal to the set of the values of t_{BW} obtained for all $\mathbf{X}_B \in \mathbb{S}_m$.

Since $\mathbf{I}_{S1} = \mathbf{Y}_{S1}\mathbf{V}_{O1}$, where $\mathbf{Y}_{S1}$ is invertible, we can assert that the set of the values of t_{AW} obtained for all nonzero $\mathbf{X}_A \in \mathbb{C}^m$, or for all $\mathbf{X}_A \in \mathbb{S}_m$, does not depend on the choice $\mathbf{X}_A = \mathbf{V}_{O1}$ or $\mathbf{X}_A = \mathbf{I}_{S1}$.

Likewise, since $m = n$ and $\mathbf{I}_{S2} = \mathbf{Y}_{S2}\mathbf{V}_{O2}$, where $\mathbf{Y}_{S2}$ is invertible, we can assert that the set of the values of t_{BW} obtained for all nonzero $\mathbf{X}_B \in \mathbb{C}^m$, or for all $\mathbf{X}_B \in \mathbb{S}_m$, does not depend on the choice $\mathbf{X}_B = \mathbf{V}_{O2}$ or $\mathbf{X}_B = \mathbf{I}_{S2}$.

We can now state and prove a reciprocal theorem on the bounds of the sets of the values of the power transfer ratios without the DUS in CA and CB.

Theorem 23. We assume $n = m$. Ignoring noise power contributions, we can assert that:

(a) the set of the values of the power transfer ratio in CA without the DUS, obtained for all nonzero $\mathbf{V}_{O1} \in \mathbb{C}^m$, or equivalently for all nonzero $\mathbf{I}_{S1} \in \mathbb{C}^m$, has a least element referred to as "minimum value" and denoted by $t_{AW\,MIN}$, and a greatest element referred to as "maximum value" and denoted by $t_{AW\,MAX}$;

(b) if $\lambda_{\min}$ and $\lambda_{\max}$ are given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{AW}$ to $\mathbf{D}_{AW}$, in the variable $\mathbf{X}_A$ according to (222) and Table 9, we have $t_{AW\,MIN} = \lambda_{\min}$ and $t_{AW\,MAX} = \lambda_{\max}$;

(c) if $\mathbf{M}$ is given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{AW}$ to $\mathbf{D}_{AW}$, in the variable $\mathbf{X}_A$, an average value of t_{AW} over m nonzero excitations is

$$t_{AW\,AVR} = \frac{\operatorname{tr}\mathbf{M}}{m} = \frac{\operatorname{tr}\left(\mathbf{N}_{AW}\mathbf{D}_{AW}^{-1}\right)}{m} \,; \tag{227}$$

(d) $t_{AW\,AVR}$ doesn't depend on the choice of $\mathbf{X}_A$, and

$$0 \leqslant t_{AW\,MIN} \leqslant t_{AW\,AVR} \leqslant t_{AW\,MAX} \leqslant 1 \,; \tag{228}$$

(e) the set of the values of the power transfer ratio in CB without the DUS, obtained for all nonzero $\mathbf{V}_{O2} \in \mathbb{C}^m$, or equivalently for all nonzero $\mathbf{I}_{S2} \in \mathbb{C}^m$, has a least element referred to as "minimum value" and denoted by $t_{BW\,MIN}$, and a greatest element referred to as "maximum value" and denoted by $t_{BW\,MAX}$;

(f) if $\lambda_{\min}$ and $\lambda_{\max}$ are given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{BW}$ to $\mathbf{D}_{BW}$, in the variable $\mathbf{X}_B$ according to (226) and Table 12, we have $t_{BW\,MIN} = \lambda_{\min}$ and $t_{BW\,MAX} = \lambda_{\max}$;

(g) if $\mathbf{M}$ is given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{BW}$ to $\mathbf{D}_{BW}$, in the variable $\mathbf{X}_B$, an average value of t_{BW} over m nonzero excitations is

$$t_{BW\,AVR} = \frac{\operatorname{tr}\mathbf{M}}{m} = \frac{\operatorname{tr}\left(\mathbf{N}_{BW}\mathbf{D}_{BW}^{-1}\right)}{m} \,; \tag{229}$$

(h) $t_{BW\,AVR}$ doesn't depend on the choice of $\mathbf{X}_B$, and

$$0 \leqslant t_{BW\,MIN} \leqslant t_{BW\,AVR} \leqslant t_{BW\,MAX} \leqslant 1\,; \quad (230)$$

(i) we have

$$t_{AW\,MAX} = t_{BW\,MAX}\,, \quad (231)$$

$$t_{AW\,AVR} = t_{BW\,AVR}\,, \quad (232)$$

and

$$t_{AW\,MIN} = t_{BW\,MIN}\,. \quad (233)$$

Proof: Since $\mathbf{D}_{AW}$ and $\mathbf{D}_{BW}$ are positive definite, assertions (a), (b), (e) and (f) directly follow from Theorem 12.

In (c), by Theorem 12 and [8, Sec. 1.3.3], we obtain $\operatorname{tr}\mathbf{M} = \operatorname{tr}\left(\mathbf{N}_{AW}\mathbf{D}_{AW}^{-1}\right)$, and the second equality of (227). By (8), each eigenvector $\mathbf{y}$ of $\mathbf{M}$ corresponds to a nonzero excitation $\mathbf{X}_A = \mathbf{D}_{AW}^{-1/2}\mathbf{y}$, and to an eigenvalue that is equal to t_{AW} for this $\mathbf{X}_A$. Using Table 9, (48) and (64), we get

$$\operatorname{rank}\left(\mathbf{N}_{AW}\mathbf{D}_{AW}^{-1}\right) = \operatorname{rank}\mathbf{N}_{AW} = m\,, \quad (234)$$

so that the number of nonzero eigenvalues of $\mathbf{M}$, counting multiplicity, is m. Since $\operatorname{tr}\mathbf{M}$ is the sum of the eigenvalues of $\mathbf{M}$, counting multiplicity, it follows that $t_{AW\,AVR}$ given by (227) is an average of t_{AW} over m nonzero excitations. This proves (c). Assertion (d) follows from (c), (372) of Appendix C, and $0 \leqslant t_{AW} \leqslant 1$.

In (g), by Theorem 12, we have $\operatorname{tr}\mathbf{M} = \operatorname{tr}\left(\mathbf{N}_{BW}\mathbf{D}_{BW}^{-1}\right)$, which allows us to write the second equality of (229). Using Table 12, (54) and (70), we get

$$\operatorname{rank}\left(\mathbf{N}_{BW}\mathbf{D}_{BW}^{-1}\right) = \operatorname{rank}\mathbf{N}_{BW} = m\,, \quad (235)$$

which can be used to prove (g) as we used (234) to prove (c). Assertion (h) follows from (g), (373) of Appendix C, and $0 \leqslant t_{BW} \leqslant 1$.

To prove (i), we can assume $\mathbf{X}_A = \mathbf{I}_{S1}$ and $\mathbf{X}_B = \mathbf{I}_{S2}$. By Theorem 12, we only need to compare the eigenvalues of $\mathbf{L} = \mathbf{N}_{AW}\mathbf{D}_{AW}^{-1}$ with the eigenvalues of $\mathbf{K} = \mathbf{N}_{BW}\mathbf{D}_{BW}^{-1}$. It follows from Table 12, (58) and (70) that

$$\begin{aligned}\mathbf{K} &= (\mathbf{Y}_{S1} + \mathbf{Y}_{S2})^{-1*} \\ &\quad\times (\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^{*})(\mathbf{Y}_{S1} + \mathbf{Y}_{S2})^{-1}(\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^{*})\,, \end{aligned} \quad (236)$$

which is of size m by m.

It follows from Table 9, (56) and (64) that

$$\begin{aligned}\mathbf{L} &= (\mathbf{Y}_{S1} + \mathbf{Y}_{S2})^{-1*} \\ &\quad\times (\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^{*})(\mathbf{Y}_{S1} + \mathbf{Y}_{S2})^{-1}(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^{*})\,, \end{aligned} \quad (237)$$

which is of size m by m. It follows from Lemma 3 of Part 2 applied to $\mathbf{A} = \mathbf{Y}_{S1}$ and $\mathbf{B} = \mathbf{Y}_{S2}$ that $\mathbf{K}$ and $\mathbf{L}$ have the same eigenvalues, counting multiplicity, which leads to the final assertions of Theorem 23. □

Theorem 23 may be viewed as a better version of Theorem 8 of Part 2.

C. BOUNDS OF THE POWER TRANSFER RATIOS AT PORT SET 1

By Observation 7, the set of the values of t_{A1} obtained for all $\mathbf{X}_A \in \mathbb{C}^m$ such that $\mathbf{X}_A \neq \mathbf{0}$ is equal to the set of the values of t_{A1} obtained for all $\mathbf{X}_A \in \mathbb{S}_m$.

If proposition $\mathcal{P}_1$ is true, the set of the values of t_{B1} obtained for all $\mathbf{X}_B \in D(\mathbf{D}_{B1})$ is equal to the set of the values of t_{B1} obtained for all $\mathbf{X}_B \in D(\mathbf{D}_{B1}) \cap \mathbb{S}_n$.

Since $\mathbf{I}_{S1} = \mathbf{Y}_{S1}\mathbf{V}_{O1}$, where $\mathbf{Y}_{S1}$ is invertible, we can assert that the set of the values of t_{A1} obtained for all nonzero $\mathbf{X}_A \in \mathbb{C}^m$, or for all $\mathbf{X}_A \in \mathbb{S}_m$, does not depend on the choice $\mathbf{X}_A = \mathbf{V}_{O1}$ or $\mathbf{X}_A = \mathbf{I}_{S1}$.

Since $\mathbf{I}_{S2} = \mathbf{Y}_{S2}\mathbf{V}_{O2}$, where $\mathbf{Y}_{S2}$ is invertible, we can demonstrate that, if proposition $\mathcal{P}_1$ is true, the set of the values of t_{B1} obtained for all $\mathbf{X}_B \in D(\mathbf{D}_{B1})$, or equivalently for all $\mathbf{X}_B \in D(\mathbf{D}_{B1}) \cap \mathbb{S}_n$, is independent of the choice $\mathbf{X}_B = \mathbf{V}_{O2}$ or $\mathbf{X}_B = \mathbf{I}_{S2}$.

We can now state and prove a new reciprocal theorem on the bounds of the sets of the values of the power transfer ratios at port set 1 in CA and CB.

Theorem 24. Ignoring noise power contributions, we can assert that:

(a) the set of the values of the power transfer ratio in CA at port set 1 of the DUS, obtained for all nonzero $\mathbf{V}_{O1} \in \mathbb{C}^m$, or equivalently for all nonzero $\mathbf{I}_{S1} \in \mathbb{C}^m$, has a least element referred to as "minimum value" and denoted by $t_{A1\,MIN}$, and a greatest element referred to as "maximum value" and denoted by $t_{A1\,MAX}$;

(b) if $\lambda_{\min}$ and $\lambda_{\max}$ are given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{A1}$ to $\mathbf{D}_{A1}$, in the variable $\mathbf{X}_A$ according to (219) and Table 7, we have $t_{A1\,MIN} = \lambda_{\min}$ and $t_{A1\,MAX} = \lambda_{\max}$;

(c) if $\mathbf{M}$ is given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{A1}$ to $\mathbf{D}_{A1}$, in the variable $\mathbf{X}_A$, an average value of t_{A1} over m nonzero excitations is

$$t_{A1\,AVR} = \frac{\operatorname{tr}\mathbf{M}}{m} = \frac{\operatorname{tr}\left(\mathbf{N}_{A1}\mathbf{D}_{A1}^{-1}\right)}{m}\,; \quad (238)$$

(d) $t_{A1\,AVR}$ doesn't depend on the choice of $\mathbf{X}_A$, and

$$0 \leqslant t_{A1\,MIN} \leqslant t_{A1\,AVR} \leqslant t_{A1\,MAX} \leqslant 1\,; \quad (239)$$

(e) if proposition $\mathcal{P}_1$ is true, we have $\ker\mathbf{D}_{B1} \subset \ker\mathbf{N}_{B1}$ so that Theorem 14 can be applied to the generalized Rayleigh ratio of $\mathbf{N}_{B1}$ to $\mathbf{D}_{B1}$;

(f) if proposition $\mathcal{P}_1$ is true, the set of the values of the power transfer ratio in CB at port set 1 of the DUS, obtained for all $\mathbf{X}_B \in D(\mathbf{D}_{B1})$, has a least element referred to as "minimum value" and denoted by $t_{B1\,MIN}$, and a greatest element referred to as "maximum value" and denoted by $t_{B1\,MAX}$;

(g) if proposition $\mathcal{P}_1$ is true, and if $\kappa_{\min}$ and $\kappa_{\max}$ are given by Theorem 14 applied to the generalized Rayleigh ratio of $\mathbf{N}_{B1}$ to $\mathbf{D}_{B1}$, in the variable $\mathbf{X}_B$ according to (223) and Table 10, then we obtain $t_{B1\,MIN} = \kappa_{\min}$ and $t_{B1\,MAX} = \kappa_{\max}$;

(h) if proposition $\mathcal{P}_1$ is true, if d is the nullity of $\mathbf{D}_{B1}$, and if $\mathbf{Q}$ and $\mathbf{R}$ are given by Theorem 14 applied to the generalized Rayleigh ratio of $\mathbf{N}_{B1}$ to $\mathbf{D}_{B1}$, in the variable $\mathbf{X}_B$, then an average value of t_{B1} over a number $N = \min\{m, n-d\}$ of nonzero excitations $\mathbf{X}_B \in D(\mathbf{D}_{B1})$ is

$$t_{B1\,AVR} = \frac{\operatorname{tr}\mathbf{Q}}{N} = \frac{\operatorname{tr}\mathbf{R}}{N}\,; \tag{240}$$

(i) if proposition $\mathcal{P}_1$ is true, we have

$$0 \leqslant t_{B1\,MIN} \leqslant t_{B1\,AVR} \leqslant t_{B1\,MAX} \leqslant 1\,; \tag{241}$$

(j) if proposition $\mathcal{P}_1$ is true, if $\mathbf{D}_{B1}$ is positive definite (this is possible only if $m \geqslant n$) and if $\lambda_{\min}$, $\lambda_{\max}$ and $\mathbf{M}$ are given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{B1}$ to $\mathbf{D}_{B1}$, in the variable $\mathbf{X}_B$, then

- we have $t_{B1\,MIN} = \lambda_{\min}$ and $t_{B1\,MAX} = \lambda_{\max}$;
- $t_{B1\,AVR}$ doesn't depend on the choice of the variable $\mathbf{X}_B$, and we have

$$t_{B1\,AVR} = \frac{\operatorname{tr}\mathbf{M}}{\min\{m,n\}} = \frac{\operatorname{tr}\left(\mathbf{N}_{B1}\mathbf{D}_{B1}^{-1}\right)}{\min\{m,n\}}\,; \tag{242}$$

(k) if proposition $\mathcal{P}_3$ is true, and if $\operatorname{rank}\mathbf{Y}_{SAM12} = m$ and/or $\operatorname{rank}\mathbf{Z}_{PAM12} = m$, we have

$$t_{A1\,MAX} = t_{B1\,MAX}\,, \tag{243}$$

and

$$t_{A1\,MIN} = t_{B1\,MIN}\,; \tag{244}$$

(ℓ) if proposition $\mathcal{P}_3$ is true, if $m = n$, and if $\mathbf{Y}_{SAM12}$ is invertible and/or $\mathbf{Z}_{PAM12}$ is invertible, then

$$t_{A1\,AVR} = t_{B1\,AVR}\,. \tag{245}$$

Proof: Since $\mathbf{D}_{A1}$ is positive definite, assertions (a) and (b) directly follow from Theorem 12.

In (c), by Theorem 12 and [8, Sec. 1.3.3], we obtain $\operatorname{tr}\mathbf{M} = \operatorname{tr}\left(\mathbf{N}_{A1}\mathbf{D}_{A1}^{-1}\right)$, and the second equality of (238). By (8), each eigenvector $\mathbf{y}$ of $\mathbf{M}$ corresponds to a nonzero excitation $\mathbf{X}_A = \mathbf{D}_{A1}^{-1/2}\mathbf{y}$, and to an eigenvalue that is equal to t_{A1} for this $\mathbf{X}_A$. Using Table 7, (44) and (60), we get

$$\operatorname{rank}\left(\mathbf{N}_{A1}\mathbf{D}_{A1}^{-1}\right) = \operatorname{rank}\mathbf{N}_{A1} \leqslant m\,, \tag{246}$$

so that the number of nonzero eigenvalues of $\mathbf{M}$, counting multiplicity, is less than or equal to m. Since $\operatorname{tr}\mathbf{M}$ is the sum of the eigenvalues of $\mathbf{M}$, counting multiplicity, it follows that $t_{A1\,AVR}$ given by (238) is an average of t_{A1} over m nonzero excitations. This, (372) of Appendix C, and $0 \leqslant t_{A1} \leqslant 1$ lead us to (c) and (d).

We have already observed that $t_{B1} \leqslant 1$. Thus, if proposition $\mathcal{P}_1$ is true, we can apply Corollary 3 to the generalized Rayleigh ratio of $\mathbf{N}_{B1}$ to $\mathbf{D}_{B1}$ in the variable $\mathbf{X}_B$, because $\mathbf{D}_{B1}$ is positive semidefinite. Thus, $\ker\mathbf{D}_{B1} \subset \ker\mathbf{N}_{B1}$. It follows that the assumptions of Theorem 14 applied to the generalized Rayleigh ratio of $\mathbf{N}_{B1}$ to $\mathbf{D}_{B1}$, in the variable $\mathbf{X}_B$, are satisfied. This proves (e), and also (f) and (g), which directly follow from Theorem 14.

In (h), by Theorem 14 and [8, Sec. 1.3.3], we obtain $\operatorname{tr}\mathbf{Q} = \operatorname{tr}\mathbf{R}$, which allows us to write the second equality of (240). Let $\mathbf{L}$, $\mathcal{L}$ and $\mathbf{P}$ be given by Theorem 14 applied to the generalized Rayleigh ratio of $\mathbf{N}_{B1}$ to $\mathbf{D}_{B1}$. By (24), each eigenvector $\mathbf{u}$ of $\mathbf{Q}$ corresponds to a nonzero excitation $\mathbf{X}_B = \mathbf{P}\mathbf{u}$, and to an eigenvalue that is equal to t_{B1} for this $\mathbf{X}_B$. We have $\operatorname{rank}\mathbf{Q} \leqslant n-d$, and $\operatorname{rank}\mathbf{Q} \leqslant \operatorname{rank}\mathbf{N}_{B1}$. Using Table 10, (52) and (68), we get

$$\operatorname{rank}\mathbf{Q} \leqslant \min\{m, n-d\}\,, \tag{247}$$

so that the number of nonzero eigenvalues of $\mathbf{Q}$, counting multiplicity, is less than or equal to $N = \min\{m, n-d\}$. Since $\operatorname{tr}\mathbf{Q}$ is the sum of the eigenvalues of $\mathbf{Q}$, counting multiplicity, it follows that $t_{B1\,AVR}$ given by (240) is an average of t_{B1} over a number N of nonzero excitations $\mathbf{X}_B \in D(\mathbf{D}_{B1})$. This and $0 \leqslant t_{B1} \leqslant 1$ lead us to (h) and (i).

If $\mathbf{D}_{B1}$ is positive definite, we have $d = 0$ and $\mathcal{L} = \mathbf{L}$, so that, according to (13), we have $\mathbf{P}\mathbf{L}^* = \mathbf{P}\mathbf{L}^{-1} = \mathbf{D}_{B1}^{-1/2}$. Consequently, it follows from (6) and (14) that

$$\mathbf{L}\mathbf{Q}\mathbf{L}^{-1} = \mathbf{D}_{B1}^{-1/2}\mathbf{N}_{B1}\mathbf{D}_{B1}^{-1/2} = \mathbf{M}\,. \tag{248}$$

Thus, if $\mathbf{D}_{B1}$ is positive definite, $\mathbf{M}$ is similar to $\mathbf{Q}$. It follows that $\mathbf{M}$ and $\mathbf{Q}$ have the same eigenvalues, counting multiplicity, by [8, Sec. 1.3.4]. This, Theorem 12, Theorem 14, and (373) of Appendix C lead us to (j).

If $\mathbf{Y}_{SAM11}$ is invertible, port set 1 behaves, in CA, as an m-port load of impedance matrix $\mathbf{Z}_{T2} = \mathbf{Y}_{SAM11}^{-1} - \mathbf{Z}_{S1}$. If follows that $H(\mathbf{Y}_{SAM11}^{-1} - \mathbf{Z}_{S1})$ is positive semidefinite. Thus, if $\mathbf{Y}_{SAM11}$ and $H(\mathbf{Y}_{SAM11}^{-1} - \mathbf{Z}_{S1})$ are invertible, $H(\mathbf{Z}_{T2})$ is positive definite. If $\mathbf{Y}_{SAM11}$ is invertible, port set 1 behaves, in CB, as an m-port generator of internal impedance matrix $\mathbf{Z}_{T2}$ and rms open-circuit voltage vector $\mathbf{V}_{T2} = -\mathbf{Y}_{SAM11}^{-1}\mathbf{Y}_{SAM12}\mathbf{V}_{O2}$.

If proposition $\mathcal{P}_3$ is true and $\operatorname{rank}\mathbf{Y}_{SAM12} = m$, then $H(\mathbf{Z}_{T2})$ is positive definite and $\mathbf{V}_{T2}$ may take on any value lying in $\mathbb{C}^m$, so that: at port set 1, configuration CA of Theorem 24 is the configuration "CA without the DUS" of Theorem 23 applied to the m-port generator of internal impedance matrix $\mathbf{Z}_{S1}$ and to an m-port load of impedance matrix $\mathbf{Z}_{T2}$; and, at port set 1, configuration CB of Theorem 24 is the configuration "CB without the DUS" of Theorem 23 applied to an m-port generator of internal impedance matrix $\mathbf{Z}_{T2}$ and to the m-port load of impedance matrix $\mathbf{Z}_{S1}$.

Thus, if proposition $\mathcal{P}_3$ is true and $\operatorname{rank}\mathbf{Y}_{SAM12} = m$, assertion (i) of Theorem 23 leads us to (243)–(244). If proposition $\mathcal{P}_3$ is true and $\operatorname{rank}\mathbf{Z}_{PAM12} = m$, a similar reasoning also allows us to use assertion (i) of Theorem 23 to obtain (243)–(244). These results lead us to (k).

We now assume that proposition $\mathcal{P}_3$ is true, $m = n$, and $\mathbf{Y}_{SAM12}$ is invertible and/or $\mathbf{Z}_{PAM12}$ is invertible. It follows from (195), (198), (204) and (206) that $\mathbf{D}_{B1}$ is positive definite and that $\mathbf{Y}_{SAM12}$ and $\mathbf{Z}_{PAM12}$ are invertible. By (d) and (j), $t_{A1\,AVR}$ doesn't depend on the choice of $\mathbf{X}_A$, and $t_{B1\,AVR}$ doesn't depend on the choice of $\mathbf{X}_B$. Consequently, to prove (ℓ), we can assume $\mathbf{X}_A = \mathbf{I}_{S1}$ and $\mathbf{X}_B = \mathbf{I}_{S2}$. By Theorem 12, we only need to compare the eigenvalues of $\mathbf{L} = \mathbf{N}_{A1}\mathbf{D}_{A1}^{-1}$ with the eigenvalues of $\mathbf{J} = \mathbf{N}_{B1}\mathbf{D}_{B1}^{-1}$.

By Table 10, (68) and (195), we have

$$\begin{aligned}\mathbf{J} = \mathbf{Z}_{PAM12}^{*}(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^{*})\mathbf{Z}_{PAM11} \\ \times \left(\mathbf{Z}_{PAM11}^{-1} + \mathbf{Z}_{PAM11}^{-1*} - \mathbf{Y}_{S1} - \mathbf{Y}_{S1}^{*}\right) \\ \times \mathbf{Z}_{PAM11}^{*}\mathbf{Z}_{PAM12}^{-1*}\,. \end{aligned} \tag{249}$$

$\mathbf{Z}_{PAM11}$ being invertible, we find that $\mathbf{J}$ is similar to

$$\begin{aligned}\mathbf{K} = \mathbf{Z}_{PAM11}^{*}(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^{*})\mathbf{Z}_{PAM11} \\ \times \left(\mathbf{Z}_{PAM11}^{-1} + \mathbf{Z}_{PAM11}^{-1*} - \mathbf{Y}_{S1} - \mathbf{Y}_{S1}^{*}\right)\,. \end{aligned} \tag{250}$$

By Table 7, (56) and (60), we have

$$\begin{aligned}\mathbf{L} = \mathbf{Z}_{PAM11}^{*}\left(\mathbf{Z}_{PAM11}^{-1} + \mathbf{Z}_{PAM11}^{-1*} - \mathbf{Y}_{S1} - \mathbf{Y}_{S1}^{*}\right) \\ \times \mathbf{Z}_{PAM11}(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^{*})\,. \end{aligned} \tag{251}$$

It follows from Lemma 3 of Part 2 applied to $\mathbf{A} = \mathbf{Y}_{S1}$ and $\mathbf{B} = \mathbf{Z}_{PAM11}^{-1} - \mathbf{Y}_{S1}$ that $\mathbf{K}$ and $\mathbf{L}$ have the same eigenvalues, counting multiplicity.

This proves (ℓ). □

D. BOUNDS OF THE POWER TRANSFER RATIOS AT PORT SET 2

By Observation 7, the set of the values of t_{B2} obtained for all $\mathbf{X}_B \in \mathbb{C}^n$ such that $\mathbf{X}_B \neq \mathbf{0}$ is equal to the set of the values of t_{B2} obtained for all $\mathbf{X}_B \in \mathbb{S}_n$.

If proposition $\mathcal{P}_2$ is true, the set of the values of t_{A2} obtained for all $\mathbf{X}_A \in D(\mathbf{D}_{A2})$ is equal to the set of the values of t_{A2} obtained for all $\mathbf{X}_A \in D(\mathbf{D}_{A2}) \cap \mathbb{S}_m$.

Since $\mathbf{I}_{S2} = \mathbf{Y}_{S2}\mathbf{V}_{O2}$, where $\mathbf{Y}_{S2}$ is invertible, we can assert that the set of the values of t_{B2} obtained for all nonzero $\mathbf{X}_B \in \mathbb{C}^n$, or for all $\mathbf{X}_B \in \mathbb{S}_n$, does not depend on the choice $\mathbf{X}_B = \mathbf{V}_{O2}$ or $\mathbf{X}_B = \mathbf{I}_{S2}$.

Since $\mathbf{I}_{S1} = \mathbf{Y}_{S1}\mathbf{V}_{O1}$, where $\mathbf{Y}_{S1}$ is invertible, we can demonstrate that, if proposition $\mathcal{P}_2$ is true, the set of the values of t_{A2} obtained for all $\mathbf{X}_A \in D(\mathbf{D}_{A2})$, or equivalently for all $\mathbf{X}_A \in D(\mathbf{D}_{A2}) \cap \mathbb{S}_m$, is independent of the choice $\mathbf{X}_A = \mathbf{V}_{O1}$ or $\mathbf{X}_A = \mathbf{I}_{S1}$.

We can now state and prove a new reciprocal theorem on the bounds of the sets of the values of the power transfer ratios at port set 2 in CA and CB.

Theorem 25. Ignoring noise power contributions, we can assert that:

(a) the set of the values of the power transfer ratio in CB at port set 2 of the DUS, obtained for all nonzero $\mathbf{V}_{O2} \in \mathbb{C}^n$, or equivalently for all nonzero $\mathbf{I}_{S2} \in \mathbb{C}^n$, has a least element referred to as "minimum value" and denoted by $t_{B2\,MIN}$, and a greatest element referred to as "maximum value" and denoted by $t_{B2\,MAX}$;

(b) if $\lambda_{\min}$ and $\lambda_{\max}$ are given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{B2}$ to $\mathbf{D}_{B2}$, in the variable $\mathbf{X}_B$ according to (225) and Table 11, we have $t_{B2\,MIN} = \lambda_{\min}$ and $t_{B2\,MAX} = \lambda_{\max}$;

(c) if $\mathbf{M}$ is given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{B2}$ to $\mathbf{D}_{B2}$, in the variable $\mathbf{X}_B$, an average value of t_{B2} over n nonzero excitations is

$$t_{B2\,AVR} = \frac{\operatorname{tr}\mathbf{M}}{n} = \frac{\operatorname{tr}\left(\mathbf{N}_{B2}\mathbf{D}_{B2}^{-1}\right)}{n}\,; \tag{252}$$

(d) $t_{B2\,AVR}$ doesn't depend on the choice of $\mathbf{X}_B$, and

$$0 \leqslant t_{B2\,MIN} \leqslant t_{B2\,AVR} \leqslant t_{B2\,MAX} \leqslant 1\,; \tag{253}$$

(e) if proposition $\mathcal{P}_2$ is true, we have $\ker\mathbf{D}_{A2} \subset \ker\mathbf{N}_{A2}$ so that Theorem 14 can be applied to the generalized Rayleigh ratio of $\mathbf{N}_{A2}$ to $\mathbf{D}_{A2}$;

(f) if proposition $\mathcal{P}_2$ is true, the set of the values of the power transfer ratio in CA at port set 2 of the DUS, obtained for all $\mathbf{X}_A \in D(\mathbf{D}_{A2})$, has a least element referred to as "minimum value" and denoted by $t_{A2\,MIN}$, and a greatest element referred to as "maximum value" and denoted by $t_{A2\,MAX}$;

(g) if proposition $\mathcal{P}_2$ is true, and if $\kappa_{\min}$ and $\kappa_{\max}$ are given by Theorem 14 applied to the generalized Rayleigh ratio of $\mathbf{N}_{A2}$ to $\mathbf{D}_{A2}$, in the variable $\mathbf{X}_A$ according to (220) and Table 8, then we obtain $t_{A2\,MIN} = \kappa_{\min}$ and $t_{A2\,MAX} = \kappa_{\max}$;

(h) if proposition $\mathcal{P}_2$ is true, if d is the nullity of $\mathbf{D}_{A2}$, and if $\mathbf{Q}$ and $\mathbf{R}$ are given by Theorem 14 applied to the generalized Rayleigh ratio of $\mathbf{N}_{A2}$ to $\mathbf{D}_{A2}$, in the variable $\mathbf{X}_A$, then an average value of t_{A2} over a number $N = \min\{m-d, n\}$ of nonzero excitations $\mathbf{X}_A \in D(\mathbf{D}_{A2})$ is

$$t_{A2\,AVR} = \frac{\operatorname{tr}\mathbf{Q}}{N} = \frac{\operatorname{tr}\mathbf{R}}{N}\,; \tag{254}$$

(i) if proposition $\mathcal{P}_2$ is true, we have

$$0 \leqslant t_{A2\,MIN} \leqslant t_{A2\,AVR} \leqslant t_{A2\,MAX} \leqslant 1\,; \tag{255}$$

(j) if proposition $\mathcal{P}_2$ is true, if $\mathbf{D}_{A2}$ is positive definite (this is possible only if $n \geqslant m$) and if $\lambda_{\min}$, $\lambda_{\max}$ and $\mathbf{M}$ are given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{A2}$ to $\mathbf{D}_{A2}$, in the variable $\mathbf{X}_A$, then
- we have $t_{A2\,MIN} = \lambda_{\min}$ and $t_{A2\,MAX} = \lambda_{\max}$;
- $t_{A2\,AVR}$ doesn't depend on the choice of the variable $\mathbf{X}_A$, and we have

$$t_{A2\,AVR} = \frac{\operatorname{tr}\mathbf{M}}{\min\{m,n\}} = \frac{\operatorname{tr}\left(\mathbf{N}_{A2}\mathbf{D}_{A2}^{-1}\right)}{\min\{m,n\}}\,; \tag{256}$$

(k) if proposition $\mathcal{P}_4$ is true, and if $\operatorname{rank}\mathbf{Y}_{SAM21} = n$ and/or $\operatorname{rank}\mathbf{Z}_{PAM21} = n$, we have

$$t_{A2\,MAX} = t_{B2\,MAX}\,, \tag{257}$$

and

$$t_{A2\,MIN} = t_{B2\,MIN}\,; \tag{258}$$

(ℓ) if proposition $\mathcal{P}_4$ is true, if $m = n$, and if $\mathbf{Y}_{SAM21}$ is invertible and/or $\mathbf{Z}_{PAM21}$ is invertible, then

$$t_{A2\,AVR} = t_{B2\,AVR}\,. \tag{259}$$

Proof: Theorem 25 corresponds to Theorem 24 with a different labeling of port sets. □

VIII. AVAILABLE POWER GAINS

We introduce two available power gains [3, Sec. 21-18]: the available power gain in CA, given by

$$G_{AA} = \frac{P_{AAVP2}}{P_{AAVG1}}\,, \tag{260}$$

and the available power gain in CB, given by

$$G_{BA} = \frac{P_{BAVP1}}{P_{BAVG2}}\,. \tag{261}$$

It follows from (28) and (31) that we have $0 \leqslant G_{AA} \leqslant 1$ and $0 \leqslant G_{BA} \leqslant 1$.

Theorem 26. Ignoring noise power contributions, we can assert that:

(a) for a specified DUS, a specified excitation and a specified $\mathbf{Y}_{S1}$ (or $\mathbf{Z}_{S1}$), G_{AA} does not depend on $\mathbf{Y}_{S2}$ (or $\mathbf{Z}_{S2}$);

(b) for a specified DUS, a specified excitation and a specified $\mathbf{Y}_{S2}$ (or $\mathbf{Z}_{S2}$), G_{BA} does not depend on $\mathbf{Y}_{S1}$ (or $\mathbf{Z}_{S1}$).

Proof: In CA, P_{AAVG1} depends on the m-port generator connected to port set 1, but neither on the DUS nor on the n-port load connected to port set 2; and P_{AAVP2} depends on the m-port generator connected to port set 1 and on the DUS, but not on the n-port load connected to port set 2. This leads us to (a).

Likewise, in CB, P_{BAVG2} depends neither on the DUS nor on the m-port load connected to port set 1; and in CB, P_{BAVP1} does not depend on the m-port load connected to port set 1. This leads us to (b). □

Observation 12. There is a similarity between Theorem 19 and Theorem 26. There are also several noteworthy differences between them.

We remark that, if t_{A2} is defined and nonzero, we have

$$G_{AA} = \frac{G_{AT}}{t_{A2}}\,. \tag{262}$$

Likewise, if t_{B1} is defined and nonzero, we have

$$G_{BA} = \frac{G_{BT}}{t_{B1}}\,. \tag{263}$$

To define the excitation in CA, let $\mathbf{X}_A$ denote one of the variables $\mathbf{V}_{O1}$ or $\mathbf{I}_{S1}$. Based on Section III.C, Section III.D and Corollary 9, we find that, if proposition $\mathcal{P}_2$ is true, G_{AA} is given by

$$G_{AA} = \frac{\mathbf{X}_A^*\,\mathbf{N}_{AA}\,\mathbf{X}_A}{\mathbf{X}_A^*\,\mathbf{D}_{AA}\,\mathbf{X}_A}\,, \tag{264}$$

where $\mathbf{N}_{AA}$ and $\mathbf{D}_{AA}$ are hermitian matrices of size m by m, and given in Table 13. We note that $\mathbf{N}_{AA}$ is positive semidefinite and $\mathbf{D}_{AA}$ is positive definite.

TABLE 13. Variable $\mathbf{X}_A$ and associated $\mathbf{N}_{AA}$ and $\mathbf{D}_{AA}$.

Variable $\mathbf{X}_A$	Applicability	$\mathbf{N}_{AA}$	$\mathbf{D}_{AA}$
$\mathbf{V}_{O1}$	proposition $\mathcal{P}_2$ is true	$\mathbf{Y}_{AAVP2}$	$\mathbf{Y}_{AAVG1}$
$\mathbf{I}_{S1}$	proposition $\mathcal{P}_2$ is true	$\mathbf{Z}_{AAVP2}$	$\mathbf{Z}_{AAVG1}$

If proposition $\mathcal{P}_2$ is true, G_{AA} is given by (264) in the form of a generalized Rayleigh ratio of $\mathbf{N}_{AA}$ to $\mathbf{D}_{AA}$, in the variable $\mathbf{X}_A$. Thus, G_{AA} depends on the excitation. Since $\mathbf{D}_{AA}$ is positive definite, G_{AA} is defined for any nonzero $\mathbf{X}_A \in \mathbb{C}^m$.

To define the excitation in CB, let $\mathbf{X}_B$ denote one of the variables $\mathbf{V}_{O2}$ or $\mathbf{I}_{S2}$. Based on Section III.C, Section III.D and Corollary 9, we find that, if proposition $\mathcal{P}_1$ is true, G_{BA} is given by

$$G_{BA} = \frac{\mathbf{X}_B^*\,\mathbf{N}_{BA}\,\mathbf{X}_B}{\mathbf{X}_B^*\,\mathbf{D}_{BA}\,\mathbf{X}_B}\,, \tag{265}$$

where $\mathbf{N}_{BA}$ and $\mathbf{D}_{BA}$ are hermitian matrices of size n by n, and given in Table 14. We note that $\mathbf{N}_{BA}$ is positive semidefinite and $\mathbf{D}_{BA}$ is positive definite.

TABLE 14. Variable $\mathbf{X}_B$ and associated $\mathbf{N}_{BA}$ and $\mathbf{D}_{BA}$.

Variable $\mathbf{X}_B$	Applicability	$\mathbf{N}_{BA}$	$\mathbf{D}_{BA}$
$\mathbf{V}_{O2}$	proposition $\mathcal{P}_1$ is true	$\mathbf{Y}_{BAVP1}$	$\mathbf{Y}_{BAVG2}$
$\mathbf{I}_{S2}$	proposition $\mathcal{P}_1$ is true	$\mathbf{Z}_{BAVP1}$	$\mathbf{Z}_{BAVG2}$

If proposition $\mathcal{P}_1$ is true, G_{BA} is given by (265) in the form of a generalized Rayleigh ratio of $\mathbf{N}_{BA}$ to $\mathbf{D}_{BA}$, in the variable $\mathbf{X}_B$. Thus, G_{BA} depends on the excitation. Since $\mathbf{D}_{BA}$ is positive definite, G_{BA} is defined for any nonzero $\mathbf{X}_B \in \mathbb{C}^n$.

If proposition $\mathcal{P}_2$ is true, the set of the values of G_{AA} obtained for all nonzero $\mathbf{X}_A \in \mathbb{C}^m$ is equal to the set of the values of G_{AA} obtained for all $\mathbf{X}_A \in \mathbb{S}_m$.

Since $\mathbf{I}_{S1} = \mathbf{Y}_{S1}\mathbf{V}_{O1}$, where $\mathbf{Y}_{S1}$ is invertible, we can assert that the set of the values of G_{AA} obtained for all nonzero $\mathbf{X}_A \in \mathbb{C}^m$, or for all $\mathbf{X}_A \in \mathbb{S}_m$, does not depend on the choice $\mathbf{X}_A = \mathbf{V}_{O1}$ or $\mathbf{X}_A = \mathbf{I}_{S1}$.

If proposition $\mathcal{P}_1$ is true, the set of the values of G_{BA} obtained for all nonzero $\mathbf{X}_B \in \mathbb{C}^n$ is equal to the set of the values of G_{BA} obtained for all $\mathbf{X}_B \in \mathbb{S}_n$.

Since $\mathbf{I}_{S2} = \mathbf{Y}_{S2}\mathbf{V}_{O2}$, where $\mathbf{Y}_{S2}$ is invertible, we can assert that the set of the values of G_{BA} obtained for all nonzero $\mathbf{X}_B \in \mathbb{C}^n$, or for all $\mathbf{X}_B \in \mathbb{S}_n$, does not depend on the choice $\mathbf{X}_B = \mathbf{V}_{O2}$ or $\mathbf{X}_B = \mathbf{I}_{S2}$.

We can now state and prove two new reciprocal theorems involving operating power gains and available power gains in CA and CB.

Theorem 27. We assume that proposition $\mathcal{P}_2$ is true. Ignoring noise power contributions, we can assert that:

(a) the set of the values of the available power gain in CA, obtained for all nonzero $\mathbf{V}_{O1} \in \mathbb{C}^m$, or equivalently for all nonzero $\mathbf{I}_{S1} \in \mathbb{C}^m$, has a least element referred to as “minimum value” and denoted by $G_{AA\,MIN}$, and a greatest element referred to as “maximum value” and denoted by $G_{AA\,MAX}$;

(b) if $\lambda_{\min}$ and $\lambda_{\max}$ are given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{AA}$ to $\mathbf{D}_{AA}$, in the variable $\mathbf{X}_A$ according to (264) and Table 13, we have $G_{AA\,MIN} = \lambda_{\min}$ and $G_{AA\,MAX} = \lambda_{\max}$;

(c) if $\mathbf{M}$ is given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{AA}$ to $\mathbf{D}_{AA}$, in the variable $\mathbf{X}_A$, an average value of G_{AA} over a number $\min\{m,n\}$ of nonzero excitations is

$$G_{AA\,AVR} = \frac{\operatorname{tr}\mathbf{M}}{\min\{m,n\}} = \frac{\operatorname{tr}\left(\mathbf{N}_{AA}\mathbf{D}_{AA}^{-1}\right)}{\min\{m,n\}}\,; \quad (266)$$

(d) $G_{AA\,AVR}$ doesn't depend on the choice of $\mathbf{X}_A$, and

$$0 \leqslant G_{AA\,MIN} \leqslant G_{AA\,AVR} \leqslant G_{AA\,MAX} \leqslant 1\,; \quad (267)$$

(e) if proposition $\mathcal{P}_4$ is true, and if the DUS and both loads are reciprocal devices, we have

$$G_{AA\,MAX} = G_{BO\,MAX}\,, \quad (268)$$

$$G_{AA\,AVR} = G_{BO\,AVR}\,, \quad (269)$$

$$(m = n) \Longrightarrow (G_{AA\,MIN} = G_{BO\,MIN})\,, \quad (270)$$

$$(m > n) \Longrightarrow (G_{AA\,MIN} = 0)\,, \quad (271)$$

and

$$(m < n) \Longrightarrow (G_{BO\,MIN} = 0)\,. \quad (272)$$

Proof: Since $\mathbf{D}_{AA}$ is positive definite, assertions (a) and (b) directly follow from Theorem 12.

In (c), by Theorem 12 and [8, Sec. 1.3.3], we obtain $\operatorname{tr}\mathbf{M} = \operatorname{tr}\left(\mathbf{N}_{AA}\mathbf{D}_{AA}^{-1}\right)$, and the second equality of (266). By (8), each eigenvector $\mathbf{y}$ of $\mathbf{M}$ corresponds to a nonzero excitation $\mathbf{X}_A = \mathbf{D}_{AA}^{-1/2}\mathbf{y}$, and to an eigenvalue that is equal to G_{AA} for this $\mathbf{X}_A$. Using Table 13, (188) and (191), we get

$$\operatorname{rank}\left(\mathbf{N}_{AA}\mathbf{D}_{AA}^{-1}\right) = \operatorname{rank}\mathbf{N}_{AA} \leqslant \min\{m,n\}\,, \quad (273)$$

so that the number of nonzero eigenvalues of $\mathbf{M}$, counting multiplicity, is less than or equal to $\min\{m,n\}$. Since $\operatorname{tr}\mathbf{M}$ is the sum of the eigenvalues of $\mathbf{M}$, counting multiplicity, it follows that $G_{AA\,AVR}$ given by (266) is an average of G_{AA} over a number $\min\{m,n\}$ of nonzero excitations. This, (372) of Appendix C, and $0 \leqslant G_{AA} \leqslant 1$ lead us to (c) and (d).

To prove (e), by Lemma 11 we can assume that $\mathbf{Z}_{PAM22}$ is invertible, $H(\mathbf{Z}_{PAM22}^{-1} - \mathbf{Y}_{S2})$ is invertible, $\mathbf{X}_A = \mathbf{I}_{S1}$ and $\mathbf{X}_B = \mathbf{I}_{S2}$. Using Table 6 and (66), we get

$$\begin{aligned}\mathbf{D}_{BO} = \mathbf{Z}_{BRP2} = \frac{1}{2}&\mathbf{Z}_{PAM22}^{*} \\ &\times \left(\mathbf{Z}_{PAM22}^{-1} + \mathbf{Z}_{PAM22}^{-1*} - \mathbf{Y}_{S2} - \mathbf{Y}_{S2}^{*}\right) \\ &\qquad \times \mathbf{Z}_{PAM22}\,,\end{aligned} \quad (274)$$

so that $\mathbf{D}_{BO} = \mathbf{Z}_{BRP2}$ is invertible, hence positive definite.

Thus, by Theorem 12 and Theorem 18, we only need to compare the eigenvalues of $\mathbf{A} = \mathbf{N}_{AA}\mathbf{D}_{AA}^{-1}$ with the eigenvalues of $\mathbf{B} = \mathbf{N}_{BO}\mathbf{D}_{BO}^{-1}$. It follows from Table 13, (56) and (189) that

$$\begin{aligned}\mathbf{A} = \mathbf{Z}_{PAM21}^{*}&\mathbf{Z}_{PAM22}^{-1*} \\ &\times \left(\mathbf{Z}_{PAM22}^{-1} + \mathbf{Z}_{PAM22}^{-1*} - \mathbf{Y}_{S2} - \mathbf{Y}_{S2}^{*}\right)^{-1} \\ &\qquad \times \mathbf{Z}_{PAM22}^{-1}\mathbf{Z}_{PAM21}\left(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^{*}\right)\,.\end{aligned} \quad (275)$$

which is of size m by m. It follows from Table 6, (68) and (274) that

$$\begin{aligned}\mathbf{B} = \mathbf{Z}_{PAM12}^{*}&(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^{*})\mathbf{Z}_{PAM12}\mathbf{Z}_{PAM22}^{-1} \\ &\times \left(\mathbf{Z}_{PAM22}^{-1} + \mathbf{Z}_{PAM22}^{-1*} - \mathbf{Y}_{S2} - \mathbf{Y}_{S2}^{*}\right)^{-1} \\ &\times \mathbf{Z}_{PAM22}^{-1*}\,,\end{aligned} \quad (276)$$

which is of size n by n.

If the DUS and both loads are reciprocal devices, $\mathbf{Z}_{PAM}$, $\mathbf{Y}_{S1}$ and $\mathbf{Y}_{S2}$ are symmetric. Thus, $\mathbf{Z}_{PAM22}$ is symmetric and the transpose of $\mathbf{Z}_{PAM12}$ is $\mathbf{Z}_{PAM21}$, so that

$$\begin{aligned}\mathbf{B}^T = \mathbf{Z}_{PAM22}^{-1*}&\left(\mathbf{Z}_{PAM22}^{-1} + \mathbf{Z}_{PAM22}^{-1*} - \mathbf{Y}_{S2} - \mathbf{Y}_{S2}^{*}\right)^{-1} \\ &\times \mathbf{Z}_{PAM22}^{-1}\mathbf{Z}_{PAM21}(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^{*})\mathbf{Z}_{PAM21}^{*}\,.\end{aligned} \quad (277)$$

By [8, Sec. 1.4.1], the eigenvalues of $\mathbf{B}^T$ are the same as those of $\mathbf{B}$, counting multiplicity. We note that, if we write

$$\begin{aligned}\mathbf{C} = \mathbf{Z}_{PAM22}^{-1*}&\left(\mathbf{Z}_{PAM22}^{-1} + \mathbf{Z}_{PAM22}^{-1*} - \mathbf{Y}_{S2} - \mathbf{Y}_{S2}^{*}\right)^{-1} \\ &\times \mathbf{Z}_{PAM22}^{-1}\mathbf{Z}_{PAM21}(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^{*})\,,\end{aligned} \quad (278)$$

the right hand sides of (275) and (277) are $\mathbf{Z}_{PAM21}^{*}\mathbf{C}$ and $\mathbf{C}\,\mathbf{Z}_{PAM21}^{*}$, respectively. Thus, using [8, Sec. 1.3.22] and the fact that $\mathbf{Z}_{PAM21}^{*}$ is of size m by n, we find that:

- if $m = n$, then $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues, counting multiplicity;
- if $m > n$, then $\mathbf{A}$ has the same eigenvalues as $\mathbf{B}$, counting multiplicity, together with $m - n$ additional eigenvalues equal to zero; and
- if $m < n$, then $\mathbf{B}$ has the same eigenvalues as $\mathbf{A}$, counting multiplicity, together with $n - m$ additional eigenvalues equal to zero.

This leads to (268)–(272). □

Theorem 28. We assume that proposition $\mathcal{P}_1$ is true. Ignoring noise power contributions, we can assert that:

(a) the set of the values of the available power gain in CB, obtained for all nonzero $\mathbf{V}_{O2} \in \mathbb{C}^n$, or equivalently for all nonzero $\mathbf{I}_{S2} \in \mathbb{C}^n$, has a least element referred to as "minimum value" and denoted by $G_{BA\,MIN}$, and a greatest element referred to as "maximum value" and denoted by $G_{BA\,MAX}$;

(b) if $\lambda_{\min}$ and $\lambda_{\max}$ are given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{BA}$ to $\mathbf{D}_{BA}$, in the variable $\mathbf{X}_B$ according to (265) and Table 14, we have $G_{BA\,MIN} = \lambda_{\min}$ and $G_{BA\,MAX} = \lambda_{\max}$;

(c) if $\mathbf{M}$ is given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{BA}$ to $\mathbf{D}_{BA}$, in the variable $\mathbf{X}_B$, an average value of G_{BA} over a number $\min\{m,n\}$ of nonzero excitations is

$$G_{BA\,AVR} = \frac{\operatorname{tr}\mathbf{M}}{\min\{m,n\}} = \frac{\operatorname{tr}\left(\mathbf{N}_{BA}\mathbf{D}_{BA}^{-1}\right)}{\min\{m,n\}}\,; \quad (279)$$

(d) $G_{BA\,AVR}$ doesn't depend on the choice of $\mathbf{X}_B$, and

$$0 \leqslant G_{BA\,MIN} \leqslant G_{BA\,AVR} \leqslant G_{BA\,MAX} \leqslant 1\,; \quad (280)$$

(e) if proposition $\mathcal{P}_3$ is true, and if the DUS and both loads are reciprocal devices, we have

$$G_{BA\,MAX} = G_{AO\,MAX}\,, \tag{281}$$

$$G_{BA\,AVR} = G_{AO\,AVR}\,, \tag{282}$$

$$(m = n) \Longrightarrow (G_{BA\,MIN} = G_{AO\,MIN})\,, \tag{283}$$

$$(n > m) \Longrightarrow (G_{BA\,MIN} = 0)\,, \tag{284}$$

and

$$(n < m) \Longrightarrow (G_{AO\,MIN} = 0)\,. \tag{285}$$

Proof: Theorem 28 corresponds to Theorem 27 with a different labeling of port sets. □

IX. UNNAMED POWER GAINS

Unnamed power gains were first introduced in [5, Sec. IV.G], for a two-port. Here, we introduce the unnamed power gain in CA, given by

$$G_{AU} = \frac{P_{AAVP2}}{P_{ARP1}}\,, \tag{286}$$

and the unnamed power gain in CB, given by

$$G_{BU} = \frac{P_{BAVP1}}{P_{BRP2}}\,. \tag{287}$$

If they exist, G_{AU} and G_{BU} are nonnegative, but they need not be less than or equal to one. We have

$$G_{AA} = G_{AU}\,t_{A1} \text{ and } G_{AO} = G_{AU}\,t_{A2} \tag{288}$$

and

$$G_{BA} = G_{BU}\,t_{B2} \text{ and } G_{BO} = G_{BU}\,t_{B1}\,, \tag{289}$$

where each equality in (288)–(289) is valid if and only if both terms of its right hand side are defined.

To define the excitation in CA, let $\mathbf{X}_A$ denote one of the variables $\mathbf{V}_{O1}$ or $\mathbf{I}_{S1}$. Based on Section III.C, Section III.D and Corollary 9, we find that, if proposition $\mathcal{P}_2$ is true, G_{AU} is given by

$$G_{AU} = \frac{\mathbf{X}_A^*\,\mathbf{N}_{AU}\,\mathbf{X}_A}{\mathbf{X}_A^*\,\mathbf{D}_{AU}\,\mathbf{X}_A}\,, \tag{290}$$

where $\mathbf{N}_{AU}$ and $\mathbf{D}_{AU}$ are hermitian matrices of size m by m, and given in Table 15. We note that $\mathbf{N}_{AU}$ and $\mathbf{D}_{AU}$ are positive semidefinite.

TABLE 15. Variable $\mathbf{X}_A$ and associated $\mathbf{N}_{AU}$ and $\mathbf{D}_{AU}$.

Variable $\mathbf{X}_A$	Applicability	$\mathbf{N}_{AU}$	$\mathbf{D}_{AU}$
$\mathbf{V}_{O1}$	proposition $\mathcal{P}_2$ is true	$\mathbf{Y}_{AAVP2}$	$\mathbf{Y}_{ARP1}$
$\mathbf{I}_{S1}$	proposition $\mathcal{P}_2$ is true	$\mathbf{Z}_{AAVP2}$	$\mathbf{Z}_{ARP1}$

G_{AU} is given by (290) in the form of a generalized Rayleigh ratio of $\mathbf{N}_{AU}$ to $\mathbf{D}_{AU}$, in the variable $\mathbf{X}_A$. Thus, G_{AU} depends on the excitation. According to the explanations provided in Section II.A, it is defined if and only if proposition $\mathcal{P}_2$ is true and $\mathbf{X}_A \in D(\mathbf{D}_{AU})$, where

$$D(\mathbf{D}_{AU}) = \{\mathbf{X}_A \in \mathbb{C}^m : \mathbf{X}_A \notin \ker \mathbf{D}_{AU}\}\,. \tag{291}$$

To define the excitation in CB, let $\mathbf{X}_B$ denote one of the variables $\mathbf{V}_{O2}$ or $\mathbf{I}_{S2}$. Based on Section III.C, Section III.D and Corollary 9, we find that, if proposition $\mathcal{P}_1$ is true, G_{BU} is given by

$$G_{BU} = \frac{\mathbf{X}_B^*\,\mathbf{N}_{BU}\,\mathbf{X}_B}{\mathbf{X}_B^*\,\mathbf{D}_{BU}\,\mathbf{X}_B}\,, \tag{292}$$

where $\mathbf{N}_{BU}$ and $\mathbf{D}_{BU}$ are hermitian matrices of size n by n, and given in Table 16. We note that $\mathbf{N}_{BU}$ and $\mathbf{D}_{BU}$ are positive semidefinite.

TABLE 16. Variable $\mathbf{X}_B$ and associated $\mathbf{N}_{BU}$ and $\mathbf{D}_{BU}$.

Variable $\mathbf{X}_B$	Applicability	$\mathbf{N}_{BU}$	$\mathbf{D}_{BU}$
$\mathbf{V}_{O2}$	proposition $\mathcal{P}_1$ is true	$\mathbf{Y}_{BAVP1}$	$\mathbf{Y}_{BRP2}$
$\mathbf{I}_{S2}$	proposition $\mathcal{P}_1$ is true	$\mathbf{Z}_{BAVP1}$	$\mathbf{Z}_{BRP2}$

G_{BU} is given by (292) in the form of a generalized Rayleigh ratio of $\mathbf{N}_{BU}$ to $\mathbf{D}_{BU}$, in the variable $\mathbf{X}_B$. Thus, G_{BU} depends on the excitation. According to the explanations provided in Section II.A, it is defined if and only if proposition $\mathcal{P}_1$ is true and $\mathbf{X}_B \in D(\mathbf{D}_{BU})$, where

$$D(\mathbf{D}_{BU}) = \{\mathbf{X}_B \in \mathbb{C}^n : \mathbf{X}_B \notin \ker \mathbf{D}_{BU}\}\,. \tag{293}$$

By Observation 7, the set of the values of G_{AU} obtained for all $\mathbf{X}_A \in D(\mathbf{D}_{AU})$ is equal to the set of the values of G_{AU} obtained for all $\mathbf{X}_A \in D(\mathbf{D}_{AU}) \cap \mathbb{S}_m$. Likewise, we can assert that the set of the values of G_{BU} obtained for all $\mathbf{X}_B \in D(\mathbf{D}_{BU})$ is equal to the set of the values of G_{BU} obtained for all $\mathbf{X}_B \in D(\mathbf{D}_{BU}) \cap \mathbb{S}_n$.

Since $\mathbf{I}_{S1} = \mathbf{Y}_{S1}\mathbf{V}_{O1}$, where $\mathbf{Y}_{S1}$ is invertible, it is possible to show that the set of the values of G_{AU} obtained for all $\mathbf{X}_A \in D(\mathbf{D}_{AU})$, or for all $\mathbf{X}_A \in D(\mathbf{D}_{AU}) \cap \mathbb{S}_m$, does not depend on the choice $\mathbf{X}_A = \mathbf{V}_{O1}$ or $\mathbf{X}_A = \mathbf{I}_{S1}$.

Likewise, since $\mathbf{I}_{S2} = \mathbf{Y}_{S2}\mathbf{V}_{O2}$, where $\mathbf{Y}_{S2}$ is invertible, we can show that the set of the values of G_{BU} obtained for all $\mathbf{X}_B \in D(\mathbf{D}_{BU})$, or for all $\mathbf{X}_B \in D(\mathbf{D}_{BU}) \cap \mathbb{S}_n$, does not depend on the choice $\mathbf{X}_B = \mathbf{V}_{O2}$ or $\mathbf{X}_B = \mathbf{I}_{S2}$.

Observation 13. Since G_{AU} and G_{BU} need not be less than or equal to one or any other number, we cannot apply Corollary 3 to the corresponding generalized Rayleigh ratios. Thus, to compute the bounds of the sets of the values of G_{AU} and G_{BU}, we cannot use Corollary 3 and Theorem 14 as we for instance did in Theorem 17 and Theorem 18 to obtain the bounds of the sets of the values of G_{AO} and G_{BO}.

We are now in a position to state and prove a new reciprocal theorem on the bounds of the sets of the values of the unnamed power gains in CA and CB.

Theorem 29. Ignoring noise power contributions, we can assert that:

(a) if $\mathcal{P}_2$ and $\mathcal{P}_3$ are true, $\mathbf{D}_{AU}$ is positive definite, and the set of the values of the unnamed power gain in CA, obtained for all nonzero $\mathbf{V}_{O1} \in \mathbb{C}^m$, or equivalently for all nonzero $\mathbf{I}_{S1} \in \mathbb{C}^m$, has a least element referred to as "minimum value" and denoted by $G_{AU\,MIN}$, and a greatest element referred to as "maximum value" and denoted by $G_{AU\,MAX}$;

(b) if $\mathcal{P}_2$ and $\mathcal{P}_3$ are true, and if $\lambda_{\min}$ and $\lambda_{\max}$ are given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{AU}$ to $\mathbf{D}_{AU}$, in the variable $\mathbf{X}_A$ according to (290) and Table 15, we find that $G_{AU\,MIN} = \lambda_{\min}$ and $G_{AU\,MAX} = \lambda_{\max}$;

(c) if $\mathcal{P}_2$ and $\mathcal{P}_3$ are true, and if $\mathbf{M}$ is given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{AU}$ to $\mathbf{D}_{AU}$, in the variable $\mathbf{X}_A$, an average value of G_{AU} over a number $\min\{m,n\}$ of nonzero excitations is

$$G_{AU\,AVR} = \frac{\operatorname{tr}\mathbf{M}}{\min\{m,n\}} = \frac{\operatorname{tr}\left(\mathbf{N}_{AU}\mathbf{D}_{AU}^{-1}\right)}{\min\{m,n\}}\,; \quad (294)$$

(d) if $\mathcal{P}_2$ and $\mathcal{P}_3$ are true, $G_{AU\,AVR}$ doesn't depend on the choice of $\mathbf{X}_A$, and

$$0 \leqslant G_{AU\,MIN} \leqslant G_{AU\,AVR} \leqslant G_{AU\,MAX}\,; \quad (295)$$

(e) if $\mathcal{P}_1$ and $\mathcal{P}_4$ are true, $\mathbf{D}_{BU}$ is positive definite, and the set of the values of the transducer power gain in CB, obtained for all nonzero $\mathbf{V}_{O2} \in \mathbb{C}^n$, or equivalently for all nonzero $\mathbf{I}_{S2} \in \mathbb{C}^n$, has a least element referred to as "minimum value" and denoted by $G_{BU\,MIN}$, and a greatest element referred to as "maximum value" and denoted by $G_{BU\,MAX}$;

(f) if $\mathcal{P}_1$ and $\mathcal{P}_4$ are true, and if $\lambda_{\min}$ and $\lambda_{\max}$ are given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{BU}$ to $\mathbf{D}_{BU}$, in the variable $\mathbf{X}_B$ according to (292) and Table 16, we find that $G_{BU\,MIN} = \lambda_{\min}$ and $G_{BU\,MAX} = \lambda_{\max}$;

(g) if $\mathcal{P}_1$ and $\mathcal{P}_4$ are true, and if $\mathbf{M}$ is given by Theorem 12 applied to the generalized Rayleigh ratio of $\mathbf{N}_{BU}$ to $\mathbf{D}_{BU}$, in the variable $\mathbf{X}_B$, an average value of G_{BU} over a number $\min\{m,n\}$ of nonzero excitations is

$$G_{BU\,AVR} = \frac{\operatorname{tr}\mathbf{M}}{\min\{m,n\}} = \frac{\operatorname{tr}\left(\mathbf{N}_{BU}\mathbf{D}_{BU}^{-1}\right)}{\min\{m,n\}}\,; \quad (296)$$

(h) if $\mathcal{P}_1$ and $\mathcal{P}_4$ are true, $G_{BU\,AVR}$ doesn't depend on the choice of $\mathbf{X}_B$, and

$$0 \leqslant G_{BU\,MIN} \leqslant G_{BU\,AVR} \leqslant G_{BU\,MAX}\,; \quad (297)$$

(i) if $\mathcal{P}_3$ and $\mathcal{P}_4$ are true, and if the DUS and both loads are reciprocal devices, then

$$G_{AU\,MAX} = G_{BU\,MAX}\,, \quad (298)$$

$$G_{AU\,AVR} = G_{BU\,AVR}\,, \quad (299)$$

$$(m = n) \Longrightarrow (G_{AU\,MIN} = G_{BU\,MIN})\,, \quad (300)$$

$$(m > n) \Longrightarrow (G_{AU\,MIN} = 0) \quad (301)$$

and

$$(m < n) \Longrightarrow (G_{BU\,MIN} = 0)\,. \quad (302)$$

Proof: If $\mathcal{P}_3$ is true, by Lemma 11 we can assume that $\mathbf{Z}_{PAM11}$ is invertible, $H(\mathbf{Z}_{PAM11}^{-1} - \mathbf{Y}_{S1})$ is invertible, and $\mathbf{X}_A = \mathbf{I}_{S1}$. Using Table 15 and (60), we get

$$\mathbf{D}_{AU} = \mathbf{Z}_{ARP1} = \frac{1}{2}\mathbf{Z}_{PAM11}^{*} \times \left(\mathbf{Z}_{PAM11}^{-1} + \mathbf{Z}_{PAM11}^{-1*} - \mathbf{Y}_{S1} - \mathbf{Y}_{S1}^{*}\right) \times \mathbf{Z}_{PAM11}\,, \quad (303)$$

so that $\mathbf{D}_{AU} = \mathbf{Z}_{ARP1}$ is invertible, hence positive definite. If $\mathcal{P}_2$ and $\mathcal{P}_3$ are true, assertions (a) and (b) directly follow from Theorem 12.

In (c), by Theorem 12 and [8, Sec. 1.3.3], we obtain $\operatorname{tr}\mathbf{M} = \operatorname{tr}\left(\mathbf{N}_{AU}\mathbf{D}_{AU}^{-1}\right)$, and then the second equality of (294). By (8), each eigenvector $\mathbf{y}$ of $\mathbf{M}$ corresponds to a nonzero excitation $\mathbf{X}_A = \mathbf{D}_{AU}^{-1/2}\mathbf{y}$, and to an eigenvalue that is equal to G_{AU} for this $\mathbf{X}_A$. Using Table 15, (189) and (192), we get

$$\operatorname{rank}\left(\mathbf{N}_{AU}\mathbf{D}_{AU}^{-1}\right) = \operatorname{rank}\mathbf{N}_{AU} \leqslant \min\{m,n\}\,, \quad (304)$$

so that the number of nonzero eigenvalues of $\mathbf{M}$, counting multiplicity, is less than or equal to $\min\{m,n\}$. Since $\operatorname{tr}\mathbf{M}$ is the sum of the eigenvalues of $\mathbf{M}$, counting multiplicity, it follows that $G_{AU\,AVR}$ given by (294) is an average of G_{AU} over a number $\min\{m,n\}$ of nonzero excitations. This proves (c). Assertion (d) follows from (c) and (372) of Appendix C.

If $\mathcal{P}_4$ is true, by Lemma 11 we can assume that $\mathbf{Z}_{PAM22}$ is invertible, $H(\mathbf{Z}_{PAM22}^{-1} - \mathbf{Y}_{S2})$ is invertible, and $\mathbf{X}_B = \mathbf{I}_{S2}$. Using Table 16 and (66), we get

$$\mathbf{D}_{BU} = \mathbf{Z}_{BRP2} = \frac{1}{2}\mathbf{Z}_{PAM22}^{*} \times \left(\mathbf{Z}_{PAM22}^{-1} + \mathbf{Z}_{PAM22}^{-1*} - \mathbf{Y}_{S2} - \mathbf{Y}_{S2}^{*}\right) \times \mathbf{Z}_{PAM22}\,, \quad (305)$$

so that $\mathbf{D}_{BU} = \mathbf{Z}_{BRP2}$ is invertible, hence positive definite. If $\mathcal{P}_1$ and $\mathcal{P}_4$ are true, assertions (e) and (f) directly follow from Theorem 12.

In (g), by Theorem 12, we have $\operatorname{tr}\mathbf{M} = \operatorname{tr}\left(\mathbf{N}_{BU}\mathbf{D}_{BU}^{-1}\right)$, which allows us to write the second equality of (296). Using Table 16, (195) and (198), we get

$$\operatorname{rank}\left(\mathbf{N}_{BU}\mathbf{D}_{BU}^{-1}\right) = \operatorname{rank}\mathbf{N}_{BU} \leqslant \min\{m,n\}\,, \quad (306)$$

which can be used to prove (g) as we used (304) to prove (c). Assertion (h) follows from (g) and (373) of Appendix C.

To prove (i), we can assume $\mathbf{X}_A = \mathbf{I}_{S1}$ and $\mathbf{X}_B = \mathbf{I}_{S2}$. By Theorem 12, we only need to compare the eigenvalues of $\mathbf{A} = \mathbf{N}_{AU}\mathbf{D}_{AU}^{-1}$ with the eigenvalues of $\mathbf{B} = \mathbf{N}_{BU}\mathbf{D}_{BU}^{-1}$. It follows from Table 15, (189) and (303) that

$$\mathbf{A} = \mathbf{Z}_{PAM21}^{*}\mathbf{Z}_{PAM22}^{-1*} \times \left(\mathbf{Z}_{PAM22}^{-1} + \mathbf{Z}_{PAM22}^{-1*} - \mathbf{Y}_{S2} - \mathbf{Y}_{S2}^{*}\right)^{-1} \times \mathbf{Z}_{PAM22}^{-1}\mathbf{Z}_{PAM21}\mathbf{Z}_{PAM11}^{-1} \times \left(\mathbf{Z}_{PAM11}^{-1} + \mathbf{Z}_{PAM11}^{-1*} - \mathbf{Y}_{S1} - \mathbf{Y}_{S1}^{*}\right)^{-1} \times \mathbf{Z}_{PAM11}^{-1*}\,, \quad (307)$$

which is of size m by m. It follows from Table 16, (195) and (305) that

$$\mathbf{B} = \mathbf{Z}^{*}_{PAM12}\mathbf{Z}^{-1*}_{PAM11} \times \left(\mathbf{Z}^{-1}_{PAM11} + \mathbf{Z}^{-1*}_{PAM11} - \mathbf{Y}_{S1} - \mathbf{Y}^{*}_{S1}\right)^{-1} \times \mathbf{Z}^{-1}_{PAM11}\mathbf{Z}_{PAM12}\mathbf{Z}^{-1}_{PAM22} \times \left(\mathbf{Z}^{-1}_{PAM22} + \mathbf{Z}^{-1*}_{PAM22} - \mathbf{Y}_{S2} - \mathbf{Y}^{*}_{S2}\right)^{-1} \times \mathbf{Z}^{-1*}_{PAM22}\,, \tag{308}$$

which is of size n by n. If the DUS and both loads are reciprocal devices, $\mathbf{Z}_{PAM}$, $\mathbf{Y}_{S1}$, $\mathbf{Y}_{S2}$ are symmetric and the transpose of $\mathbf{Z}_{PAM12}$ is $\mathbf{Z}_{PAM21}$, so that

$$\mathbf{B}^{T} = \mathbf{Z}^{-1*}_{PAM22}\left(\mathbf{Z}^{-1}_{PAM22} + \mathbf{Z}^{-1*}_{PAM22} - \mathbf{Y}_{S2} - \mathbf{Y}^{*}_{S2}\right)^{-1} \times \mathbf{Z}^{-1}_{PAM22}\mathbf{Z}_{PAM21}\mathbf{Z}^{-1}_{PAM11} \times \left(\mathbf{Z}^{-1}_{PAM11} + \mathbf{Z}^{-1*}_{PAM11} - \mathbf{Y}_{S1} - \mathbf{Y}^{*}_{S1}\right)^{-1} \times \mathbf{Z}^{-1*}_{PAM11}\mathbf{Z}^{*}_{PAM21}\,. \tag{309}$$

By [8, Sec. 1.4.1], the eigenvalues of $\mathbf{B}^{T}$ are the same as those of $\mathbf{B}$, counting multiplicity. We note that, if we write

$$\mathbf{C} = \mathbf{Z}^{-1*}_{PAM22}\left(\mathbf{Z}^{-1}_{PAM22} + \mathbf{Z}^{-1*}_{PAM22} - \mathbf{Y}_{S2} - \mathbf{Y}^{*}_{S2}\right)^{-1} \times \mathbf{Z}^{-1}_{PAM22}\mathbf{Z}_{PAM21}\mathbf{Z}^{-1}_{PAM11} \times \left(\mathbf{Z}^{-1}_{PAM11} + \mathbf{Z}^{-1*}_{PAM11} - \mathbf{Y}_{S1} - \mathbf{Y}^{*}_{S1}\right)^{-1} \times \mathbf{Z}^{-1*}_{PAM11}\,, \tag{310}$$

the right hand sides of (307) and (309) are $\mathbf{Z}^{*}_{PAM21}\mathbf{C}$ and $\mathbf{C}\,\mathbf{Z}^{*}_{PAM21}$, respectively. Thus, using [8, Sec. 1.3.22] and the fact that $\mathbf{Z}^{*}_{PAM21}$ is of size m by n, we find that:

- if $m = n$, then $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues, counting multiplicity;
- if $m > n$, then $\mathbf{A}$ has the same eigenvalues as $\mathbf{B}$, counting multiplicity, together with $m - n$ additional eigenvalues equal to zero; and
- if $m < n$, then $\mathbf{B}$ has the same eigenvalues as $\mathbf{A}$, counting multiplicity, together with $n - m$ additional eigenvalues equal to zero.

This leads to the final assertions of Theorem 29. □

X. SOME INEQUALITITES

The following corollary states inequalities which supplement (k) of Theorem 24 and (k) of Theorem 25.

Corollary 10. Ignoring noise power contributions, we assert that:

(a) if proposition $\mathcal{P}_3$ is true, we have

$$t_{A1\,MAX} \geqslant t_{B1\,MAX}\,, \tag{311}$$

and

$$t_{A1\,MIN} \leqslant t_{B1\,MIN}\,; \tag{312}$$

(b) if proposition $\mathcal{P}_4$ is true, we have

$$t_{A2\,MAX} \leqslant t_{B2\,MAX}\,, \tag{313}$$

and

$$t_{A2\,MIN} \geqslant t_{B2\,MIN}\,. \tag{314}$$

Proof: According to the explanations provided above to prove (k) of Theorem 24, if $\mathbf{Y}_{SAM11}$ and $H(\mathbf{Y}^{-1}_{SAM11} - \mathbf{Z}_{S1})$ are invertible, then the set of the values of t_{B1} obtained for all $\mathbf{X}_B \in D(\mathbf{D}_{B1})$ equals the set of the values of t_{B1} obtained for all nonzero $\mathbf{V}_{T2} \in \operatorname{range}(\mathbf{Y}^{-1}_{SAM11}\mathbf{Y}_{SAM12})$. This set is a subset of the set A of the values of t_{B1} which would be obtained if $\mathbf{V}_{T2}$ could take on any value lying in $\mathbb{C}^m$. Assertion (a) follows from the fact that, by Theorem 23 applied to port set 1, A has a minimum value that is equal to $t_{A1\,MIN}$, and a maximum value that is equal to $t_{A1\,MAX}$.

Assertion (b) corresponds to assertion (a) with a different labeling of port sets. □

Some equalities between power ratios lead to inequalities between their bounds. For instance, since Theorem 15, Theorem 17, Theorem 18, Theorem 24 and Theorem 25 ensure that the maximum and minimum values of G_{AT}, G_{BT}, G_{AO}, G_{BO}, t_{A1} and t_{B2} are defined and computable, (217) leads us to:

$$G_{AO\,MIN}\,t_{A1\,MIN} \leqslant G_{AT\,MIN} \leqslant \min\{G_{AO\,MIN}\,t_{A1\,MAX},\, G_{AO\,MAX}\,t_{A1\,MIN}\} \leqslant \min\{G_{AO\,MIN},\, t_{A1\,MIN}\}\,, \tag{315}$$

$$G_{AO\,MAX}\,t_{A1\,MAX} \geqslant G_{AT\,MAX} \geqslant \max\{G_{AO\,MIN}\,t_{A1\,MAX},\, G_{AO\,MAX}\,t_{A1\,MIN}\}\,, \tag{316}$$

$$G_{BO\,MIN}\,t_{B2\,MIN} \leqslant G_{BT\,MIN} \leqslant \min\{G_{BO\,MIN}\,t_{B2\,MAX},\, G_{BO\,MAX}\,t_{B2\,MIN}\} \leqslant \min\{G_{BO\,MIN},\, t_{B2\,MIN}\}\,, \tag{317}$$

and

$$G_{BO\,MAX}\,t_{B2\,MAX} \geqslant G_{BT\,MAX} \geqslant \max\{G_{BO\,MIN}\,t_{B2\,MAX},\, G_{BO\,MAX}\,t_{B2\,MIN}\}\,. \tag{318}$$

Similar inequalities can be obtained from (218), (262)–(263) and (288)–(289).

XI. SPECIAL CASE OF A LOSSLESS DUS

The DUS is lossless only if, for any positive definite $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$, we have

$$P_{ADP2} = P_{ARP1} \text{ and } P_{BDP1} = P_{BRP2}\,. \tag{319}$$

In this Section XI, we now assume that the DUS is lossless. It follows from (71)–(72), (211), (215) and (319) that, for nonzero excitations,

$$G_{AT} = t_{A1} \text{ and } G_{BT} = t_{B2}\,. \tag{320}$$

If $P_{ARP1} \neq 0$ W, it follows from (107) and (319) that we have

$$G_{AO} = 1\,. \tag{321}$$

Also, if $P_{BRP2} \neq 0$ W, it follows from (108) and (319) that we have

$$G_{BO} = 1\,. \tag{322}$$

Theorem 30. We assume that the DUS is lossless, and that the DUS and both loads are reciprocal devices. Ignoring noise power contributions, we assert that:

$$t_{A1\,MAX} = t_{B2\,MAX}\,, \tag{323}$$

$$m\,t_{A1\,AVR} = n\,t_{B2\,AVR}\,, \tag{324}$$

$$(m = n) \Longrightarrow (t_{A1\,MIN} = t_{B2\,MIN})\,, \tag{325}$$

$$(m > n) \Longrightarrow (t_{A1\,MIN} = 0) \tag{326}$$

and

$$(m < n) \Longrightarrow (t_{B2\,MIN} = 0)\,. \tag{327}$$

Proof: Theorem 15 and (320) allow us to directly obtain (323), (325), (326) and (327), but not (324).

By Table 1, Table 7 and (319), in the case $\mathbf{X}_A = \mathbf{I}_{S1}$ and in the case $\mathbf{X}_A = \mathbf{V}_{O1}$, for any $\mathbf{X}_A \in \mathbb{C}^m$, we have

$$\mathbf{X}_A^* \mathbf{N}_{AT} \mathbf{X}_A = P_{ADP2} = P_{ARP1} = \mathbf{X}_A^* \mathbf{N}_{A1} \mathbf{X}_A\,. \tag{328}$$

Thus, using (367) of Appendix C, we may conclude that $\mathbf{N}_{AT} = \mathbf{N}_{A1}$. Since, according to Table 1 and Table 7, we also have $\mathbf{D}_{AT} = \mathbf{D}_{A1}$, it follows from (c) of Theorem 15 and (c) of Theorem 24 that

$$m\,t_{A1\,AVR} = \min\{m,n\}\,G_{AT\,AVR}\,. \tag{329}$$

By Table 2, Table 11 and (319), in the case $\mathbf{X}_B = \mathbf{I}_{S2}$ and in the case $\mathbf{X}_B = \mathbf{V}_{O2}$, for any $\mathbf{X}_B \in \mathbb{C}^n$, we have

$$\mathbf{X}_B^* \mathbf{N}_{BT} \mathbf{X}_B = P_{BDP1} = P_{BRP2} = \mathbf{X}_B^* \mathbf{N}_{B2} \mathbf{X}_B\,. \tag{330}$$

This allows us to conclude that $\mathbf{N}_{BT} = \mathbf{N}_{B2}$. Since, according to Table 2 and Table 11, we also have $\mathbf{D}_{BT} = \mathbf{D}_{B2}$, it follows from (g) of Theorem 15 and (c) of Theorem 25 that

$$n\,t_{B2\,AVR} = \min\{m,n\}\,G_{BT\,AVR}\,. \tag{331}$$

Consequently, (324) follows from (j) of Theorem 15. ☐

In Section VIII of Part 2, it was assumed that $\mathcal{P}_3$ and $\mathcal{P}_4$ are true, and we defined: t_{MAX1}, t_{MAX2}, t_{MIN1} and t_{MIN2}, which correspond to $t_{A1\,MAX}$, $t_{B2\,MAX}$, $t_{A1\,MIN}$ and $t_{B2\,MIN}$, respectively. Thus, Theorem 30 may be viewed as an improved version of Theorem 9 of Part 2.

Theorem 31. We assume that the DUS is lossless, and that $\mathcal{P}_3$ and $\mathcal{P}_4$ are true. Ignoring noise power contributions, we assert that:

(a) if $\mathbf{Z}_{PAM12}$ is of rank m and/or $\mathbf{Y}_{SAM12}$ is of rank m, then, for any nonzero excitation in CA, we have

$$G_{AA} = 1 \text{ and } t_{A1} = t_{A2}\,; \tag{332}$$

(b) if $\mathbf{Z}_{PAM12}$ is of rank m and/or $\mathbf{Y}_{SAM12}$ is of rank m, then,

$$t_{A1\,MAX} = t_{A2\,MAX}\,, \tag{333}$$

$$t_{A1\,AVR} = t_{A2\,AVR} \tag{334}$$

and

$$t_{A1\,MIN} = t_{A2\,MIN}\,; \tag{335}$$

(c) if $\mathbf{Z}_{PAM21}$ is of rank n and/or $\mathbf{Y}_{SAM21}$ is of rank n, then, for any nonzero excitation in CB, we have

$$G_{BA} = 1 \text{ and } t_{B1} = t_{B2}\,; \tag{336}$$

(d) if $\mathbf{Z}_{PAM21}$ is of rank n and/or $\mathbf{Y}_{SAM21}$ is of rank n, then,

$$t_{B1\,MAX} = t_{B2\,MAX}\,, \tag{337}$$

$$t_{B1\,AVR} = t_{B2\,AVR} \tag{338}$$

and

$$t_{B1\,MIN} = t_{B2\,MIN}\,; \tag{339}$$

(e) if $\mathbf{Z}_{PAM12}$ is of rank m and/or $\mathbf{Y}_{SAM12}$ is of rank m, and if $\mathbf{Z}_{PAM21}$ is of rank n and/or $\mathbf{Y}_{SAM21}$ is of rank n, then,

$$G_{AT\,MAX} = G_{BT\,MAX}\,, \tag{340}$$

$$G_{AT\,MIN} = G_{BT\,MIN}\,, \tag{341}$$

$$G_{AU\,MAX} = G_{BU\,MAX} \tag{342}$$

and

$$G_{AU\,MIN} = G_{BU\,MIN}\,. \tag{343}$$

Proof: Since $\mathcal{P}_3$ and $\mathcal{P}_4$ are assumed to be true, Lemma 11 tells us that $\mathbf{Y}_{SAM11}$ and $H(\mathbf{Y}_{SAM11}^{-1} - \mathbf{Z}_{S1})$ are invertible, and $\mathbf{Y}_{SAM22}$ and $H(\mathbf{Y}_{SAM22}^{-1} - \mathbf{Z}_{S2})$ are invertible. Thus, port set 1 behaves, in CA, as an m-port load of impedance matrix $\mathbf{Z}_{T2} = \mathbf{Y}_{SAM11}^{-1} - \mathbf{Z}_{S1}$ such that $H(\mathbf{Z}_{T2})$ is positive definite, and port set 2 behaves, in CB, as an n-port load of impedance matrix $\mathbf{Z}_{T1} = \mathbf{Y}_{SAM22}^{-1} - \mathbf{Z}_{S2}$ such that $H(\mathbf{Z}_{T1})$ is positive definite.

$\mathbf{Z}_{PAM22}$ being invertible by Lemma 11, port set 2 may be viewed, in CA, as an n-port generator of internal impedance matrix $\mathbf{Z}_{T1}$ and rms short-circuit current vector

$$\mathbf{I}_{T1} = \mathbf{Z}_{PAM22}^{-1} \mathbf{Z}_{PAM21} \mathbf{I}_{S1}\,. \tag{344}$$

In the proof of Theorem 10 of Part 2, it is shown that, if $\operatorname{rank} \mathbf{Z}_{PAM12} = m$, then

$$(\mathbf{Z}_{S2} = \mathbf{Z}_{T1}^*) \Longrightarrow (\mathbf{Z}_{S1} = \mathbf{Z}_{T2}^*)\,. \tag{345}$$

Let us assume that $\operatorname{rank} \mathbf{Z}_{PAM12} = m$. By the maximum power transfer theorem for multiports [4], [13], for any excitation $\mathbf{I}_{S1} \in \mathbb{C}^m$ in CA, the circumstance $\mathbf{Z}_{S2} = \mathbf{Z}_{T1}^*$ entails: $P_{ADP2} = P_{AAVP2}$; and $P_{ARP1} = P_{AAVG1}$ by (345). It follows from (319) that, for the excitation $\mathbf{I}_{S1}$, we have $P_{AAVP2} = P_{AAVG1}$. This result is independent of the value of $\mathbf{Z}_{S2}$, because $\mathbf{Z}_{S2}$ has no effect on P_{AAVG1} and no effect on P_{AAVP2}. Thus, for any value of $\mathbf{Z}_{S2}$ and any nonzero excitation $\mathbf{I}_{S1} \in \mathbb{C}^m$ in CA, we have $G_{AA} = 1$, and we have $t_{A1} = t_{A2}$ because $P_{ADP2} = P_{ARP1}$ by (319).

If, instead of assuming that $\operatorname{rank} \mathbf{Z}_{PAM12} = m$, we assume that $\operatorname{rank} \mathbf{Y}_{SAM12} = m$, we can likewise show that, for any value of $\mathbf{Y}_{S2}$ and any nonzero excitation $\mathbf{V}_{O1} \in \mathbb{C}^m$ in CA, we have $G_{AA} = 1$ and $t_{A1} = t_{A2}$.

This leads us to (332), and to (333) and (335) using (a) of Theorem 24 and (f) of Theorem 25. A different approach is needed to prove (334). According to Table 7 and Table 8, in the case $\mathbf{X}_A = \mathbf{I}_{S1}$ and in the case $\mathbf{X}_A = \mathbf{V}_{O1}$, for any $\mathbf{X}_A \in \mathbb{C}^m$, we have

$$\mathbf{X}_A^* \mathbf{N}_{A1} \mathbf{X}_A = P_{ARP1} = P_{ADP2} = \mathbf{X}_A^* \mathbf{N}_{A2} \mathbf{X}_A \quad (346)$$

by (319), and we have

$$\mathbf{X}_A^* \mathbf{D}_{A1} \mathbf{X}_A = P_{AAVG1} = P_{AAVP2} = \mathbf{X}_A^* \mathbf{D}_{A2} \mathbf{X}_A \quad (347)$$

because $G_{AA} = 1$.

Thus, utilizing (367) of Appendix C, we may conclude that $\mathbf{N}_{A1} = \mathbf{N}_{A2}$ and $\mathbf{D}_{A1} = \mathbf{D}_{A2}$. This proves (334).

If, instead of assuming that $\mathbf{Z}_{PAM12}$ or $\mathbf{Y}_{SAM12}$ are of rank m, we assume that $\operatorname{rank} \mathbf{Z}_{PAM21} = n$ or that $\operatorname{rank} \mathbf{Y}_{SAM21} = n$, we can likewise show that, for any value of $\mathbf{Y}_{S1}$ and any nonzero excitation in CB, we have $G_{BA} = 1$ and $t_{B1} = t_{B2}$, and then obtain (337)–(339).

If we now assume that $\mathbf{Z}_{PAM12}$ or $\mathbf{Y}_{SAM12}$ are of rank m, and that $\mathbf{Z}_{PAM21}$ or $\mathbf{Y}_{SAM21}$ are of rank n, we can for instance use (a) and (e) of Theorem 15, (320), (333), (335) and (k) of Theorem 25, to obtain (340) and (341).

Since $\mathcal{P}_3$ and $\mathcal{P}_4$ are assumed to be true, it follows from Theorem 29 that G_{AU} and G_{BU} are defined, and that $\mathbf{D}_{AU}$ and $\mathbf{D}_{BU}$ are positive definite. Thus, (342)–(343) follow from (288)–(289), (320) and (332)–(341). $\square$

Theorem 31 is new. It shows that, in the case of a lossless DUS, we can obtain reciprocal relations on the bounds of the sets of the values of the transducer power gains in CA and CB, if some conditions are satisfied, which do not require that the DUS and/or the loads are reciprocal devices.

XII. SOME EXAMPLES

A. FIRST EXAMPLE

In a first example, such that $m = n = 2$ and already used in Section VI.B of Part 1, we assume that

$$\mathbf{Z}_{S1} = \begin{pmatrix} 51+39j & 19+79j \\ 27+56j & 37+61j \end{pmatrix} \Omega\,, \quad (348)$$

$$\mathbf{Z}_{S2} = \begin{pmatrix} 32+87j & 11+41j \\ 23+37j & 73+13j \end{pmatrix} \Omega\,, \quad (349)$$

and that the DUS has an impedance matrix given by

$$\mathbf{Z} = \begin{pmatrix} 89+25j & 31+11j & 31+5j & 17+40j \\ 21+3j & 59+35j & 3+62j & 40+17j \\ 3+21j & 41+29j & 73+41j & 21+49j \\ 33+13j & 7+7j & 23+42j & 49+21j \end{pmatrix} \Omega\,. \quad (350)$$

$\mathbf{Z}_{S1}$, $\mathbf{Z}_{S2}$ and $\mathbf{Z}$ are not symmetric and have each a positive definite hermitian part. Thus, the DUS and the loads are passive, not reciprocal and not lossless. $\mathbf{Z}_{PAM}$ can be computed using (35).

The maximum, average and minimum values of the power ratios defined above have been computed a first time using Theorem 12, a second time using Theorem 14, and a third time using an extremum-seeking algorithm (as explained in Section VI.A of Part 1). The three methods give exactly the same values, shown in Table 17 to Table 19.

TABLE 17. Maximum values for the first example.

Quantity	CA	CB
power transfer ratio at port set 1 of the DUS	0.786206	0.786206
power transfer ratio at port set 2 of the DUS	0.950923	0.950923
transducer power gain	0.084966	0.171115
operating power gain	0.132830	0.183025
available power gain	0.098264	0.320845
unnamed power gain	0.165136	0.372990
power transfer ratio without the DUS	0.864763	0.864763
insertion power gain	0.126970	0.291078

TABLE 18. Average values for the first example.

Quantity	CA	CB
power transfer ratio at port set 1 of the DUS	0.608507	0.608507
power transfer ratio at port set 2 of the DUS	0.626807	0.626807
transducer power gain	0.049283	0.100428
operating power gain	0.079257	0.139813
available power gain	0.069560	0.183837
unnamed power gain	0.118456	0.256470
power transfer ratio without the DUS	0.539976	0.539976
insertion power gain	0.087939	0.192515

TABLE 19. Minimum values for the first example.

Quantity	CA	CB
power transfer ratio at port set 1 of the DUS	0.430809	0.430809
power transfer ratio at port set 2 of the DUS	0.302691	0.302691
transducer power gain	0.013600	0.029740
operating power gain	0.025685	0.096601
available power gain	0.040856	0.046829
unnamed power gain	0.071777	0.139950
power transfer ratio without the DUS	0.215189	0.215189
insertion power gain	0.048907	0.093953

The computed values are compatible with the reciprocal power relations stated in (i) of Theorem 23, (k)–(ℓ) of Theorem 24 and (k)–(ℓ) of Theorem 25. We also find that the reciprocal power relations stated in (i) and (j) of Theorem 15, (i) of Theorem 16, (e) of Theorem 27, (e) of Theorem 28 and (i) of Theorem 29 need not be true in a case where the DUS is not reciprocal and not lossless, and where the loads are not reciprocal.

B. SECOND EXAMPLE

In a second example, such that $m = n = 2$ and already used in Section VI.C of Part 1, we assume that

$$\mathbf{Z}_{S1} = \begin{pmatrix} 51+39j & 23+68j \\ 23+68j & 37+61j \end{pmatrix} \Omega\,, \quad (351)$$

$$\mathbf{Z}_{S2} = \begin{pmatrix} 32+87j & 17+39j \\ 17+39j & 73+13j \end{pmatrix} \Omega\,, \quad (352)$$

and that the DUS has an impedance matrix given by

$$\mathbf{Z} = \begin{pmatrix} 89+25j & 26+7j & 17+13j & 25+27j \\ 26+7j & 59+35j & 22+46j & 24+12j \\ 17+13j & 22+46j & 73+41j & 22+46j \\ 25+27j & 24+12j & 22+46j & 49+21j \end{pmatrix} \Omega\,. \quad (353)$$

TABLE 20. Maximum values for the second example.

Quantity	CA	CB
power transfer ratio at port set 1 of the DUS	0.814625	0.814625
power transfer ratio at port set 2 of the DUS	0.984960	0.984960
transducer power gain	0.065234	0.065234
operating power gain	0.118331	0.090104
available power gain	0.090104	0.118331
unnamed power gain	0.189454	0.189454
power transfer ratio without the DUS	0.876240	0.876240
insertion power gain	0.159534	0.141010

TABLE 21. Average values for the second example.

Quantity	CA	CB
power transfer ratio at port set 1 of the DUS	0.642041	0.642041
power transfer ratio at port set 2 of the DUS	0.643069	0.643069
transducer power gain	0.041627	0.041627
operating power gain	0.072153	0.067040
available power gain	0.067040	0.072153
unnamed power gain	0.122072	0.122072
power transfer ratio without the DUS	0.551352	0.551352
insertion power gain	0.098332	0.091509

TABLE 22. Minimum values for the second example.

Quantity	CA	CB
power transfer ratio at port set 1 of the DUS	0.469456	0.469456
power transfer ratio at port set 2 of the DUS	0.301177	0.301177
transducer power gain	0.018019	0.018019
operating power gain	0.025975	0.043977
available power gain	0.043977	0.025975
unnamed power gain	0.054690	0.054690
power transfer ratio without the DUS	0.226464	0.226464
insertion power gain	0.037131	0.042008

TABLE 23. Maximum values for the third example.

Quantity	CA	CB
power transfer ratio at port set 1 of the DUS	0.785846	0.785846
power transfer ratio at port set 2 of the DUS	0.526631	0.526631
transducer power gain	0.049441	0.049441
operating power gain	0.067408	0.131088
available power gain	0.131088	0.067408
unnamed power gain	0.176905	0.176905
power transfer ratio without the DUS	0.286756	0.286756
insertion power gain	0.172413	0.172413

TABLE 24. Average values for the third example.

Quantity	CA	CB
power transfer ratio at port set 1 of the DUS	0.754678	0.754678
power transfer ratio at port set 2 of the DUS	0.410713	0.410713
transducer power gain	0.033257	0.033257
operating power gain	0.044716	0.086282
available power gain	0.086282	0.044716
unnamed power gain	0.115480	0.115480
power transfer ratio without the DUS	0.286756	0.286756
insertion power gain	0.115976	0.115976

TABLE 25. Minimum values for the third example.

Quantity	CA	CB
power transfer ratio at port set 1 of the DUS	0.723510	0.723510
power transfer ratio at port set 2 of the DUS	0.294795	0.294795
transducer power gain	0.017073	0.017073
operating power gain	0.022024	0.041476
available power gain	0.041476	0.022024
unnamed power gain	0.054056	0.054056
power transfer ratio without the DUS	0.286756	0.286756
insertion power gain	0.059538	0.059538

Here, $\mathbf{Z}_{S1}$, $\mathbf{Z}_{S2}$ and $\mathbf{Z}$ are symmetric and have a positive definite hermitian part. Thus, the DUS and the loads are passive, reciprocal and not lossless. $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are neither circulant nor in the form of a complex number times $\mathbf{1}_2$.

The maximum, average and minimum values of the power ratios defined above were computed a first time using Theorem 12, a second time using Theorem 14, and a third time using an extremum-seeking algorithm. The three methods give exactly the same values, shown in Table 20 to Table 22.

We find that the computed values are compatible with the reciprocal power relations stated in (i) and (j) of Theorem 15, (i) of Theorem 23, (k)–(ℓ) of Theorem 24, (k)–(ℓ) of Theorem 25, (e) of Theorem 27, (e) of Theorem 28 and (i) of Theorem 29. We also find that the reciprocal power relations stated in (i) of Theorem 16 need not be true in a case where we cannot say that $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are each in the form of a complex number times an identity matrix, and where we cannot say that $\mathbf{Z}_{S1}$, $\mathbf{Z}_{S2}$ and $\mathbf{Z}_{PAM21}$ are circulant.

C. THIRD EXAMPLE

In a third example, already used in Section VI.D of Part 1, we assume that

$$\mathbf{Z}_{S1} = (51 + 39j) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Omega\,, \tag{354}$$

$$\mathbf{Z}_{S2} = (32 + 87j) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Omega\,, \tag{355}$$

and that the DUS has an impedance matrix given by (353).

Here, $\mathbf{Z}_{S1}$, $\mathbf{Z}_{S2}$ and $\mathbf{Z}$ are symmetric and have a positive definite hermitian part. Thus, the DUS and the loads are passive, reciprocal and not lossless. Also, $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are each in the form of a complex number times the identity matrix $\mathbf{1}_2$.

The maximum, average and minimum values of the power ratios defined above were computed a first time using Theorem 12, a second time using Theorem 14, and a third time using an extremum-seeking algorithm. The three methods give exactly the same values, shown in Table 23 to Table 25. These values are compatible with the reciprocal power relations stated in (i)–(j) of Theorem 15, (i) of Theorem 16, (i) of Theorem 23, (k)–(ℓ) of Theorem 24, (k)–(ℓ) of Theorem 25, (e) of Theorem 27, (e) of Theorem 28 and (i) of Theorem 29.

D. FOURTH EXAMPLE

In a fourth example, already used in Section VI.E of Part 1, we assume that

$$\mathbf{Z}_{S1} = \begin{pmatrix} 51 - 39j & 7 + 16j \\ 7 + 16j & 51 - 39j \end{pmatrix} \Omega\,, \tag{356}$$

TABLE 26. Maximum values for the fourth example.

Quantity	CA	CB
power transfer ratio at port set 1 of the DUS	0.994997	0.994997
power transfer ratio at port set 2 of the DUS	0.453406	0.453406
transducer power gain	0.120251	0.120251
operating power gain	0.165995	0.309882
available power gain	0.309882	0.165995
unnamed power gain	0.427761	0.427761
power transfer ratio without the DUS	0.691529	0.691529
insertion power gain	0.215581	0.215581

TABLE 27. Average values for the fourth example.

Quantity	CA	CB
power transfer ratio at port set 1 of the DUS	0.859712	0.859712
power transfer ratio at port set 2 of the DUS	0.420730	0.420730
transducer power gain	0.065159	0.065159
operating power gain	0.088056	0.166041
available power gain	0.166041	0.088056
unnamed power gain	0.225037	0.225037
power transfer ratio without the DUS	0.624666	0.624666
insertion power gain	0.115068	0.115068

TABLE 28. Minimum values for the fourth example.

Quantity	CA	CB
power transfer ratio at port set 1 of the DUS	0.724428	0.724428
power transfer ratio at port set 2 of the DUS	0.388055	0.388055
transducer power gain	0.010066	0.010066
operating power gain	0.010116	0.022200
available power gain	0.022200	0.010116
unnamed power gain	0.022312	0.022312
power transfer ratio without the DUS	0.557803	0.557803
insertion power gain	0.014556	0.014556

$$\mathbf{Z}_{S2} = \begin{pmatrix} 32+47j & 11+41j \\ 11+41j & 32+47j \end{pmatrix} \Omega\,, \tag{357}$$

and that the DUS has an impedance matrix, given by

$$\mathbf{Z} = \begin{pmatrix} 54+25j & 6+7j & 20+13j & -10-5j \\ 6+7j & 54+25j & -10-5j & 20+13j \\ 20+13j & -10-5j & 25-25j & 6+17j \\ -10-5j & 20+13j & 6+17j & 25-25j \end{pmatrix} \Omega\,. \tag{358}$$

Here, $\mathbf{Z}_{S1}$, $\mathbf{Z}_{S2}$ and $\mathbf{Z}$ are symmetric and have a positive definite hermitian part. Also, $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are circulant, and $\mathbf{Z}$ is a 2-by-2 block matrix, the blocks of which are of size 2 by 2 and circulant. Thus, the DUS and the loads are passive, reciprocal and not lossless, and it follows from Observation 4 of Part 1 that $\mathbf{Z}_{PAM21}$ is circulant.

The maximum, average and minimum values of the power ratios defined above were computed a first time using Theorem 12, a second time using Theorem 14, and a third time using an extremum-seeking algorithm. The three methods give exactly the same values, shown in Table 26 to Table 28. These values are compatible with the reciprocal power relations stated in (i)–(j) of Theorem 15, (i) of Theorem 16, (i) of Theorem 23, (k)–(ℓ) of Theorem 24, (k)–(ℓ) of Theorem 25, (e) of Theorem 27, (e) of Theorem 28 and (i) of Theorem 29.

TABLE 29. Maximum values for the fifth example.

Quantity	CA	CB
power transfer ratio at port set 1 of the DUS	0.924578	0.924578
power transfer ratio at port set 2 of the DUS	0.924578	0.924578
transducer power gain	0.924578	0.924578
operating power gain	1.000000	1.000000
available power gain	1.000000	1.000000
unnamed power gain	14.48862	14.48862
power transfer ratio without the DUS	0.864763	0.864763
insertion power gain	1.187487	1.238501

TABLE 30. Average values for the fifth example.

Quantity	CA	CB
power transfer ratio at port set 1 of the DUS	0.496799	0.496799
power transfer ratio at port set 2 of the DUS	0.496799	0.496799
transducer power gain	0.496799	0.496799
operating power gain	1.000000	1.000000
available power gain	1.000000	1.000000
unnamed power gain	7.785100	7.785100
power transfer ratio without the DUS	0.539976	0.539976
insertion power gain	0.738134	0.757694

TABLE 31. Minimum values for the fifth example.

Quantity	CA	CB
power transfer ratio at port set 1 of the DUS	0.069020	0.069020
power transfer ratio at port set 2 of the DUS	0.069020	0.069020
transducer power gain	0.069020	0.069020
operating power gain	1.000000	1.000000
available power gain	1.000000	1.000000
unnamed power gain	1.081575	1.081575
power transfer ratio without the DUS	0.215189	0.215189
insertion power gain	0.288782	0.276887

E. FIFTH EXAMPLE

In a fifth example, the DUS has an impedance matrix

$$\mathbf{Z} = \begin{pmatrix} 25j & 31+11j & 31+5j & 17+40j \\ -31+11j & 35j & 3+62j & 40+17j \\ -31+5j & -3+62j & 41j & 21+49j \\ -17+40j & -40+17j & -21+49j & 21j \end{pmatrix} \Omega\,. \tag{359}$$

In this fifth example, already used in Section IX.A of Part 2, $\mathbf{Z}_{S1}$ is given by (348), and $\mathbf{Z}_{S2}$ is given by (349). $\mathbf{Z}_{S1}$, $\mathbf{Z}_{S2}$ and $\mathbf{Z}$ are not symmetric, so that the DUS and the loads are not reciprocal devices. $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ have each a positive definite hermitian part. We have $H(\mathbf{Z}) = 0$ because $\mathbf{Z}$ is the impedance matrix of a lossless DUT. We have computed $\mathbf{Z}_{PAM}$ and found that $\operatorname{rank} \mathbf{Z}_{PAM12} = \operatorname{rank} \mathbf{Z}_{PAM21} = 2$.

The maximum, average and minimum values of the power ratios defined above were computed a first time using Theorem 12, a second time using Theorem 14, and a third time using an extremum-seeking algorithm. The three methods give exactly the same values, shown in Table 29 to Table 31. These values are compatible with the reciprocal power relations stated in Theorem 31.

This example also allows us to observe unnamed power gains and insertion power gains that are greater than one.

F. A NOTE ABOUT THE FIVE EXAMPLES

In Table 17 to Table 31, all values could be computed using Theorem 12 and Theorem 14, and it was possible to use the simple formulas (189) and (195) to obtain $\mathbf{Z}_{AAVP2}$ and $\mathbf{Z}_{BAVP1}$, and compute the maximum, average and minimum values of t_{A2}, t_{B1}, G_{AA}, G_{BA}, G_{AU} and G_{BU}.

In other problems, this need not be possible, especially when $m \neq n$.

XIII. ABOUT THE FRIIS TRANSMISSION FORMULA

The original Friis transmission formula is about “a radio circuit made up of a transmitting antenna and a receiving antenna in free space” [7]. It is about two single-port antennas, and it assumes that the antennas are polarization matched, that the distance between the antennas, denoted by d, is sufficiently large (far-field condition), and that the transmitting antenna is reciprocal [5, Sec. VII]. The original Friis transmission formula is:

$$\frac{P_{avr}}{P_t} = \frac{A_r A_t}{d^2 \lambda^2}\,, \tag{360}$$

where: P_{avr} is the power available at the port of the receiving antenna; P_t is the power fed into the transmitting antenna at its port; A_r is the effective area of the receiving antenna, in the direction of the transmitting antenna; A_t is the effective area of the transmitting antenna, in the direction of the receiving antenna; and λ is the wavelength.

In [5, Sec. VII], we observed that one of the teachings of (360) is that, if the roles of the antennas are reversed (i.e., the receiving antenna becomes the transmitting antenna and vice versa) without moving the antennas, then the unnamed power gain does not change. In [5, Sec. VII], we also explained that this teaching can be generalized to two single-port LTI and reciprocal antennas lying in an LTI and reciprocal medium, neither assuming polarization-matched antennas, nor a large value of d, nor an homogeneous free space environment.

We can now propose a broader generalization. To this end, we now consider the DUS shown in Fig. 2, comprising: a first multiport antenna array (MAA) denoted by “MAA 1”; a second MAA denoted by “MAA 2”; and whatever lies around MAA 1 and MAA 2. Let d be the distance between the MAAs. We neither assume a large value d, nor a free space environment, nor any form of polarization-matching between the MAAs. In CA, MAA 1 is used for emission and MAA 2 for reception. In CB, MAA 2 is used for emission and MAA 1 for reception. If we assume that both MAAs are reciprocal and that the medium surrounding them is reciprocal [18, Sec. 13.06], we can use theorem II of [19], known as the “Rayleigh-Carson reciprocity theorem” and corresponding to [18, eq. (13-40)], to assert that $\mathbf{Z}_{PAM}$ and $\mathbf{Y}_{SAM}$ are symmetric. The DUS being in this case reciprocal, (i) of Theorem 29 on unnamed power gain in CA and CB can be used if $\mathcal{P}_3$ and $\mathcal{P}_4$ are true, and if both loads shown in Fig. 2 are reciprocal devices. The wanted generalization of the Friis transmission formula is (298)–(302) of Theorem 29.

The reciprocal relations (298)–(302) generalize the above-mentioned teaching of the original Friis transmission formula (360), according to which the unnamed power gain does not change if the receiving antenna becomes the transmitting antenna and vice versa. Other reciprocal relations obtained above for a reciprocal DUS may also be used, such as: (79)–(83) of Theorem 15; (97)–(99) of Theorem 16 if $m = n$; (268)–(272) of Theorem 27; and (281)–(285) of Theorem 28. Each reciprocal relation is based on the assumptions stated in the corresponding theorem, the most general being the reciprocal relations on the transducer power gain, that is (79)–(83). Note that ionospheric propagation may involve a significant Faraday rotation, which makes the propagation medium non-reciprocal [18, Sec. 17.10], [19], [20, Sec. 6.6].

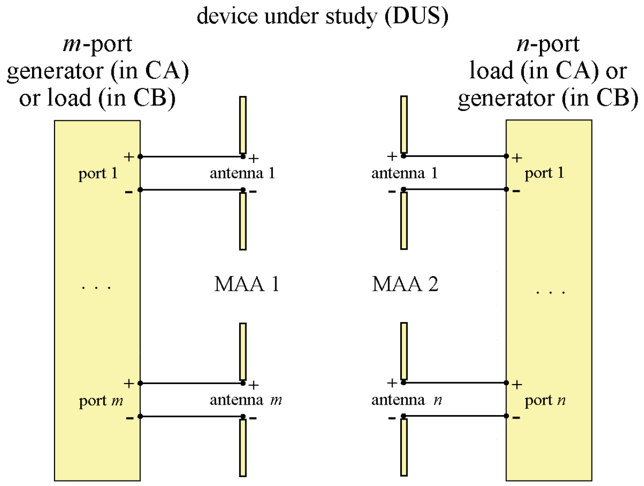

FIGURE 2. The configurations considered in Section XIII, in which the DUS comprises: MAA 1 having m antenna ports; and MAA 2 having n antenna ports.

Lossless MAAs operating in a lossless medium do not lead to a lossless DUS, in the meaning of Section XI, except in the theoretical case of two lossless MAAs installed inside a closed lossless enclosure that does not allow radiation outside its boundary, the enclosure containing only the MAAs and a lossless medium. Such an enclosure may for instance be made of a perfect electric conductor (PEC). In this theoretical case, we note that $G_{AO} = G_{BO} = 1$, regardless of the distance between the MAAs (inside the enclosure).

Let us assume that $\mathcal{P}_3$ and $\mathcal{P}_4$ are true. In CA, P_{AAVP2} depends on the generator connected to port set 1 and on the DUS, but not on $\mathbf{Z}_{S2}$. Except in the theoretical case discussed above, for sufficiently large values of d, the interaction between MAA 1 and MAA 2 will typically be very small, so that P_{ARP1} depends very little on $\mathbf{Z}_{S2}$, and G_{AU} depends very little on $\mathbf{Z}_{S2}$. Likewise, for sufficiently large values of d, G_{BU} depends very little on $\mathbf{Z}_{S1}$. It follows that, in the far field, $G_{AU\,MAX}$, $G_{AU\,AVR}$ and $G_{AU\,MIN}$ depend very little on $\mathbf{Z}_{S2}$; and $G_{BU\,MAX}$, $G_{BU\,AVR}$ and $G_{BU\,MIN}$ depend very little on $\mathbf{Z}_{S1}$. Thus, in the far field, if the DUS and the loads shown in Fig. 2 are reciprocal, then: $G_{AU\,MAX} = G_{BU\,MAX}$ and $G_{AU\,AVR} = G_{BU\,AVR}$ depend very little on $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$; and, if $m = n$, then $G_{AU\,MIN} = G_{BU\,MIN}$ depends very little on $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$.

This is what makes $G_{AU\,MAX}$, $G_{BU\,MAX}$, $G_{AU\,AVR}$, $G_{BU\,AVR}$, $G_{AU\,MIN}$ and $G_{BU\,MIN}$ attractive metrics of a radio link.

XIV. CONCLUSION

We have improved two reciprocal theorems on the bounds of the sets of the values of transducer power gains and insertion power gains for all nonzero excitations, which had been treated in Part 1. We have established new results about the computation of the power available at output ports, which generalize the maximum power transfer theorem for multiports stated in [4] and [13], to a port set having an immittance matrix whose hermitian part need not be invertible. We have established new results about the bounds of the sets of the values of power transfer ratios, operating power gains, available power gains and unnamed power gains for all relevant excitations. The new results include five reciprocal theorems (theorems 24, 25, 27, 28 and 29), two of them (theorems 24 and 25) being fully applicable to a DUS that need not be a reciprocal device.

We have also established other results about power ratios and their bounds, including some inequalities and some reciprocal relations applicable to a lossless DUS that need not be a reciprocal device.

We appreciate that the reciprocal relations and theorems treated in this article are comparable to, but much more complex than, the reciprocal relations covered in [5]. This is why [5] can advantageously be read before this article, as a prologue.

One of the new reciprocal theorems, relating to unnamed power gains, was used to generalize the Friis transmission formula.

This article is applicable to many problems in which bidirectional transmission may occur in an LTI system having more than two ports, and particularly relevant to problems in which the location of $\mathbf{X}_A/||\mathbf{X}_A||_2$ on $\mathbb{S}_m$ and/or the location of $\mathbf{X}_B/||\mathbf{X}_B||_2$ on $\mathbb{S}_n$ are not constant and/or not known. Such circumstances for instance happen in the technical areas of radio communication, wireless power transmission, and electromagnetic compatibility (EMC).

APPENDIX A

This Appendix A is about corrections to known errors in Part 1, Part 2 and [5].

The only known error in Part 1 [1] is that the title shown in the header of even pages ("Some Theorems on Power in Passive Linear Time-Invariant Multiports, Part 1") should be replaced with the correct title ("Some Results on Power in Passive Linear Time-Invariant Multiports, Part 1").

In Part 2 [2], page 11 column 2, "$\mathbf{I}_{T1} \in \mathbb{C}^N$" should be replaced with "$\mathbf{I}_{T1} \in \mathbb{C}^n$" (2 occurrences), and "$\mathbf{I}_{S1} \in \mathbb{C}^N$" should be replaced with "$\mathbf{I}_{S1} \in \mathbb{C}^m$" (2 occurrences).

Several errors have been detected in [5]: in page 5, column 2, in the first sentence of Section IV.E, "operating power gains is" should be replaced with "operating power gain is"; in page 6, column 2, in the first sentence of Section V, "is lossless if and only if" should be replaced with "is lossless only if"; in page 6, column 2, in Section V, "$\mathbf{Z}_{APP2}$" should be replaced with "$\mathbf{Z}_{BPP2}$" (4 occurrences); and in page 10, column 2, the title of reference [17] should be in italics instead of between quotation marks.

APPENDIX B

In this Appendix B, we want to show that (189) can be directly derived from (188). If $\mathbf{Z}_{PAM22}$ is invertible, it follows from (128) and (129) that

$$\mathbf{Z}_{E2} = \mathbf{Z}^*_{PAM22} \times \left[\mathbf{Z}^{-1}_{PAM22} + \mathbf{Z}^{-1*}_{PAM22} - \mathbf{Y}_{S2} - \mathbf{Y}^*_{S2}\right]\mathbf{Z}_{PAM22} \quad (361)$$

and

$$\mathbf{K}_{E2} = -\mathbf{Z}^{-1*}_{PAM22}(\mathbf{Z}_{E2} - \mathbf{Z}_{PAM22})\,. \quad (362)$$

In this Appendix B, we now assume that $\mathbf{Z}_{PAM22}$ and $H(\mathbf{Z}^{-1}_{PAM22} - \mathbf{Y}_{S2})$ are invertible. Thus, it follows from (361) that $\mathbf{Z}_{E2}$ is invertible, so that $\mathbf{Z}^{\dagger}_{E2} = \mathbf{Z}^{-1}_{E2}$. By (362), we have

$$\begin{aligned}\mathbf{K}_{E2}\mathbf{Z}^{\dagger}_{E2}\mathbf{K}^*_{E2} = \mathbf{Z}^{-1*}_{PAM22}(\mathbf{Z}_{E2} - \mathbf{Z}_{PAM22}) \\ \times \mathbf{Z}^{-1}_{E2}(\mathbf{Z}_{E2} - \mathbf{Z}_{PAM22})^*\mathbf{Z}^{-1}_{PAM22} = \\ \mathbf{Z}^{-1*}_{PAM22}\Big[\mathbf{Z}_{E2} - \mathbf{Z}_{PAM22} - \mathbf{Z}^*_{PAM22} \\ + \mathbf{Z}_{PAM22}\mathbf{Z}^{-1}_{E2}\mathbf{Z}^*_{PAM22}\Big]\mathbf{Z}^{-1}_{PAM22}\,.\end{aligned} \quad (363)$$

Using (128) in (363), we obtain

$$\mathbf{Y}_{S2} + \mathbf{Y}^*_{S2} + \mathbf{K}_{E2}\mathbf{Z}^{\dagger}_{E2}\mathbf{K}^*_{E2} = \mathbf{Z}^{-1*}_{PAM22}\mathbf{Z}_{PAM22}\mathbf{Z}^{-1}_{E2}\mathbf{Z}^*_{PAM22}\mathbf{Z}^{-1}_{PAM22}\,. \quad (364)$$

Using (361) in (364), we get

$$\mathbf{Y}_{S2} + \mathbf{Y}^*_{S2} + \mathbf{K}_{E2}\mathbf{Z}^{\dagger}_{E2}\mathbf{K}^*_{E2} = \mathbf{Z}^{-1*}_{PAM22} \times \left[\mathbf{Z}^{-1}_{PAM22} + \mathbf{Z}^{-1*}_{PAM22} - \mathbf{Y}_{S2} - \mathbf{Y}^*_{S2}\right]^{-1} \times \mathbf{Z}^{-1}_{PAM22}\,. \quad (365)$$

Using (365) in (188), we obtain (189).

APPENDIX C

Let $\mathbf{A}$ and $\mathbf{B}$ be hermitian matrices of size ν by ν, where ν is a positive integer. The hermitian matrix $\mathbf{A}$ can be used to define the hermitian quadratic form $\alpha : \mathbb{C}^\nu \to \mathbb{R}$ such that $\alpha(\mathbf{x}) = \mathbf{x}^*\mathbf{A}\mathbf{x}$. The hermitian matrix $\mathbf{B}$ can be used to define the hermitian quadratic form $\beta : \mathbb{C}^\nu \to \mathbb{R}$ such that $\beta(\mathbf{x}) = \mathbf{x}^*\mathbf{B}\mathbf{x}$.

An important result is

$$(\alpha = \beta) \iff (\mathbf{A} = \mathbf{B})\,, \quad (366)$$

that is to say

$$\big(\forall\, \mathbf{x} \in \mathbb{C}^\nu,\ \mathbf{x}^*\mathbf{A}\mathbf{x} = \mathbf{x}^*\mathbf{B}\mathbf{x}\big) \iff \big(\mathbf{A} = \mathbf{B}\big)\,. \quad (367)$$

To prove this, we observe that $\forall\, \mathbf{x} \in \mathbb{C}^\nu,\ \mathbf{x}^*\mathbf{A}\mathbf{x} = \mathbf{x}^*\mathbf{B}\mathbf{x}$ if and only if $\forall\, \mathbf{x} \in \mathbb{C}^\nu,\ \mathbf{x}^*(\mathbf{A} - \mathbf{B})\mathbf{x} = \mathbf{0}$. It follows from Observation 6 that $\forall\, \mathbf{x} \in \mathbb{C}^\nu,\ \mathbf{x}^*\mathbf{A}\mathbf{x} = \mathbf{x}^*\mathbf{B}\mathbf{x}$ if and only if $\forall\, \mathbf{x} \in \mathbb{C}^\nu,\ \mathbf{x} \in \ker(\mathbf{A} - \mathbf{B})$, that is to say $\mathbf{A} = \mathbf{B}$.

Other proofs of (366)–(367) exist [8, Sec. 4.1.P6], [9, p. 174 Problem 6], [10, Sec. 3.2.4], but are more complicated.

Let $\mathbf{Y}_{X1}$ be an hermitian admittance matrix of size m by m and $\mathbf{Z}_{X1}$ be an hermitian impedance matrix of size m by m. If an average power is given by $P_{X1} = \mathbf{V}^*_{O1}\mathbf{Y}_{X1}\mathbf{V}_{O1}$ for any $\mathbf{V}_{O1} \in \mathbb{C}^m$ and $P_{X1} = \mathbf{I}_{S1}^* \mathbf{Z}_{X1} \mathbf{I}_{S1}$ for any $\mathbf{I}_{S1} \in \mathbb{C}^m$, it follows from $\mathbf{I}_{S1} = \mathbf{Y}_{S1} \mathbf{V}_{O1}$ and (367) that:

$$\mathbf{Y}_{X1} = \mathbf{Y}_{S1}^* \mathbf{Z}_{X1} \mathbf{Y}_{S1} \tag{368}$$

and

$$\mathbf{Z}_{X1} = \mathbf{Z}_{S1}^* \mathbf{Y}_{X1} \mathbf{Z}_{S1} \,. \tag{369}$$

Likewise, $\mathbf{Y}_{X2}$ being an hermitian admittance matrix of size n by n and $\mathbf{Z}_{X2}$ an hermitian impedance matrix of size n by n, if an average power is $P_{X2} = \mathbf{V}_{O2}^* \mathbf{Y}_{X2} \mathbf{V}_{O2}$ for any $\mathbf{V}_{O2} \in \mathbb{C}^n$ and $P_{X2} = \mathbf{I}_{S2}^* \mathbf{Z}_{X2} \mathbf{I}_{S2}$ for any $\mathbf{I}_{S2} \in \mathbb{C}^n$, then

$$\mathbf{Y}_{X2} = \mathbf{Y}_{S2}^* \mathbf{Z}_{X2} \mathbf{Y}_{S2} \tag{370}$$

and

$$\mathbf{Z}_{X2} = \mathbf{Z}_{S2}^* \mathbf{Y}_{X2} \mathbf{Z}_{S2} \,. \tag{371}$$

It follows that, if a power ratio is defined as a generalized Rayleigh ratio of $\mathbf{Y}_{N1}$ to $\mathbf{Y}_{D1}$, in the variable $\mathbf{V}_{O1}$, and as a generalized Rayleigh ratio of $\mathbf{Z}_{N1}$ to $\mathbf{Z}_{D1}$, in the variable $\mathbf{I}_{S1}$, then $\mathbf{Y}_{N1} = \mathbf{Y}_{S1}^* \mathbf{Z}_{N1} \mathbf{Y}_{S1}$ and $\mathbf{Y}_{D1} = \mathbf{Y}_{S1}^* \mathbf{Z}_{D1} \mathbf{Y}_{S1}$. Thus, if $\mathbf{Y}_{D1}$ and $\mathbf{Z}_{D1}$ are invertible, $\mathbf{Y}_{N1} \mathbf{Y}_{D1}^{-1}$ and $\mathbf{Z}_{N1} \mathbf{Z}_{D1}^{-1}$ are similar. It follows that

$$\operatorname{tr}\left(\mathbf{Y}_{N1} \mathbf{Y}_{D1}^{-1}\right) = \operatorname{tr}\left(\mathbf{Z}_{N1} \mathbf{Z}_{D1}^{-1}\right) \,. \tag{372}$$

Likewise, if a power ratio is defined as a generalized Rayleigh ratio of $\mathbf{Y}_{N2}$ to $\mathbf{Y}_{D2}$, in the variable $\mathbf{V}_{O2}$, and as a generalized Rayleigh ratio of $\mathbf{Z}_{N2}$ to $\mathbf{Z}_{D2}$, in the variable $\mathbf{I}_{S2}$, and if $\mathbf{Y}_{D2}$ and $\mathbf{Z}_{D2}$ are invertible, then $\mathbf{Y}_{N2} \mathbf{Y}_{D2}^{-1}$ and $\mathbf{Z}_{N2} \mathbf{Z}_{D2}^{-1}$ are similar. It follows that

$$\operatorname{tr}\left(\mathbf{Y}_{N2} \mathbf{Y}_{D2}^{-1}\right) = \operatorname{tr}\left(\mathbf{Z}_{N2} \mathbf{Z}_{D2}^{-1}\right) \,. \tag{373}$$

REFERENCES

[1] F. Broydé and E. Clavelier, “Some Results on Power in Passive Linear Time-Invariant Multiports, Part 1,” *Excem Research Papers in Electronics and Electromagnetics*, no. 2, doi: 10.5281/zenodo.4419637, Jan. 2021.

[2] F. Broydé and E. Clavelier, “Some Results on Power in Passive Linear Time-Invariant Multiports, Part 2”, *Excem Research Papers in Electronics and Electromagnetics*, no. 3, doi: 10.5281/zenodo.4683896, Apr. 2021.

[3] F.E. Terman, *Electronic and Radio Engineering*, 4th ed., International Student Edition, New York, NY, USA: McGraw-Hill, 1955.

[4] H. Baudrand, “On the generalizations of the maximum power transfer theorem”, *Proc. IEEE*, vol. 58, no. 10, pp. 1780-1781, Oct. 1970.

[5] F. Broydé and E. Clavelier, “About the Power Ratios Relevant to a Passive Linear Time-Invariant 2-Port”, *Excem Research Papers in Electronics and Electromagnetics*, no. 5, doi: 10.5281/zenodo.6555682, May 2022.

[6] F. Broydé and E. Clavelier, “The Radiation and Transducer Efficiencies of a Multiport Antenna Array”, *Excem Research Papers in Electronics and Electromagnetics*, no. 4, doi: 10.5281/zenodo.5816837, Jan. 2022.

[7] H.T. Friis, “A note on a simple transmission formula”, *Proceedings of the I.R.E. and Waves and Electrons*, vol. 34, no. 5, pp. 254-256, May 1946.

[8] R.A. Horn and C.R. Johnson, *Matrix analysis*, 2nd ed., New York, NY, USA: Cambridge University Press, 2013.

[9] R.A. Horn and C.R. Johnson, *Matrix analysis*, New York, NY, USA: Cambridge University Press, 1985.

[10] E. Ramis, C. Deschamps and J. Odoux, *Cours de mathématiques spéciales — 2 — Algèbre et applications à la géométrie*, Paris, France: Masson, 1979.

[11] R.F. Harrington, *Field Computation by Moment Methods*, Piscataway, NJ, USA: IEEE Press 1993.

[12] F.R. Gantmacher, *The Theory of Matrices — vol. 1*, New York, NY, USA: Chelsea Publishing Company, 1977.

[13] C.A. Desoer, “The maximum power transfer theorem for n-ports,” *IEEE Trans. Circuit Theory*, vol. 20, no. 3, pp. 328-330, May 1973.

[14] G. Gonzalez, *Microwave Transistor Amplifiers*, 2nd ed., Upper Saddle River, NJ, USA: Prentice Hall, 1997.

[15] J.M. Ortega, *Matrix Theory*, New York, NY, USA: Plenum Press, 1987.

[16] L. Han and M. Neumann, “Orthogonal projection, least squares, and singular value decomposition” in L. Hogben, Ed., *Handbook of Linear Algebra*, Boca Raton, FL, USA: Chapman & Hall/CRC, 2007.

[17] C.A. Desoer and E.S. Kuh, *Basic Circuit Theory*, New York, NY, USA: McGraw-Hill, 1969.

[18] E.C. Jordan and K.G. Balmain, *Electromagnetic Waves and Radiating Systems*, 2nd Edition, Englewood Cliffs, NJ, USA: Prentice-Hall, 1968.

[19] J.R. Carson, “Reciprocal theorems in radio communication”, *Proceedings of the Institute of Radio Engineers*, vol. 17, no. 6, pp. 952-956, Jun. 1929.

[20] R.E. Collin, *Antennas and Radiowave Propagation*, International Edition, New York, NY, USA: McGraw-Hill, 1985.

FRÉDÉRIC BROYDÉ was born in France in 1960. He received the M.S. degree in physics engineering from the Ecole Nationale Supérieure d’Ingénieurs Electriciens de Grenoble (ENSIEG) and the Ph.D. in microwaves and microtechnologies from the Université des Sciences et Technologies de Lille (USTL).

He co-founded the Excem corporation in May 1988, a company providing engineering and research and development services. He is president of Excem since 1988. He is now also president and CTO of Eurexcem, a subsidiary of Excem. Most of his activity is allocated to engineering and research in electronics, radio, antennas, electromagnetic compatibility (EMC) and signal integrity.

Dr. Broydé is author or co-author of about 100 technical papers, and inventor or co-inventor of about 90 patent families, for which 71 patents of France and 49 patents of the USA have been granted. He is a Senior Member of the IEEE since 2001. He is a licensed radio amateur (F5OYE).

EVELYNE CLAVELIER was born in France in 1961. She received the M.S. degree in physics engineering from the Ecole Nationale Supérieure d’Ingénieurs Electriciens de Grenoble (ENSIEG).

She is co-founder of the Excem corporation, based in Maule, France, and she is currently CEO of Excem. She is also president of Tekcem, a company selling or licensing intellectual property rights to foster research. She is an active engineer and researcher.

Her current research areas are radio communications, antennas, matching networks, EMC and circuit theory.

Prior to starting Excem in 1988, she worked for Schneider Electrics (in Grenoble, France), STMicroelectronics (in Grenoble, France), and Signetics (in Mountain View, CA, USA).

Ms. Clavelier is the author or a co-author of about 90 technical papers. She is co-inventor of about 90 patent families. She is a Senior Member of the IEEE since 2002. She is a licensed radio amateur (F1PHQ).

 Published by Excem. ISBN 978-2-909056-27-2
Excem — 12, chemin des Hauts de Clairefontaine — 78580 Maule — France