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The Role of Counterexample and Paradox in Teaching Statistics and Probability

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Abstract: The process of teaching and learning often suffers from its effectiveness for lack of appropriate rational challenges. Counterexamples and paradoxes as two powerful and effective tools are interesting examples that can excite and inspire learners and stimulate their enthusiasm for finding a better way to learn. Furthermore, thought-provoking counterexamples and paradoxes can motivate students and propose them a great opportunity to understand more about the ways of solving a mathematics or statistics problem. This paper explains the importance and the role of these tools for teaching statistics and probability. The results show that counterexamples and paradoxes can be thought concepts in a better way modify the wrong vision of students and improve their understanding of probabilities problems.

Keywords: Counterexample, Teaching Statistics and Probability, Paradox, Mathematics

INTRODUCTION

Counterexamples can be used to show that the proposed assumption is incorrect. However, finding counterexamples or paradoxes requires lots of efforts, thoughts and time that is not often considered in teaching in scientific schools. There are several publications related to the use of counterexamples in teaching and learning mathematics, especially in calculus. In addition, there are three well-known books on counterexamples in advanced statistics and probability; but no article or book has been published about the use of counterexamples in teaching and learning in statistics and probability for students in schools; except Kachapova *et al.*, (2007)[1] and Klymchuk, S and Kachapova, F (2011)[2]. The purpose of this paper is to present some of these examples as an educational helpful policy. In fact, the main goal of this study is to investigate the existing assumptions about how counterexamples as well as paradoxes affect deeper learning of concepts, correcting students' misconceptions, and improving the learning of statistics and probability.

Theoretical Foundation and Review of the Literature

Theoretical Framework

The word paradox comes from the Greek word paradox on which means unexpected. Several usages of this word exist, including those that allow for contradiction[3]. In logical form, a paradox is an apparently valid argument whose logical form can be used to derive an apparently false conclusion from apparently true premises.

In this regard, the meaning of counterexample is an example that refutes or disproves a proposition or theory[4]. They are good indicators showing that a suggested hypothesis or a chosen direction of research is wrong[5]. Before trying to prove a conjecture or a hypothesis, it is valuable to look for a possible counterexample or paradox that it can save lots of time and effort.

Creating examples and counterexamples is not related to any algorithm or procedure and requires advanced thinking which is not often taught at school (Selden & Selden, 1998; Tall, 1991; Tall, *et al.*, 2001). Many students are used to concentrate on techniques, manipulations, familiar procedures and do not pay much attention to concepts, conditions of theorems and rules, reasoning, and justifications. As Seldens argue, coming up with examples requires different cognitive skills from carrying out algorithms.

In this paper, the word paradox will be used to indicate an unexpected, counter-intuitive statement that looks invalid but in fact is true in examples related to statistics and probabilities.

Literature Review

There are several publications on using counter-examples in teaching and learning of mathematics, statistics and probability (Gelbaum and Olmstead, 1964; Peled & Zaslavski, 1997; Zaslavski & Ron, 1998; Bermudez, 2004; Gruenwald & Klymchuk, 2003; Klymchuk, 2004 & 2005). There are three well-known books on counter-examples in statistics at an advanced level (Stoyanov, 1997; Romano, 1986; Wise & Hall, 1986). But we could not find any publication on using counter-examples in learning of Probability for bachelor students in engineering majors.

Some studies in mathematics education at secondary level (Swan, 1993; Irwin, 1997) found conflict to be more effective than direct instruction. 'Provoking cognitive conflict to help students understand areas of mathematics is often recommended' (Irwin, 1997). Swedosh and Clark (1997) used this method to help undergraduate students to extinguish their misconceptions. Another study by (Horiguchi & Hirashima, 2001) used a similar approach in providing discovery learning environment in their mechanics classes to learn from mistakes. Mason and Watson (2001) used a method of so-called boundary examples, which suggested creating by students' examples to correct statements and questions that satisfied their conditions. (Mason and Watson, 2001). In this method, the students were actively involved in creative discovery learning that stimulated development of their advanced statistical thinking.

Methodology

Counterexamples and Paradox in the Classroom

In this paper we apply counter-examples along with Paradoxes as a pedagogical strategy in engineering probability course.

The Bachelor students in some engineering majors were given premises in statistics and probability and they were asked to disprove the premises with counterexamples. This type of activity was completely new and very challenging for most of them. At first, some of the students did not understand the difference between proving and disproving a proposition using a counterexample. In fact, most students fail to find just a counterexample to disprove their conjecture. It can be useful to first use non-mathematical examples to explain the idea of disproving a proposition with a counterexample.

As an assignment, students can be asked to use a counterexample to disprove the following false propositions:

- Pairwise independence of events would result in their independence.
- If events A and B are conditionally independent, then they are independent.
- Uncorrelated random variables are independent. A. Consider discrete random variables. B. Consider continuous random variables.
- Pairwise independence of random variables requires their mutual independence.

On the other hand, the correct answer to some problems contradicts our intuitive understanding. To clear such doubts, giving some examples as paradoxes will be of great help to students to understand such problems. In the following, some famous paradoxes are given.

- Galton's paradox. We flip three fair coins. Suppose that at least two are alike, and it is an even chance that the third is a head or a tail. Therefore, the probability that all three are the same is 0.5. Do you agree?
- Simpson's paradox. A clinical trial was conducted by a physician to determine the relative effect of two drugs, the results of which are as follows:

The success rate of drug 1: 0.108 and the success rate of drug 2: 0.459. So, the success rate is higher for drug 2.

Among women, the success rate is:

For drug 1: $\frac{200}{2000} \approx 0{\cdot}1$

For drug 2: $\frac{10}{200} \approx 0{\cdot}05$

Among men, the success rate is:

For drug 1: $$\frac{19}{20} = 0{\cdot}95$$

For drug 2: $$\frac{1000}{2000} = 0{\cdot}5$$

Therefore, when the ratios are calculated separately for women and men, the success rate is higher for drug 1. Which drug is better?

- ❖ Prisoner's dilemma. Consider three prisoners A, B and C. The prison guard tells them that two of them will be released and one will be executed. But he is not allowed to tell them who is supposed to be executed. Prisoner A asks the guard to tell him whom, between B and C, will be released and the guard says: "B will be released". Suppose that the guard tells the truth to prisoner A.

A. What is the probability that A and C will be executed, respectively?
B. Can A change his fate with C?

Despite what is evident, the conditional probabilities of being executed for A and C are different, that is, 0.33 and 0.67, respectively.

- ❖ St. Petersburg paradox. In a game of chance, the player pays a fixed amount initially and then flips a fair coin repeatedly until a tail appears. If the first tail appears after n^{th} toss, then the player wins \$2.

A. What is the player's expected win?
B. How much is the fair entrance fee?

The answer to both questions is infinite. It seems, contrary to common sense, any high value is worth paying to enter this game.

Research Method

A questionnaire[6] was designed to investigate the role of using counterexamples and paradoxes in learning and teaching statistics and probability. The result of a sample of this survey is as follows:

Most of the students found the counterexamples useful for learning and understanding statistics and probability. In the following, some examples of the explanations of these students are given:

- ❖ Counterexamples are fun and informative.
- ❖ Counterexamples are useful, because they help us to analyze our findings more efficiently in homework problems.
- ❖ Counterexamples and paradoxes help us to understand better statistics and probability.
- ❖ They improve our understanding of probability.
- ❖ They develop our logical and reasoning skills.
- ❖ They strengthen our ability to think.

About 64% of students found counterexamples effective and 36% considered it ineffective. One of those who considered them ineffective has given the following explanations:

- ❖ Counterexamples are problematic.
- ❖ Sometimes they are confusing.
- ❖ We should practice them more.

CONCLUSIONS

The statistical results of this study show that 91% of the students considered the use of counterexamples and paradoxes as an effective teaching strategy in the course of probability. Many of the students made positive comments that using counterexamples helped them to eliminate misconceptions and prevent making mistakes in future.

This study was conducted only on a limited number of students of a particular class, many of them were the second- and third-year students, so they had a better mathematical background than the first-year students, which makes the study a little biased. The results of this study can attract the attention of teachers to the proposed strategy. Conducting this study on their students, the teachers can see the results on them. In the first step, to introduce counterexamples, it is suggested that a counterexample and paradox be provided, and students be asked to explain or justify it. In the next step, the students are asked to come up with their own counterexample for the false statement. Finally, for a given mathematical statement, the students should decide whether it is true or show with a counterexample that the mathematical statement is false.

The Results indicate that counterexamples and paradoxes in Statistics and probabilities encourage students and teachers to examine them for these purposes:

- To provide deeper conceptual understanding
- To reduce or eliminate common misconceptions
- To advance probabilities and mathematical thinking beyond algorithmic or procedural reasoning
- To enhance baseline critical thinking skills—analyzing, justifying, verifying, and checking
- To expand the example set of noteworthy statistical ideas
- To stimulate students in more active and creative learning
- To encourage students in doing further investigation of mathematical and statistical topics

Further Study

This study can be expanded to investigate the effect of counterexamples on the strategy of teaching by asking students through preparing understandable questions. To this end, two groups of students with similar background are compared with each other. Counterexamples and paradoxes are widely used in the first group, and the other group is considered as a control group. Then, using statistical methods, we can show if there is a significant difference between these two groups or not.

Appendix

A Survey questionnaire is prepared on students' perception on the use of counterexamples and paradoxes for learning statistics and probability. This survey is taken for 55 undergraduate students from Amirkabir University (Poly-Technique) in Tehran-Iran in 2018. Students are on different years of their educational program consists of first, second, third, fourth or more year.

1. To learn statistical problems, have you ever used counterexamples?
A) Yes B) No
2. To solve statistical problems, are you familiar with using counterexamples?
A) Yes B) No
3. How much the use of counterexamples can help you in solving statistical problems?
Very low low moderate high very high
4. In your opinion, can counterexamples be used to solve statistical problems?
A) Yes B) No
5. Do counterexamples and paradoxes help you understand and learn statistics and probability?
A) Yes B) No
6. Do you want the use of counterexamples and paradoxes to be considered as a part of your course evaluation?
A) Yes B) No
7. In your opinion, what kind of perspective in understanding and solving statistical problems can be developed for students by using counterexamples and paradoxes?
8. Do the use of counterexamples and paradoxes make problems be more understandable?
A) Yes B) No

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