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Distribution of a Global CO_2 Budget – A Mathematical Description of Resource Sharing Models

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1 Introduction

Resource sharing models directly address the allocation of a remaining global CO_2 budget (cf. Sargl, et al., 2022). This paper shines the spotlight on the mathematical formulae of resource sharing models. It contributes to greater transparency and comparability through a uniform mathematical representation, by showing generalisations as well as relations, similarities and differences between these models. It also contains mathematical proofs for specified properties of the models.

For the models in Chapter 2 a global pathway that meets a remaining global budget is a prerequisite. Chapter 2.1 considers convergence models that allocate global emissions to countries, with increasing weight given to population. At the end of a convergence period the global emissions are allocated to countries according to population only. The Emission Probability Model in Chapter 2.2 determines country specific emission density functions and caps the emissions of individuals in order to limit the sum of national emissions to global emissions. Chapter 2.3 shows properties of resulting national budgets.

The approaches in Chapter 3 are based on the Extended Smooth Pathway Model (ESPM), in which national budgets are derived from a global budget in a first step and plausible emission paths that adhere to this national budget are determined in a second step. Different mathematical solutions are offered for the second step.

We offer Excel tools and web apps to calculate national emission pathways for all countries in the world that are compatible with the Paris Agreement, using the Regensburg Formula (see Chapter 2.1.2) or the Extended Smooth Pathway Model (see Chapter 3). The Excel tools can be downloaded from our homepage http://www.save-the-climate.info. An overview of the web apps and Excel tools is given here: https://www.climate-calculator.info.

2 Models based on the allocation of a global emissions pathway[1]

2.1 Convergence models

All convergence models presented here start with a global pathway that meets a remaining global budget until the last year of consideration *T* usually corresponding to a certain degree of global warming.[2] Then the models break down the annual global emissions on country level, transforming the actual emissions in a base year (*BY*) into emissions based on a per capita allocation in a convergence year (*CY*) at the end of a limited convergence period. This per capita allocation is also used after the convergence year.

2.1.1 Models breaking down the global pathway in a simple way

Contraction & Convergence Model

The Global Commons Institute already propounded the following Contraction & Convergence Model (C&C Model) in the early 1990s. This model defines the emissions of the country *i* in the year *t* $\left(\widehat{E_t^i}\right)$ recursively (cf. Meyer, No date):

$$\widehat{E_t^i} := \begin{cases} \left(\left(1-\widehat{C_t}\right) * \frac{\widehat{E_{t-1}^i}}{E_{t-1}} + \widehat{C_t} * \frac{P_t^i}{P_t}\right) * E_t, \text{for } BY+1 \leq t < CY \\ \frac{P_t^i}{P_t} * E_t, \text{ for } CY \leq t \leq T, i.\ e.\ \widehat{C_t} = 1 \text{ for } CY \leq t \leq T \end{cases} \quad (1)$$

where

E_t global emissions in the year *t*,

P_t global population in the year *t* and

P_t^i population of the country *i* in the year *t*.

$\widehat{C_t}$ denotes the weight of the population when allocating global emissions to countries.

The Global Commons Institute considered two specifications of $\widehat{C_t}$:

- exponential (C&C-exp): $\widehat{\mathrm{C_t}} = \exp\left(-\mathrm{a}\left(1-\frac{\mathrm{t-BY}}{\mathrm{CY-BY}}\right)\right)$ with the parameter $\mathrm{a} > 0$ to be

[1] See also our paper comparing the results of these models: (Sargl, et al., 2022).

[2] The approaches in Chapter 3 for deriving national emission pathways that adhere to a specified budget can also be used to derive global pathways. In our Regensburg Model [cf. (Sargl, et al., 2017) and (Sargl, et al., 2023c)], which uses the Regensburg Formula to calculate national pathways (see Chapter 2.1.2), we use the RM Scenario Types [cf. (Wolfsteiner & Wittmann, 2023a) and Chapter 3.4] to determine global pathways.

determined. "*The higher the value [a], the more the convergence happens towards the end of the convergence period, and vice-versa. Choosing a = 4 gives an even balance.*" (Meyer, 1998, p. 21)

- linear (C&C-lin): $\widehat{C}_t = \frac{t - BY}{CY - BY}$.

LIMITS Model

LIMITS, a research project funded by the EU, uses the following formula for the emissions of the country *i* in the year *t* $(\widetilde{E_t^i})$ (cf. Tavoni, et al., 2013):

$$\widetilde{E}_t^i := \begin{cases} \left((1 - \widetilde{C}_t) * \frac{E_{BY}^i}{E_{BY}} + \widetilde{C}_t * \frac{P_t^i}{P_t} \right) * E_t \quad \text{for } BY + 1 \leq t < CY \\ \frac{P_t^i}{P_t} * E_t \text{ for } CY \leq t \leq T, \quad i.\ e.\ \widehat{C}_t = 1 \text{ for } CY \leq t \leq T \end{cases} \tag{2}$$

$\widetilde{C}_t$ denotes the weight of the population when allocating global emissions to countries. LIMITS considered only the linear specification of $\widetilde{C}_t$ ($\widetilde{C}_t = \frac{t - BY}{CY - BY}$).

The LIMITS Model (LIMITS) uses formula (2) to determine emissions pathways for different regions of the world.

Generalised C&C Model and Generalised LIMITS Model

C&C and LIMITS consider only certain specifications of C_t. However, any non-decreasing weighting function C_t that takes the value 1 in the convergence year (*CY*) can be used. Numerous such weighting functions are conceivable. Thus, we obtain the Generalised Contraction & Convergence Model (G-C&C) and the Generalised LIMITS Model (G-LIMITS).

Exemplary weighting functions

National emissions pathways with weighting functions that take the value 0 (or approximately 0) in the base year (*BY*) normally do not have a step after the base year. Therefore, we only list the most intuitive weighting functions with this property:

- linear (lin): $C_t = \frac{t - BY}{CY - BY}$ (C&C-lin and LIMITS)
- exponential (exp_a): $C_t = exp\left(-a\left(1 - \frac{t - BY}{CY - BY}\right)\right)$ with the parameter $a > 0$ to be determined (C&C-exp)
- convex quadratic (conv quadr): $C_t = \left(\frac{t - BY}{CY - BY}\right)^2$
- concave quadratic (conc quadr): $C_t = 1 - \left(1 - \frac{t - BY}{CY - BY}\right)^2$
- general quadratic: $C_t = a(t - BY)^2 + b(t - BY) + c$, where a, b and c are parameters to be determined in such a way that $C_{BY} = 0$, $C_{CY} = 1$ and with a third constraint, e. g. a given value for the year after the base year. The linear, the convex quadratic and the concave quadratic specifications of C_t are special cases of the general quadratic specification.
- cubic: $C_t = -2\left(\frac{t - BY}{CY - BY}\right)^3 + 3\left(\frac{t - BY}{CY - BY}\right)^2$
- convex polynomial (conv pol_n): $C_t = \left(\frac{t - BY}{CY - BY}\right)^n$, where *n* is a natural number
- concave polynomial (conc pol_n): $C_t = 1 - \left(1 - \frac{t - BY}{CY - BY}\right)^n$, where *n* is a natural number

The weighting functions above depend directly on the year (*t*). Another class of weighting functions is obtained by introducing the emissions in the year *t* (E_t). Thus, these weighting functions depend on the global emissions and only indirectly on the year. We only show the linear specification as an example: linear in E_t (lin_E_t): $C_t = \frac{E_{BY} - E_t}{E_{BY} - E_{CY}}$.

Figure 1 depicts the trajectories of some weighting functions.

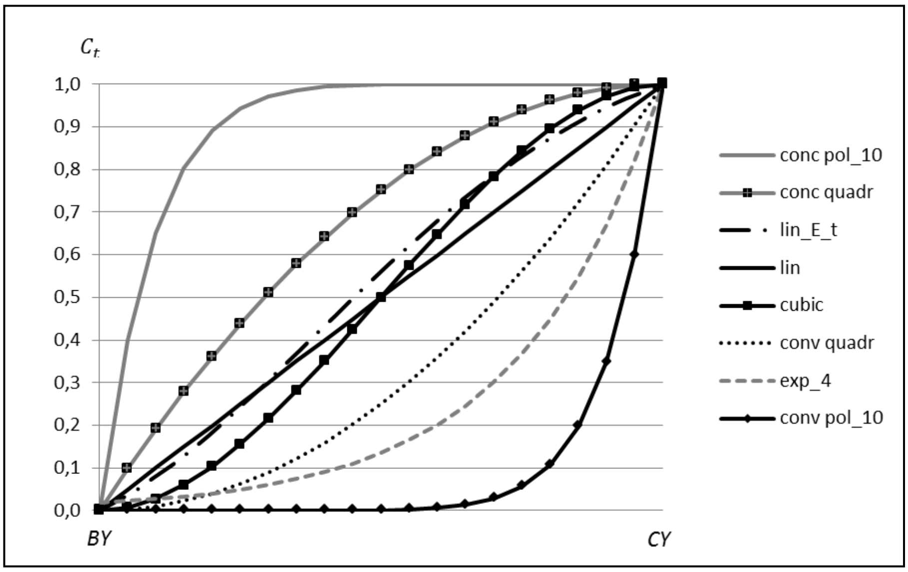

Figure 1: Trajectories of the different specifications of C_t

Figure 1 shows that, if *n* is great enough, the allocation key "population"

- in the concave polynomial specification comes fully into effect already in the first year after the base year (equity, immediate climate justice).
- in the convex polynomial specification comes into effect only in the convergence year (inertia).

Common but Differentiated Convergence Model

The Common but Differentiated Convergence Model (CDC Model) is described in (cf. Höhne, et al., 2006). This source does not contain any formulae, so the formulae presented here are our interpretation of the description of the CDC Model.

First a threshold TH_t in the year *t* is defined, which decreases if the global emissions decrease:

$$TH_t := \frac{E_t}{P_t} * PT,$$

where PT is a given percentage, e. g. 0.95. If the average emissions of the country *i* in the year *t* in a business as usual scenario $\left(\frac{E_t^{i\_bau}}{P_t^i}\right)$ are below or equal to the threshold, i. e. $\frac{E_t^{i\_bau}}{P_t^i} \leq TH_t$, the country is allocated emissions according to the business as usual scenario and we define

$$E_t^i := E_t^{i\_bau}.$$

Otherwise, if the average emissions of the country *i* in the year *t* in the business as usual scenario are above the threshold ($\frac{E_t^{i\_bau}}{P_t^i} > TH_t$), the country is allocated emissions according to the C&C formula and we define

$$E_t^i \coloneqq \left((1-\widehat{C}_t) * \frac{E_{t-1}^i}{E_{t-1}^{oTH\_t}} + \widehat{C}_t * \frac{P_t^i}{P_t^{oTH}}\right) * E_t^{oTH},$$

where

$\widehat{C}_t$ weighting of per capita emissions in the year *t*,

E_t^{oTH} remaining emissions in the year *t* for the countries over the threshold in the year *t*, i. e.

$$E_t^{oTH} = E_t - \sum_{\substack{i \\ \text{if } \frac{E_t^{i\_bau}}{P_t^i} \le TH_t}} E_t^i ,$$

$E_{t-1}^{oTH\_t}$ emissions in the year *t*-1 of the countries over the threshold in the year *t*, i. e.

$$E_{t-1}^{oTH\_t} = \sum_{\substack{i \\ \text{if } \frac{E_t^{i\_bau}}{P_t^i} > TH_t}} E_{t-1}^i \text{ and}$$

P_t^{oTH} population in the year *t* of the countries over the threshold in the year *t*, i. e.

$$P_t^{oTH} = \sum_{\substack{i \\ \text{if } \frac{E_t^{i\_bau}}{P_t^i} > TH_t}} P_t^i.$$

Remark: Obviously the equation

$$E_t^{oTH} = \sum_{\substack{i \\ \text{if } \frac{E_t^{i\_bau}}{P_t^i} > TH_t}} E_t^i$$

holds, but this equation cannot be used to define E_t^{oTH}, because E_t^i is defined with the help of E_t^{oTH}.

2.1.2 The Regensburg Formula (RF)

We will present three equivalent notations of the Regensburg Formula[3]

- as a weighting function with an annual degree of achieving the global convergence amount
- as a straight line with a conversion factor for the reduction of emissions
- as a recursion with an annual rate of change

and show how they are derived from each other.

The RF as a weighting function

The notation of the **RF as a weighting function** (cf. Sargl, et al., 2017) uses the annual degree of achieving the global convergence amount E_{CY} in year t

$$\overline{\boldsymbol{C_t}} := \frac{\boldsymbol{E_{BY} - E_t}}{\boldsymbol{E_{BY} - E_{CY}}}$$

as weighting factor for the national convergence amount E_{CY}^i (in case of the national convergence amount being directly proportional to the population, it is also a per-capita weighting factor) for the calculation of emissions of the country i in year t:

$$\boxed{\overline{\boldsymbol{E_t^i}} := (\boldsymbol{1} - \overline{\boldsymbol{C_t}}) * \boldsymbol{E_{BY}^i} + \overline{\boldsymbol{C_t}} * \boldsymbol{E_{CY}^i}, \; BY + 1 \leq t \leq CY}$$

Directly from this definition of the RF we obtain the following results:

[3] The Regensburg Formula is part of the Regensburg Model [cf. (Sargl, et al., 2017) and (Sargl, et al., 2023c)], in which global pathways are derived using the RM Scenario Types [cf. (Wolfsteiner & Wittmann, 2023a) and Chapter 3.4].

Remark 1 (equal proportions in all countries and the world)

In each year *t*, the proportion of emissions still to be reduced and the proportion of emissions already reduced in relation to the emissions to be reduced altogether are equal in all countries and globally:

$$\frac{E_t - E_{CY}}{E_{BY} - E_{CY}} = \frac{\overline{E_t^i} - E_{CY}^i}{E_{BY}^i - E_{CY}^i} \; (= 1 - \overline{C_t}) \text{ and}$$

$$\frac{E_{BY} - E_t}{E_{BY} - E_{CY}} = \frac{E_{BY}^i - \overline{E_t^i}}{E_{BY}^i - E_{CY}^i} \; (= \overline{C_t}).$$

In each year *t*, therefore, the degree of achieving the global convergence amount and the degree of achieving the national convergence amount are identical.

Remark 2 (national convergence amounts in all countries in CY)

In *CY* emissions calculated with the RF and the national convergence amount are the same in each country.

Remark 3 (Uniqueness of $\overline{C_t}$)

There is only one weighting function $\boldsymbol{\overline{C_t}}$ so that the equation

$$\boldsymbol{E_t^i = (1 - \overline{C_t}) * E_{BY}^i + \overline{C_t} * E_{CY}^i}$$

holds for each country. This weighting function is $\boldsymbol{\overline{C_t}} := \frac{\boldsymbol{E_{BY} - E_t}}{\boldsymbol{E_{BY} - E_{CY}}}$. This can be shown by summing up the equation across all countries, yielding an equation that can be solved for $\boldsymbol{\overline{C_t}}$.

The RF as a straight line

Theorem 1 (notation of the RF as a straight line)

The emissions of each country *i* as a function of the global emissions are on a straight line:

$$\boxed{\boldsymbol{\overline{E_t^i} = (E_t - E_{CY}) * a^i + E_{CY}^i}, \; BY + 1 \leq t \leq CY,}$$

with the conversion factor for the reduction: $\boldsymbol{a^i} := \dfrac{\boldsymbol{E_{BY}^i - E_{CY}^i}}{\boldsymbol{E_{BY} - E_{CY}}}$.

Proof:

$$E_t^i =$$

$$= E_{BY}^i * (1 - \overline{C}_t) + \overline{C}_t * E_{CY}^i =$$

$$= E_{BY}^i * \left(1 - \frac{E_{BY} - E_t}{E_{BY} - E_{CY}}\right) + \left(\frac{E_{BY} - E_t}{E_{BY} - E_{CY}}\right) * E_{CY}^i =$$

$$= E_{BY}^i * \left(\frac{E_t - E_{CY}}{E_{BY} - E_{CY}}\right) + \left(1 - \frac{E_t - E_{CY}}{E_{BY} - E_{CY}}\right) * E_{CY}^i =$$

$$= (E_t - E_{CY}) * \frac{E_{BY}^i - E_{CY}^i}{E_{BY} - E_{CY}} + E_{CY}^i =$$

$$= (E_t - E_{CY}) * a^i + E_{CY}^i$$

□

Remark 4 (stepwise approximation)

By presenting the RF as a straight line, it becomes clear that a stepwise approximation of the global emission pathway to the global convergence amount is transmitted to all national emission pathways.

Remark 5 (construction of national graphs)

This theorem also shows that, when applying the RF, the national graph (t, $\overline{E_t^i}$) for country *i* with a reduction amount ($E_{BY}^i > E_{CY}^i$) can be derived from the global graph (t, E_t) by changing the scaling on the ordinate and by vertically shifting the abscissa. For countries with a national convergence amount permitting increasing annual emissions ($E_{BY}^i < E_{CY}^i$), the global graph additionally needs to be reflected across the abscissa to obtain the national graph.

Remark 6 (factor for converting reductions = proportional factor)

Because of $\sum_i a^i = 1$ the factor for converting the reduction is also called “proportional factor”.

Corollary 1 (constant factor for converting reductions)

For each country *i* there is a constant proportional factor α^i that allows converting annual global reductions to annual reductions of the country *i*:

$$\overline{E_t^i} - \overline{E_{t-1}^i} = (E_t - E_{t-1}) * a^i.$$

Factor a^i for converting reductions can be determined by the ratio between emissions that remain to be reduced by country *i* in year *t* and emissions which remain to be reduced globally:

$$a^i = \frac{\overline{E_t^i} - E_{CY}^i}{E_t - E_{CY}} \quad (BY \leq t \leq CY - 1).$$

Remark 7 (monotonicity)

This corollary also shows that monotonicity of the global emission pathway is transferred to the national emission pathways.

Corollary 2 (complete distribution of global emissions)

The emissions determined according to the RF of all countries together sum up to the amount of global emissions:

$$\textstyle\sum_i \overline{E_t^i} = E_t \text{ for every year } t$$

Proof by the notation of the RF as a straight line:

$$\sum_i \overline{E_t^i} =$$

$$= \sum_i \left((E_t - E_{CY}) * a^i + E_{CY}^i \right) =$$

$$= (E_t - E_{CY}) * \sum_i a^i + \sum_i E_{CY}^i =$$

$$= (E_t - E_{CY}) * 1 + E_{CY} = E_t$$

□

The RF as a recursion

Theorem 2 (notation of the RF as a recursion)

We have:[4]

$$\boxed{\boldsymbol{\overline{E_t^i} = \overline{E_{t-1}^i} - CR_{t-1} * (\overline{E_{t-1}^i} - E_{CY}^i)}, \qquad BY + 1 \ \leq t \leq CY}$$

with the annual rate of change $\boldsymbol{CR_{t-1}} := \frac{\boldsymbol{E_{t-1} - E_t}}{\boldsymbol{E_{t-1} - E_{CY}}}$.

Proof:

CR_{t-1} is well defined, because $E_{t-1} \neq E_{CY}$ for $BY + 1 \ \leq t \leq CY$.

By using corollary 1 for the factor for converting reductions, we can say:

$$\overline{E_t^i} == \ \overline{E_{t-1}^i} + (E_t - E_{t-1}) \ * \ a^i =$$

$$= \overline{E_{t-1}^i} - \frac{E_{t-1} - E_t}{E_{t-1} - E_{CY}} * \left(\overline{E_{t-1}^i} - E_{CY}^i\right) =$$

$$\overline{= E_{t-1}^i} - CR_{t-1} * (\overline{E_{t-1}^i} - E_{CY}^i)$$

□

Remark 8 (identical annual rates of change)

The notation as a recursion offers another interpretation of the RF: The annual emissions of the country *i* in the year *t* are determined by transferring the rates of change which are derived from the global emission pathway to national emission pathways. Therefore, in each year t, the national and global annual rates of change are identical.

Remark 9 (national convergence amounts in all countries in the convergence year)

From the notation of the RF as a recursion, you can see that the convergence amounts are achieved in all countries in the year CY, if you take into consideration that the rate of change CR_{CY-1} takes value 1.

---

[4] Alternative notation with TA $:= E_{CY}, TA^i := E_{CY}^i$ and $\widetilde{CR}_{t-1} := \ - CR_{t-t} = \frac{E_t - E_{t-1}}{E_{t-1} - TA}$: $\overline{E_t^i} = \overline{E_{t-1}^i} + \widetilde{CR}_{t-1} * (\overline{E_{t-1}^i} - TA^i)$.

2.1.3 Convertibility of the convergence models

Equivalence of the Generalised C&C and LIMITS Model

If the population is frozen in both models, the convergence amount of a country *i* is defined by

$$E_{CY}^{i} = \frac{P^{i}}{P} * E_{CY}.$$

G-C&C is given by

$$\widehat{E_t^i} := \left((1 - \widehat{C}_t) * \frac{\widehat{E_{t-1}^i}}{E_{t-1}} + \widehat{C}_t * \frac{P^i}{P} \right) * E_t, \text{for } BY + 1 \leq t \leq CY$$

with a weighting function $\widehat{C}_t$ that takes the value 0 (or approximately 0) in *BY* and the value 1 in the *CY*. Here $\widehat{E_t^i}$ is defined recursively.

The G-Limits is given by

$$\widetilde{E_t^i} := \left(\left(1 - \widetilde{C}_t\right) * \frac{E_{BY}^i}{E_{BY}} + \widetilde{C}_t * \frac{P^i}{P} \right) * E_t, \text{for } BY + 1 \leq t \leq CY$$

with a weighting function $\widetilde{C}_t$ that takes the value 0 (or approximately 0) in *BY* and the value 1 in *CY*.

Theorem 3 (equivalence of G-C&C and G-LIMITS)

For any weighting function $\widehat{C}_t$ of G-C&C there is a weighting function $\widetilde{C}_t$ for G-LIMITS, so that the results of G-C&C and G-LIMITS are the same.
For any weighting function $\widetilde{C}_t$ of G-LIMITs there is a weighting function $\widehat{C}_t$ for G-C&C, so that the results of G-C&C and G-LIMITS are the same.

Proof:

If we know the weighting function $\widehat{C}_t$ of G-C&C, the weighting function $\widetilde{C}_t$ of G-LIMITS is given by

$$\widetilde{C}_t := 1 - \prod_{l=BY+1}^{t} \left(1 - \widehat{C}_l\right) \text{for } BY + 1 \leq t \leq CY .$$

We proof the first part of the theorem by aid of mathematical induction.

Base case: For *t* = *BY* + 1 we obtain $\widetilde{C_{BY+1}} = \widehat{C_{BY+1}}$ and

$$\widetilde{E^{i}_{BY+1}} := \left(\left(1-\widetilde{C_{BY+1}}\right) * \frac{E^{i}_{BY}}{E_{BY}} + \widetilde{C_{BY+1}} * \frac{P^{i}}{P}\right) * E_{BY+1}$$

$$= \left((1-\widehat{C_{BY+1}}) * \frac{E^{i}_{BY}}{E_{BY}} + \widehat{C_{BY+1}} * \frac{P^{i}}{P}\right) * E_{BY+1} = \widehat{E^{i}_{BY+1}}$$

Inductive step: Assuming that if $\widehat{E^{i}_{t-1}} = \widetilde{E^{i}_{t-1}} = \left(\left(1-\widetilde{C_{t-1}}\right) * \frac{E^{i}_{BY}}{E_{BY}} + \widetilde{C_{t-1}} * \frac{P^{i}}{P}\right) * E_{t-1}$, we show that $\widehat{E^{i}_{t}} = \widetilde{E^{i}_{t}}$. Algebraically

$$\widehat{E^{i}_{t}} = \left((1-\widehat{C}_{t}) * \frac{\widehat{E^{i}_{t-1}}}{E_{t-1}} + \widehat{C}_{t} * \frac{P^{i}}{P}\right) * E_{t},$$

$$= \left((1-\widehat{C}_{t}) * \frac{\left(\left(1-\widetilde{C_{t-1}}\right) * \frac{E^{i}_{BY}}{E_{BY}} + \widetilde{C_{t-1}} * \frac{P^{i}}{P}\right) * E_{t-1}}{E_{t-1}} + \widehat{C}_{t} * \frac{P^{i}}{P}\right) * E_{t}$$

$$= \left((1-\widehat{C}_{t}) * \left(\left(1-\widetilde{C_{t-1}}\right) * \frac{E^{i}_{BY}}{E_{BY}} + \widetilde{C_{t-1}} * \frac{P^{i}}{P}\right) + \widehat{C}_{t} * \frac{P^{i}}{P}\right) * E_{t}$$

$$= \left((1-\widehat{C}_{t}) * \left(\left(1-\left(1-\prod_{l=BY+1}^{t-1}\left(1-\widehat{C}_{l}\right)\right)\right) * \frac{E^{i}_{BY}}{E_{BY}} + \widetilde{C_{t-1}} * \frac{P^{i}}{P}\right) + \widehat{C}_{t} * \frac{P^{i}}{P}\right) * E_{t}$$

$$= \left((1-\widehat{C}_{t}) * \left(\left(\prod_{l=BY+1}^{t-1}\left(1-\widehat{C}_{l}\right)\right) * \frac{E^{i}_{BY}}{E_{BY}} + \widetilde{C_{t-1}} * \frac{P^{i}}{P}\right) + \widehat{C}_{t} * \frac{P^{i}}{P}\right) * E_{t}$$

$$= \left(\prod_{l=BY+1}^{t}\left(1-\widehat{C}_{l}\right) * \frac{E^{i}_{BY}}{E_{BY}} + (1-\widehat{C}_{t}) * \left(1-\prod_{l=BY+1}^{t-1}\left(1-\widehat{C}_{l}\right)\right) * \frac{P^{i}}{P} + \widehat{C}_{t} * \frac{P^{i}}{P}\right) * E_{t}$$

$$= \left(\left(1-\widetilde{C}_{t}\right) * \frac{E^{i}_{BY}}{E_{BY}} + \left((1-\widehat{C}_{t}) - \left(1-\widetilde{C}_{t}\right)\right) * \frac{P^{i}}{P} + \widehat{C}_{t} * \frac{P^{i}}{P}\right) * E_{t}$$

$$= \left(\left(1-\widetilde{C}_{t}\right) * \frac{E^{i}_{BY}}{E_{BY}} + \widetilde{C}_{t} * \frac{P^{i}}{P}\right) * E_{t} = \widetilde{E^{i}_{t}}$$

Second part of the theorem: If we know the weighting function $\widetilde{C}_t$ of G-LIMITS, we solve the definition of $\widetilde{C}_t$ for $\widehat{C}_t$ and obtain recursively the weighting function $\widehat{C}_t$ of G-C&C:

$$\widehat{C}_t = 1 - \frac{1 - \widetilde{C}_t}{\prod_{l=BY+1}^{t-1}\left(1 - \widehat{C}_l\right)} \text{ for } BY + 1 \leq t \leq CY$$

$\widehat{C}_t$ is well defined because *CY* is by definition the year when the convergence amount is reached.

□

RF as a special case of the Generalised C&C and LIMITS Model

Theorem 4 (The RF as a special case of G-LIMITS)

With the weighting function

$$\widetilde{C}_t = \frac{\frac{\overline{E_t^i}}{E_t} - \frac{E_{BY}^i}{E_{BY}}}{\frac{E_{CY}^i}{E_{CY}} - \frac{E_{BY}^i}{E_{BY}}}$$

the results of G-LIMITS and the RF are the same.

Proof:

The weighting function $\widetilde{C}_t$ is obtained by transforming G-LIMITS for country *i* using $\frac{P^i}{P} = \frac{E_{CY}^i}{E_{CY}}$ and assuming that $\widetilde{E_t^i} = \overline{E_t^i}$. Thus, we must proof that we obtain the same weighting function $\widetilde{C}_t$ for any other country *j:*

$$\frac{\frac{E_t^i}{E_t} - \frac{\overline{E_{BY}^i}}{E_{BY}}}{\frac{E_{CY}^i}{E_{CY}} - \frac{E_{BY}^i}{E_{BY}}} = \frac{\frac{E_t^j}{E_t} - \frac{\overline{E_{BY}^j}}{E_{BY}}}{\frac{E_{CY}^j}{E_{CY}} - \frac{E_{BY}^j}{E_{BY}}}$$

$$\frac{\overline{E_t^i} * E_{BY} - E_{BY}^i * E_t}{E_t * E_{BY}} * \frac{E_{CY} * E_{BY}}{E_{CY}^i * E_{BY} - E_{BY}^i * E_{CY}} = \frac{\overline{E_t^j} * E_{BY} - E_{BY}^j * E_t}{E_t * E_{BY}} * \frac{E_{CY} * E_{BY}}{E_{CY}^j * E_{BY} - E_{BY}^j * E_{CY}}$$

$$0 = \left(\overline{E_t^i} * E_{BY} - E_{BY}^i * E_t\right) * \left(E_{CY}^j * E_{BY} - E_{BY}^j * E_{CY}\right)$$
$$-\left(\overline{E_t^j} * E_{BY} - E_{BY}^j * E_t\right) * \left(E_{CY}^i * E_{BY} - E_{BY}^i * E_{CY}\right).$$

Since $\overline{E_t^i}$ and $\overline{E_t^j}$ can be seen as a function of E_t whose images are on a straight line (theorem 1), the right side of this equation can be seen as a function of E_t whose image is on a straight line. Therefore, it is sufficient to proof that two points of the image are 0. These two points are obviously E_{BY} and E_{CY}.

□

Remark 10 (The RF as a special case of G-C&C)

Since the results of G-LIMITS can be obtained with G-C&C using an appropriate weighting function (theorem 3), the RF is also a special case of G-C&C.

2.2 Emission Probability Model (EPM)

Chakravarty et al. (cf. Chakravarty, et al., 2009) described three steps to obtaining and cutting an emission probability density function (PDF) starting with the points of a Lorenz curve. We hence summarize how to obtain a Lorenz Curve from a PDF in Chapter 2.2.1, show the results for a gamma PDF in Chapter 2.2.2 and describe the Emission Probability Model (EPM) in Chapter 2.2.3.

2.2.1 General case: The Lorenz Curve obtained from a PDF

Let *f* be an income PDF.

Then

- the cumulative population share *x* is given by the cumulative distribution function (CDF) *F*, i. e. the probability of an income equal to z or less is $x = F(z) \; = \int_{-\infty}^{z} f(t)\, dt$
- the cumulative income share *y* is given by $y = \frac{\int_{-\infty}^{z} t\, f(t)\, dt}{\int_{-\infty}^{\infty} t\, f(t)\, dt}$

$\int_{-\infty}^{z} t\, f(t)\, dt$: average income of the persons with an income equal to *z* or less

$\int_{-\infty}^{\infty} t\, f(t)\, dt$: average income of the population

Thus a parametric representation of the Lorenz curve $\overline{L}$ is given by

$$\overline{L}(z) = \begin{pmatrix} x \; = \; F(z) \\ y = \dfrac{\int_{-\infty}^{z} t\, f(t)\, dt}{\int_{-\infty}^{\infty} t\, f(t)\, dt} \end{pmatrix} \tag{3}$$

If the inverse function F^{-1} of the CDF *F* exists, the Lorenz curve *L* is directly given by

$$\mathrm{y = L(x)} = \frac{\int_{-\infty}^{F^{-1}(\mathrm{x})} \mathrm{t\, f(t)\, dt}}{\int_{-\infty}^{\infty} \mathrm{t\, f(t)\, dt}}. \tag{4}$$

Substituting $t = \; F^{-1}(\check{t})$ yields $\frac{dt}{d\check{t}} = \; (F^{-1})'(\check{t}) = \frac{1}{F\prime(F^{-1}(\check{t}))} = \frac{1}{f(F^{-1}(\check{t}))}$ and the Lorenz curve can be written as

$$\mathrm{y = L(x)} = \frac{\int_0^{\mathrm{x}} \mathrm{F}^{-1}\,(\check{\mathrm{t}})\mathrm{d}\check{\mathrm{t}}}{\int_0^{1} \mathrm{F}^{-1}\,(\check{\mathrm{t}})\mathrm{d}\check{\mathrm{t}}}. \tag{5}$$

Theorem 5 (Scaling)

The Lorenz curve is independent of the scaling of the z-axis.

Proof: With a scaling factor $s \neq 0$ the scaled PDF $\tilde{f}$ for a PDF f is given by

$$\tilde{f}(\tilde{z}) = s\, f(s\tilde{z}).$$

For the CDF $\tilde{F}$ we obtain

$$\tilde{F}(\tilde{z}) = \int_{-\infty}^{\tilde{z}} \tilde{f}(\tilde{t})\, d\tilde{t} = s \int_{-\infty}^{\tilde{z}} f(s\tilde{t})\, d\tilde{t} = \int_{-\infty}^{s\tilde{z}} f(t) dt = F(s\tilde{z}).$$

Thus $\tilde{F}^{-1}$, the inverse function of the CDF $\tilde{F}$, is given by

$$\tilde{F}^{-1} = \frac{1}{s}\, F^{-1}.$$

With the help of the representation (5). of the Lorenz curve we see that, the Lorenz curve from the PDF f and the PDF $\tilde{f}$ are the same.

2.2.2 *Special case: The Lorenz Curve obtained from a gamma probability distribution*

In general, the evaluation of the integrals in equation (1) or (2) can cause trouble. However, if Z is a gamma distributed random variable all this work can be done by a spreadsheet programme, such as EXCEL.

Let Z be a gamma distributed random variable. Then the PDF g is given by

$$g(z;a,b) = \begin{cases} 0 & \text{for z } < 0 \\ \dfrac{1}{b^a\,\Gamma(a)} z^{a-1} e^{-\frac{z}{b}} & \text{for } z \geq 0 \end{cases}$$

with parameters a, $b > 0$ and $\Gamma(a) = \int_0^\infty z^{a-1} e^{-z}\, dz$.

The CDF is denoted by

$$G(z;a,b) = \int_0^z g(t;\ a,b)\, dt = \int_0^z \frac{1}{b^a\,\Gamma(a)} t^{a-1} e^{-\frac{t}{b}}\, dt$$

Since $\Gamma(a+1) = a\ \Gamma(a)$, the equation $t\ g(t;a,b) = ab\ g(t;a+1,b)$ holds. Thus

- the expected value (or mean) of Z is given by

$$E[Z] = \int_0^\infty t\ g(t;a,b)\, dt = ab \int_0^\infty g(t;a+1,b)dt = ab$$

  and

- using the representation (4). the Lorenz curve is given by

$$L(x) = \frac{\int_0^{G^{-1}(x;a,b)} t\ g(t;a,b)\, dt}{\int_0^\infty t\ g(t,a,b)\, dt} = \frac{ab\ \int_0^{G^{-1}(x;a,b)} g(t;a+1,b)\, dt}{ab} = G(G^{-1}(x;a,b);a+1,b).$$

<u>Scaling</u>

With a scaling factor $s \neq 0$ we easily find

$$\tilde{g}(\tilde{z};a,b) = s\ g(s\tilde{z};a,b) = g(\tilde{z};a,\frac{b}{s})$$

This equation shows that the scaling of a gamma distribution with parameters a, b leads to another gamma distribution with parameters a, $\frac{b}{s}$. Since the Lorenz curve does not depend on scaling, the Lorenz curve must be independent of the parameter b.

2.2.3 Description of the EPM

In a base year let there be $\left(x_j^i, y_j^i\right)$ points of the Lorenz curve $\check{L}^i$ of the country i, i. e. $y_j^i = \check{L}^i\ (x_j^i)$.

In the first step, an income PDF $f^i(z; p^i)$ for each country i is determined. For this purpose, the parameters p^i are estimated by adapting the Lorenz curves $L^i(z; p^i)$ with a least square fit:

$\min_{p^i} \left\{\sum_j \left(L^i\left(x_j^i; p^i\right) - y_j^i\right)^2\right\}$.

In the second step, for each country *i* an emission PDF $\tilde{f}^i$ is obtained by scaling the income PDF f^i.

$$\tilde{f}^i\left(\tilde{z}; p^i\right) = s^i * f^i(s^i * \tilde{z}; p^i)$$

with the scaling factor $s^i \coloneqq \frac{\text{average emissions in country } i}{\text{average income in country } i}$ of the country i.

In the third step, in each year *t* a cap CA_t is determined in such a way that the emissions in all countries yield the underlying global emissions in the year *t* (E_t):

$$\sum_i E_t^i = \sum_i P_t^i \left(\int_{-\infty}^{CA_t} z\, \tilde{f}^i(z; p^i)\, dz + CA_t \int_{CA_t}^{\infty} \tilde{f}^i(z; p^i)\, dz\right) = E_t.$$

Usually, it is assumed that each person earns a non-negative income. That is why the scaling in the second step is possible. However, when global emissions are negative a different transformation, which converts an income PDF, which is zero for negative incomes, into an emission PDF that addresses negative emissions, must be found. Such transformations are conceivable, but they are not indisputable.

2.3 Properties of resulting national budgets

We refer to the emissions of the country *i* until the year *t* as the national budget of the country *i* until the year *t*

$$B_t^i := \sum_{l=BY+1}^{t} E_l^i.$$

and to the global emissions until the year *t* as the global budget until the year *t*:

$$B_t := \sum_{l=BY+1}^{t} E_t \left(= \sum_{l=BY+1}^{t} \sum_i E_l^i = \sum_i \sum_{l=BY+1}^{t} E_l^i = \sum_i B_t^i \right)$$

In Chapter 2.3.1 we show properties of national budgets with regard to the Regensburg Formula. After a definition of the comparative value implicit weighting of the population we show its properties in Chapter 2.3.2.

2.3.1 National budgets using the RF

If the RF is used the national budget until the convergence year can be easily calculated.

Theorem 6 (national budget until the convergence year using the RF)

Using the RF the national budget of the country *i* until the convergence year is given by:

$$B_{CY}^i = E_{CY}^i * (CY - BY) + (B - E_{CY} * (CY - BY)) * \ a^i,$$

$$\text{with the factor } a^i = \frac{E_{BY}^i - E_{CY}^i}{E_{BY} - E_{CY}} \text{ for converting reductions.}$$

Proof:

According to the notation of the RF as a straight line, the following applies to the emissions of the country *i* in year *t*:

$$\overline{E_t^i} = (E_t - E_{CY}) * a^i + E_{CY}^i.$$

By summing up these emissions across all years, we obtain the national budget of the country *i* in the convergence period:

$$B_{CY}^i = \sum_{t=BY+1}^{CY} \overline{E_t^i} =$$

$$= \sum_{t=BY+1}^{CY} E_{CY}^i + \sum_{t=BY+1}^{CY} (E_t - E_{CY}) * a^i$$

$$= E_{CY}^{i} * (CY - BY) + (B - E_{CY} * (CY - BY)) * \ a^{i}$$

□

Remark 11 (national budget depending only on the global budget)

This theorem also shows that the national budget of the country *i* until the convergence year only depends on – besides the national emissions of the country *i* and the global emissions in *BY* and in *CY* – the global budget until the convergence year, but not on the global emissions E_{BY+1}, E_{BY+2}, …, E_{CY-2}, E_{CY-1}. Under certain conditions, this results in an implicit national budget that is independent of the global pathway chosen.

2.3.2 *Implicit weighting of the population*

Each convergence model allocates a country *i* until the year *t* a national budget B_t^i that can – if the share of the emissions of country *i* in global emissions is different from the share of the frozen population of country *i* in the frozen global population $\left(\frac{E_{BY}^i}{E_{BY}} \neq \frac{P^i}{P}\right)$ – be considered as a weighting of the two extreme allocations "emissions in the past" and "frozen population":

$$B_t^i = \left(\left(1 - IC_t^i\right) * \frac{E_{BY}^i}{E_{BY}} + IC_t^i * \frac{P^i}{P}\right) * B_t \qquad (6)$$

where

B_t^i	$\left(= \sum_{l=BY+1}^{t} E_l^i\right)$ emissions of the country *i* until the year *t* (national budget of the country *i* until the year *t*),
B_t	$\left(= \sum_{l=BY+1}^{t} E_l\right)$ global emissions until the year *t* (global budget until the year *t*),
IC_t^i	weighting of the population of the country *i* in the year *t* (this parameter is defined implicitly),
E_{BY}	global emissions in the base year,
E_{BY}^i	emissions of the country *i* in the base year,
P	(frozen) global population and
P^i	(frozen) population of the country *i*.

Remark 12 (First explicit specification of the weighting of population)

Solving equation (6) for IC_t^i leads to

$$IC_t^i = \frac{B_t^i - B_t * \frac{E_{BY}^i}{E_{BY}}}{B_t * \left(\frac{P^i}{P} - \frac{E_{BY}^i}{E_{BY}}\right)} \qquad (7)$$

Remark 13 (Independence of the implicit weighting of population of the global budget)

The implicit weighting of population only depends on the convergence model, but not on the global budget as long as changes of the global budget are passed on to countries in proportion to the previous national budgets.

Theorem 7 (Identical weighting of the population in all countries)

If the population is frozen, then for any convergence model, the implicit weighting of the population is the same for each country: $IC_t^i = IC_t^j$ for any countries *i* and *j*. Convergence models thus show an implicit weighting of the population.[5]

Proof:

We proof this theorem for the G-LIMITS showing that the implicit weighting of the population does not depend on the country *i* by aid of mathematical induction. The rest follows from the equivalence of G-C&C and G-LIMITS (theorem 3) and the fact that the RF is a special case of G-LIMITS (theorem 4).

Base case: For $t = BY + 1$ the national budget of the country *i* until the year $BY + 1$ is E_{BY+1}^i and the global budget is E_{BY+1}. By comparing equation (2) with equation (6) we obtain $IC_{BY+1}^i = \widetilde{C_{BY+1}}$ and notice that IC_{BY+1}^i does not depend on *i*.

Inductive step: Assuming that if $IC_{t-1}^i = IC_{t-1}$ for each country *i* we show that $IC_t^i = IC_t$.

For the national budget of each country *i* until the year *t* we obtain

$$B_t^i = E_t^i + B_{t-1}^i = E_t^i + \left(\left(1 - IC_{t-1}^i\right) * \frac{E_{BY}^i}{E_{BY}} + IC_{t-1}^i * \frac{P^i}{P}\right) * B_{t-1} =$$

$$= \left(\left(1 - \widetilde{C}_t\right) * \frac{E_{BY}^i}{E_{BY}} + \widetilde{C}_t * \frac{P^i}{P}\right) * E_t + \left(\left(1 - IC_{t-1}\right) * \frac{E_{BY}^i}{E_{BY}} + IC_{t-1} * \frac{P^i}{P}\right) * B_{t-1} =$$

$$= \left(E_t + B_{t-1} - \widetilde{C}_t * E_t - IC_{t-1} * B_{t-1}\right) * \frac{E_{BY}^i}{E_{BY}} + \left(\widetilde{C}_t * E_t + IC_{t-1} * B_{t-1}\right) * \frac{P^i}{P}.$$

We define $IC_t := \frac{\widetilde{C}_t * E_t + IC_{t-1} * B_{t-1}}{B_t}$ and obtain

[5] In our Excel tool for the Regensburg Model, this implicit weighting can be calculated for different framework data (Wolfsteiner & Wittmann, 2023b).

$$B_t^i = (B_t - IC_t * B_t) * \frac{E_{BY}^i}{E_{BY}} + (IC_t * B_t) * \frac{P^i}{P} = \left((1 - IC_t) * \frac{E_{BY}^i}{E_{BY}} + IC_t * \frac{P^i}{P}\right) * B_t.$$

□

Remark 14 (Generalisation of theorem 7)

The proof of theorem 6 only uses the assumption that the emissions of the country *i* in the year *t* are given by

$$E_t^i = \left((1 - C_t) * \frac{E_{BY}^i}{E_{BY}} + C_t * \frac{P^i}{P}\right) * E_t \tag{8}$$

Therefore, if the population is frozen, then for any convergence model, the implicit weighting of the population resulting from a national budget at any year *t* is the same for each country: $IC_t^i = IC_t$. This result even holds for non-convergence models as long as they allocate emissions according to equation (8).[6]

Theorem 8 (Second explicit specification of the weighting of population)

Given a model that allocates emissions according to equation (8). Then the implicit weighting of the population in the year *t* is given by

$$IC_t = \frac{\sum_{l=BY+1}^{t} C_l * E_l}{B_t}$$

Proof:

Substituting the national budget of the country *i* until the year *t* $\left(B_t^i\right)$ in equation (7) yields:

$$IC_t^i = \frac{B_t^i - B_t * \frac{E_{BY}^i}{E_{BY}}}{B_t * \left(\frac{P^i}{P} - \frac{E_{BY}^i}{E_{BY}}\right)} =$$

[6] This implicit weighting of the population therefore applies in principle to the entire period of our Regensburg Model, even if the Regensburg Formula is not applied throughout. For details, please refer to the Excel tool: (Wolfsteiner & Wittmann, 2023b).

$$= \frac{\sum_{l=BY+1}^{t} E_l^i - B_t * \frac{E_{BY}^i}{E_{BY}}}{B_t * \left(\frac{P^i}{P} - \frac{E_{BY}^i}{E_{BY}}\right)} =$$

$$= \frac{\sum_{l=BY+1}^{t} \left((1 - C_l) * \frac{E_{BY}^i}{E_{BY}} + C_l * \frac{P^i}{P}\right) * E_l - B_t * \frac{E_{BY}^i}{E_{BY}}}{B_t * \left(\frac{P^i}{P} - \frac{E_{BY}^i}{E_{BY}}\right)} =$$

$$= \frac{\frac{E_{BY}^i}{E_{BY}} * B_t + \sum_{l=BY+1}^{t} C_l * E_l \left(\frac{P^i}{P} - \frac{E_{BY}^i}{E_{BY}}\right) * E_l - B_t * \frac{E_{BY}^i}{E_{BY}}}{B_t * \left(\frac{P^i}{P} - \frac{E_{BY}^i}{E_{BY}}\right)} =$$

$$= \frac{\sum_{l=BY+1}^{t} C_l * E_l}{B_t}$$

□

Remark 15 (Identical weighting of the population in all countries)

Theorem 8 also shows, that if the population is frozen, then for any convergence model, the implicit weighting of the population does not depend on a country.

Theorem 9 (More ambitious in the beginning means higher implicit weighting)

Let two global pathways be given, both of which adhere to the same global budget until year the *t* and intersect only once. Then, for any model for which equation (8) holds with monotonically increasing *C_t*, the more ambitious pathway after the base year has a higher implicit weighting of the population.

Proof:

First, we set $D_l \coloneqq \tilde{E}_l - E_l$ for $l = BY+1, BY+2, \ldots, t$. Then we consider the global pathway E_{BY+1}, $E_{BY+2}, \ldots, E_t$ with the implicit weight IC and a more ambitious global path $\tilde{E}_{BY+1}, \tilde{E}_{BY+2}, \ldots, \tilde{E}_t$ with the implicit weight $\widetilde{IC}$. According to the assumptions, the following applies

$$\tilde{E}_{BY+1} - E_{BY+1} = D_{BY+1} < 0 \text{ and}$$

$$\sum_{l=BY+1}^{t} \tilde{E}_l = \sum_{l=BY+1}^{t} E_l = B_t. \text{ Hence } \sum_{l=BY+1}^{t} D_l = 0$$

We must show that applies $\widetilde{IC} - IC > 0$.

Using theorem 8 the last inequation can be written as

$$\sum_{l=BY+1}^{t} \left(C_l * \tilde{E}_l - C_l * E_l\right) = \sum_{l=BY+1}^{t} C_l * \left(\tilde{E}_l - E_l\right) = \sum_{l=BY+1}^{t} C_l * D_l > 0.$$

We consider the numbers $D_{BY+1}, D_{BY+2}, \ldots, D_t$. Since the emission pathways intersect only once and since the number D_{BY+1} is negative, the first numbers D_{BY+1}, ..., D_N are negative and the last numbers D_{N+1}, ..., D_t are positive.

We now split or combine the positive numbers so that we obtain the positive numbers $-D_{BY+1}$, ..., $-D_N$ for the negative numbers D_{BY+1}, ..., D_N. When splitting a number D_l $\left(= D_{l_1} + D_{l_2}\right)$ into the numbers D_{l_1} and D_{l_2} we assign to the numbers D_{l_1} and D_{l_2} the weighting factor C_l, when combining the numbers D_{l_1} and D_{l_2} we assign to the sum $D_l = D_{l_1} + D_{l_2}$ the smaller weighting factor $C_l = \min\left\{C_{l_1}; C_{l_2}\right\}$. Now for every negative number there is a positive number with the opposite sign, but with a larger weighting factor. Thus, the sum $\sum_{l=BY+1}^{t} C_t * D_l$ is positive.

□

3 Emission pathways that adhere to a predefined budget (ESPM)

In this chapter we consider models consisting of two steps. In the first step, national budgets are derived from a global CO2 budget. In the second step, national emission pathways are determined that comply with this budget (cf. i.a. Sargl, et al., 2023b).

In chapter 3.1 we present Raupach's weighting formula for allocating a global remaining budget to countries. The national budgets can be transformed into a pathway in a simple (chapter 3.2), a smooth (chapter 3.3) or a smart way (chapter 3.4). Hence these models are also called Extended Smooth[7] Pathway Model (ESPM).

The approaches used for transforming a national budget can also be applied on the global budget.

3.1 Distribution of a global budget: Weighting model from Raupach

A variety of distribution keys are conceivable for deriving national budgets from a global budget.[8]

Raupach et al. propose a simple weighting formula for distributing a global remaining budget to countries (cf. Raupach, et al., 2014):

$$RB^i = \left(C * \frac{P_{BY}^i}{P_{BY}} + (1 - C) * \frac{E_{BY}^i}{E_{BY}}\right) * RB,$$

where

E_{BY} resp. E_{BY}^i global emissions resp. emissions of the country *i* in the base year

P_{BY} resp. P_{BY}^i global population resp. population of the country *i* in the base year

RB resp. RB^i global remaining budget resp. remaining budget of the country *i*

C weighting of the population

[7] In this model designation, "smooth" is used as a generic term for "simple", "mathematical smooth" and "smart".

[8] See corresponding excursus in: (Sargl, et al., 2023b).

3.2 Linear Pathway Models

3.2.1 Linear Pathway Model (LPM)

The Linear Pathway Models (LPM) derives a national or global emission pathway from a given national or global remaining budget in an obvious way. After a base year, the emission power $\dot{E}^i$, i. e. the derivative of emissions with respect to time (the emissions per unit of time) of a country *i* is supposed to decrease on a straight line until the point of time T^i when the emission power is zero. After that point, the emission power shall continue to be 0. This leads to the following approach for the emission power:

$$\dot{E}^i(z) := \begin{cases} a^i\, z + b^i \text{ for } BY \le z \le T^i \\ 0 \text{ for } T^i \le z \end{cases}, \tag{9}$$

with parameters a^i and b^i of the straight line to be determined.

Theorem 10 (calculation of the emissions with the help of the emission power function)

The emissions of the country *i* in the period from the point of time t_1 to the point of time $t_2 \le T^i$ is given by

$$\int_{t_1}^{t_2} \dot{E}^i(z)\, dz = (t_2 - t_1) * \dot{E}^i\left(\frac{t_1 + t_2}{2}\right) = (t_2 - t_1) * \frac{\dot{E}^i(t_1) + \dot{E}^i(t_2)}{2}$$

Proof:

For the emissions of the country *i* in the period from t_1 to $t_2 \le T^i$ we obtain

$$\int_{t_1}^{t_2} \dot{E}^i(z)\, dz = \int_{t_1}^{t_2} a^i\, z + b^i \;\, dz =$$

$$= \left[0.5 * a^i * z^2 + b^i * z\right]_{z=t_1}^{z=t_2} =$$

$$= 0.5 * a^i * ({t_2}^2 - {t_1}^2) + b^i * (t_2 - t_1) =$$

$$= (t_2 - t_1) * \left[0.5 * a^i * (t_2 + t_1) + b^i\right] =$$

$$= (t_2 - t_1) * \left[0.5 * a^i * t_1 + 0.5 * b^i + 0.5 * a^i * t_2 + 0.5 * b^i\right] =$$

$$(t_2 - t_1) * \dot{E}^i\left(\frac{t_1 + t_2}{2}\right) = (t_2 - t_1) * \frac{\dot{E}^i(t_1) + \dot{E}^i(t_2)}{2}.$$

□

Corollary 3 (national emissions)

The emission of the country *i* in the year *t* can be calculated with the help of the emission power function by

$$E_t^i = \int_t^{t+1} \dot{E}^i(z)\,dz = \begin{cases} \dot{E}^i(t+0.5) \text{ for } t \le \lfloor T^i \rfloor - 1 \\ 0.5 * \dot{E}^i(\lfloor T^i \rfloor) * (T^i - \lfloor T^i \rfloor) \text{ for } t = \lfloor T^i \rfloor \\ 0 \text{ for } \lfloor T^i \rfloor + 1 \le t \end{cases},$$

where $\lfloor\ \rfloor$ is the floor function for delimiting the definition areas of E_t^i. The floor function takes as an input a real number T^i and gives as an output $\lfloor T^i \rfloor$ the greatest integer less than or equal to T^i.

Proof:

For $t \le \lfloor T^i \rfloor - 1$ we obtain

$$E_t^i = \int_t^{t+1} \dot{E}^i(z)\,dz = \dot{E}^i(t+0.5).$$

For $t = \lfloor T^i \rfloor$ we get

$$E_T^i = \int_{\lfloor T^i \rfloor}^{\lfloor T^i \rfloor + 1} \dot{E}^i(z)\,dz = \int_{\lfloor T^i \rfloor}^{T^i} \dot{E}^i(z)\,dz =$$

$$= 0.5 * (T^i - \lfloor T^i \rfloor) * \dot{E}^i(\lfloor T^i \rfloor).$$

For $\lfloor T^i \rfloor + 1 \le t$ we have

$$E_t^i = \int_t^{t+1} \dot{E}^i(z)\,dz = \int_t^{t+1} 0\,dz = 0.$$

□

Theorem 11 (explicit specification of the emission power function)

The emission power function for a country *i* is given by

$$\dot{E}^i(z) :=$$

$$\begin{cases} \dfrac{E_{BY}^i}{T^i - BY - 0.5} * (T^i - z) \text{ for } BY \le z \le T^i \\ 0 \text{ for } T^i \le z \end{cases}$$

with

$$T^i = BY + 1 + \frac{RB^i + \sqrt{RB^i * (RB^i + E_{BY}^i)}}{E_{BY}^i}.$$

Proof:

$\dot{E}^i(T^i)$ is obviously zero and with the help of theorem 10 we obtain

$$\int_{BY}^{BY+1} \dot{E}^i(z)\, dz = \dot{E}^i(BY + 0.5) = E_{BY}^i.$$

It remains to be shown that

$$\int_{BY+1}^{T^i} \dot{E}^i(z)\, dz = RB^i. \tag{10}$$

We use theorem 10 again and obtain

$$\left(T^i - BY - 1\right) * \dot{E}^i(BY + 1) = 2 * RB^i.$$

Using the definition of the emission power function $\dot{E}^i$, we get

$$\left(T^i - BY - 1\right) * E_{BY}^i * \left(T^i - BY - 1\right) = 2 * RB^i * \left(T^i - BY - 0.5\right).$$

We rearrange this quadratic equation for T^i:

$$E_{BY}^i * \left(T^i - BY - 1\right)^2 - 2 * RB^i * \left(T^i - BY - 0.5\right) = 0$$

$$E_{BY}^i * \left(T^i\right)^2 + T^i * \left[-2 * E_{BY}^i * (BY + 1) - 2 * RB^i\right] + E_{BY}^i * (BY + 1)^2 + RB^i * (2 * BY + 1) = 0.$$

We calculate the discriminant and find out that is positive:

$$\left[-2 * E_{BY}^i * (BY + 1) - 2 * RB^i\right]^2 - 4 * E_{BY}^i * \left[E_{BY}^i * (BY + 1)^2 + RB^i * (2 * BY + 1)\right] =$$

$$= 4 * \left(E_{BY}^i\right)^2 * (BY + 1)^2 + 8 * E_{BY}^i * (BY + 1) * RB^i + 4 * \left(RB^i\right)^2$$

$$- 4 * \left(E_{BY}^i\right)^2 * (BY + 1)^2 - 4 * E_{BY}^i * (2 * BY + 1) * RB^i =$$

$$= 4 * E_{BY}^i * [2 * BY + 2 - 2 * BY - 1] * RB^i + 4 * \left(RB^i\right)^2 =$$

$$= 4 * RB^i * \left[E_{BY}^i + RB^i\right] > 0.$$

Using the quadratic formula, we obtain the expression

$$T_{1/2}^i = \frac{2 * E_{BY}^i * (BY + 1) + 2 * RB^i \pm \sqrt{4 * RB^i * [E_{BY}^i + RB^i]}}{2 * E_{BY}^i} =$$

$$= BY + 1 + \frac{RB^i \pm \sqrt{RB^i * (E_{BY}^i + RB^i)}}{E_{BY}^i}.$$

Since we are interested in a solution later than the end of the base year, only "+" of "±" remains as a solution. □

Remark 16 (year when the emission power function is zero, year of emission neutrality)

The year when the emission power function is zero is given by

$$\lfloor T^i \rfloor,$$

the year of emission neutrality, i. e. the year, when emissions are zero (or negative), is given by

$$\lceil T^i \rceil,$$

where $\lceil\ \rceil$ is the ceiling function. The ceiling function takes as an input a real number T^i and gives as an output $\lceil T^i \rceil$ the least integer greater than or equal to T^i.

3.2.2 Simplified Linear Pathway Model (SLPM)[9]

We avoid solving a quadratic equation for T^i (cf. Theorem 11) and discuss two easier approaches. The first approach is obtained by using an approximation of T^i in the LPM, the second by correcting the remaining budget in the LPM. Both approaches lead to the Simplified Linear Pathway Model (SLPM).

Using an approximation of T^i

This approach uses $\tilde{T}^i$, an approximation of T^i. This approximation can be found by considering the area under the emission power function of the LPM.

Theorem 12 (explicit specification of $\tilde{T}^i$)

An approximation of T^i is given by

$$\tilde{T}^i = BY + 1.5 + \frac{2 * RB^i}{E_{BY}^i} \approx T^i.$$

Proof:

We integrate the emission from *BY+0.5* to T^i and obtain with the help of theorem 10

$$\int_{BY+0.5}^{T^i} \dot{E}^i(z)\, dz = 0.5 * \left(T^i - BY - 0.5\right) * E_{BY}^i =$$

$$= \int_{BY+0.5}^{BY+1} \dot{E}^i(z)\, dz + \int_{BY+1}^{T^i} \dot{E}^i(z)\, dz \approx$$

$$\approx 0.5 * E_{BY}^i + RB^i$$

Thus, we have

$$T^i - BY - 0.5 \approx \frac{E_{BY}^i + 2 * RB^i}{E_{BY}^i} = 1 + \frac{2 * RB^i}{E_{BY}^i}$$

or

9 We use this model in the following web app: http://m-national-budgets.climate-calculator.info.

$$T^i \approx BY + 1.5 + \frac{2 * RB^i}{E^i_{BY}} = \tilde{T}^i.$$

□

Remark 17 (Error of the approximation)

The SLPM uses for these emissions of the country *i* in the period of time from *BY*+0.5 to *BY*+1 the area of the rectangular defined by the points (BY+0.5, 0), (BY+0.5, $\dot{E}^i(BY + 0.5)$), (BY+1, $\dot{E}^i(BY + \mathbf{0.5})$) and (*BY*+1, 0), where $\dot{E}^i$ is the emission power function of the SLPM. But the actual emission in this period would be represented by area of the right trapezium defined by the points (BY+0.5, 0), (BY+0.5, $\dot{E}^i(BY + 0.5)$), (BY+1, $\dot{E}^i(BY + \mathbf{1})$) and (BY+1, 0). Thus, the error is represented by the area of the triangle defined by the points (*BY*+0.5, $\dot{E}^i(BY + 0.5)$), (BY+1, $\dot{E}^i(BY + 0.5)$) and (*BY*+1, $\dot{E}^i(BY + 1)$).

Remark 18 (Another approach for $\tilde{T}^i$)

Since $\sqrt{1+x} \approx 1 + 0.5 * x$ for $x \ll 1$, we can simplify the expression for T^i given in theorem 11 and obtain the same result as in theorem 12:

$$T^i = BY + 1 + \frac{RB^i + \sqrt{RB^i * \left(RB^i + E^i_{BY}\right)}}{E^i_{BY}} =$$

$$= BY + 1 + \frac{RB^i + RB^i * \sqrt{1 + \frac{E^i_{BY}}{RB^i}}}{E^i_{BY}} \approx$$

$$\approx BY + 1 + \frac{RB^i + RB^i * \left(1 + 0{,}5 * \frac{E^i_{BY}}{RB^i}\right)}{E^i_{BY}} =$$

$$= BY + 1.5 + \frac{2 * RB^i}{E^i_{BY}} = \tilde{T}^i.$$

Correcting the remaining budget

The following approach, developed through trial and error, also provides useful results.

We assume

- that firstly a straight line through the points (BY, E^i_{BY}) and (T^i, 0) evaluated at (at the discrete point) t describes the emissions of a country i in the year t as long as the emissions are not negative during the whole year t:

$$E^i_t = -\frac{E^i_{BY}}{T^i - BY} * (t - BY) + E^i_{BY}, \text{ if } t \leq \lfloor T^i \rfloor - 1 \text{ and}$$

- that secondly the remaining budget of the country i is given by

$$RB^i = 0.5 * \left(T^i - BY\right) * E^i_{BY}.$$

We rearrange the last equation and obtain the point of emission neutrality as

$$T^i = BY + \frac{2 * RB^i}{E^i_{BY}}.$$

Substituting $T^i - BY = \frac{2*RB^i}{E^i_{BY}}$ in the upper equation for the emissions of a country i in the year t yields

$$E^i_t = -\frac{\left(E^i_{BY}\right)^2}{2 * RB^i} * (t - BY) + E^i_{BY}, \text{ if } t \leq \lfloor T^i \rfloor - 1.$$

The first assumption should describe the average emissions of a year of emissions decreasing on a straight line. Two adjustments are needed to properly convert this assumption into a formula. Firstly, we evaluate the straight line (=the emission power function) for the year t at the middle of the year, i. e. at time t+0.5, in order to obtain a representative value for the emissions of this year. Evaluating our straight line for the base year at BY+0.5, however, does not yield E^i_{BY}. Therefore, secondly, we shift our straight line by 0.5 to the right, whereby the point of time of emission neutrality is also shifted to the right by 0.5. If both the straight line and the points evaluated with this straight line are shifted by 0.5 to the right, the results do not change.

The second assumption should describe the remaining budget from time BY+1 onwards. However, our second assumption considers the period from time BY to time T^i. We notice that the area under the shifted straight line through the point (BY+0.5, E^i_{BY}) from time BY+0.5 to time BY+1 is approximately $0.5 * E^i_{BY}$. Thus, a corrected remaining budget is

$$RB^i_{cor} \approx RB^i + 0.5 * E^i_{BY}.$$

Using this corrected budget, we obtain for the emissions of a country i in the year t by

$$E_t^i = -\frac{\left(E_{BY}^i\right)^2}{2*RB_{cor}^i} * (t - BY) + E_{BY}^i, \text{ if } t \leq \lfloor T^i \rfloor - 1$$

and the point of time of emission neutrality $\tilde{T}^i$ is given by

$$\tilde{T}^i = BY + 0.5 + \frac{2*RB_{cor}^i}{E_{BY}^i}.$$

Remark 19 (Another emission power function)

Another elegant way to avoid solving a quadratic equation for T^i is to assume that the emission power function is constant in the base year and then decreases as a straight line through the points (*BY+1*, E_{BY}^i*)* and (T^i, 0). Thus, the remaining budget will be given by

$$RB^i = 0.5 * \left(T^i - BY - 1\right) * \ E_{BY}^i.$$

This equation can be easily solved for T^i. However, the emission reduction after the base year is only half as big as the following ones

$$2 * \left(E_{BY}^i - E_{BY+1}^i\right) = \ E_t^i - E_{t+1}^i \text{ for } t\text{=}BY\text{+1}, \ldots, \lfloor T^i \rfloor - 2.$$

Theorem 13 (equivalent approaches)

Using the approximation $\tilde{T}^i$ and correcting the remaining budget and gives the same results in the years $t \leq \lfloor T^i \rfloor - 1$.

Proof:

We have to show that

$$E_t^i = \ -\frac{\left(E_{BY}^i\right)^2}{2 * RB_{cor}^i} * (t - BY) + E_{BY}^i = \frac{E_{BY}^i}{\tilde{T}^i - BY - 0.5} * \left(\tilde{T}^i - t - 0.5\right)$$

with the corrected national remaining budget $RB_{cor}^i = RB^i + 0.5 * E_{BY}^i$ and

$$\tilde{T}^i = BY + 0.5 + \frac{2*RB_{cor}^i}{E_{BY}^i} = BY + 1.5 + \frac{2*RB^i}{E_{BY}^i}.$$

First, we notice that

$$BY + 0.5 + \frac{2 * RB_{cor}^i}{E_{BY}^i} =$$

$$= BY + 0.5 + \frac{2 * \left(RB^i + 0.5 * E_{BY}^i\right)}{E_{BY}^i} =$$

$$= BY + 0.5 + \frac{2 * RB^i + E_{BY}^i}{E_{BY}^i} =$$

$$= BY + 0.5 + \frac{2 * RB^i}{E_{BY}^i} + \frac{E_{BY}^i}{E_{BY}^i} =$$

$$= BY + 1.5 + \frac{2 * RB^i}{E_{BY}^i} = \tilde{T}^i.$$

Then we combine Corollary 3 with Theorem 11 and obtain for $t \leq \lfloor \tilde{T}^i \rfloor - 1$

$$E_t^i = \frac{E_{BY}^i}{\tilde{T}^i - BY - 0.5} * \left(\tilde{T}^i - t - 0.5\right)$$

with $\tilde{T}^i = BY + 1.5 + \frac{2*RB^i}{E_{BY}^i}$. Substituting $\tilde{T}^i$ yields

$$E_t^i = \frac{E_{BY}^i}{1 + \frac{2 * RB^i}{E_{BY}^i}} * \left(BY + 1 + \frac{2 * RB^i}{E_{BY}^i} - t\right) =$$

$$= \frac{\left(E_{BY}^i\right)^2}{E_{BY}^i + 2 * RB^i} * \left(BY + 1 + \frac{2 * RB^i}{E_{BY}^i} - t\right) =$$

$$= -\frac{\left(E_{BY}^i\right)^2}{E_{BY}^i + 2 * RB^i} * (t - BY) + \frac{\left(E_{BY}^i\right)^2}{E_{BY}^i + 2 * RB^i} * \left(\frac{E_{BY}^i + 2 * RB^i}{E_{BY}^i}\right) =$$

$$= -\frac{\left(E_{BY}^i\right)^2}{E_{BY}^i + 2 * RB^i} * (t - BY) + E_{BY}^i.$$

Finally, we use the fact that $RB_{cor}^i = RB^i + 0.5 * E_{BY}^i$ by definition and obtain

$$E_t^i = -\frac{\left(E_{BY}^i\right)^2}{2 * RB_{cor}^i} * (t - BY) + E_{BY}^i.$$

□

3.2.3 Generalised Linear Pathway Model (GLPM)[10]

In order to allow for net negative emissions, we generalise the emission power function in equation (9). After a base year, the emission power $\dot{E}^i$, i. e. the derivative of emissions with respect to time (the emissions per unit of time) of a country *i* is supposed to decrease on a straight line until the point of time U^i when the emission power reaches its minimum $E^i_{min} \leq 0$. After that point, the emission power shall continue to be E^i_{min} until the end of the year 2100. This leads to the following approach for the emission power:

$$\dot{E}^i(z) := \begin{cases} E^i_{BY} - \frac{E^i_{BY} - E^i_{min}}{U^i - BY - 0.5} * (z - BY - 0{,}5) \text{ for } BY \leq z \leq U^i \\ E^i_{min} \text{ for } U^i \leq z \leq 2101 \end{cases}, \tag{11}$$

The idea is to determine U^i such, that the country *i* meets a remaining budget RB^i by the end of the year 2100: $\int_{BY+1}^{2101} \dot{E}^i(z)\, dz = RB^i$. We note that theorem 10 also holds if $\dot{E}^i(z)$ is negative and obtain:

$$\int_{BY+1}^{2101} \dot{E}^i(z)\, dz = \int_{BY+1}^{U^i} \dot{E}^i(z)\, dz + \int_{U^i}^{2101} \dot{E}^i(z)\, dz =$$

$$= \left(U^i - BY - 1\right) * \dot{E}^i\left(\frac{U^i + BY + 1}{2}\right) + E^i_{min} * \left(2101 - U^i\right) =$$

$$= \left(U^i - BY - 1\right) * \left\{E^i_{BY} - \frac{E^i_{BY} - E^i_{min}}{U^i - BY - 0{,}5} * \frac{U^i - BY}{2}\right\} + E^i_{min} * \left(2101 - U^i\right) = RB^i$$

We rearrange the last equation and obtain a quadratic equation for U^i:

$$2 * E^i_{BY} * \left(U^i - BY - 1\right) * \left(U^i - BY - 0{,}5\right) - \left(E^i_{BY} - E^i_{min}\right) * \left(U^i - BY\right) * \left(U^i - BY - 1\right) +$$

$$+2 * E^i_{min} * \left(2101 - U^i\right) * \left(U^i - BY - 0.5\right) - 2 * RB^i * \left(U^i - BY - 0{,}5\right) = 0$$

$$2 * E^i_{BY} * \left((U^i)^2 + U^i * (-2 * BY - 1.5) + (BY + 1) * (BY + 0{,}5)\right) +$$

$$- \left(E^i_{BY} - E^i_{min}\right) * \left((U^i)^2 + U^i * (-2 * BY - 1) + BY * (BY + 1)\right) +$$

$$+2 * E^i_{min} * \left(-(U^i)^2 + U^i * (BY + 2101{,}5) - 2101 * (BY + 0{,}5)\right) -$$

$$-2 * RB^i * \left(U^i - BY - 0{,}5\right) = 0$$

[10] We use this model in the following web app: http://national-budgets.climate-calculator.info.

$$(U^i)^2 * \left[2 * E^i_{BY} + E^i_{min} - E^i_{BY} - 2 * E^i_{min}\right] +$$

$$U^i * \begin{bmatrix} 2 * E^i_{BY} * (-2 * BY - 1.5) + \left(E^i_{BY} - E^i_{min}\right) * (2 * BY + 1) \\ +2 * E^i_{min} * (BY + 2101.5) - 2 * RB^i \end{bmatrix} +$$

$$+2 * E^i_{BY} * (BY + 1) * (BY + 0,5) - \left(E^i_{BY} - E^i_{min}\right) * BY * (BY + 1) -$$

$$-2 * E^i_{min} * 2101 * (BY + 0,5) + 2 * RB^i * (BY + 0,5) = 0$$

$$(U^i)^2 * \left[E^i_{BY} - E^i_{min}\right] +$$

$$+U^i * \left[-2 * E^i_{BY} * (BY + 1) + 4202 * E^i_{min} - 2 * RB^i\right] +$$

$$+E^i_{BY} * (BY + 1) * (BY + 1) + E^i_{min} * \left(BY * (BY + 1) - 2101 * (2 * BY + 1)\right) +$$

$$+RB^i * (2 * BY + 1) = 0$$

This equation may have no real solutions or only solutions with $U^i < BY$ or $U^i > 2101$. In these cases we do not get any useful solutions.

Theorem 14 (approximation of U^i)

$$U^i \approx \frac{2 * RB^i + E^i_{BY} * (BY + 1,5) + E^i_{min} * (BY - 4201,5)}{E^i_{BY} - E^i_{min}}.$$

Proof:

We integrate the emission from *BY+0.5* to the end of the year 2100 and obtain with the help of theorem 10

$$\int_{BY+0.5}^{2101} \dot{E}^i(z)\, dz =$$

$$= \int_{BY+0.5}^{U^i} \dot{E}^i(z)\, dz + \int_{U^i}^{2101} \dot{E}^i(z)\, dz = \left(U^i - BY - 0.5\right) * \frac{E^i_{BY} + E^i_{min}}{2} + E^i_{min} * \left(2101 - U^i\right)$$

$$= \int_{BY+0.5}^{BY+1} \dot{E}^i(z)\, dz + \int_{BY+1}^{2101} \dot{E}^i(z)\, dz \approx 0.5 * E^i_{BY} + RB^i.$$

Thus, we have

$$U^i * \left[\frac{E^i_{BY} - E^i_{min}}{2}\right] - (BY + 0.5) * \frac{E^i_{BY} + E^i_{min}}{2} + 2101 * \ E^i_{min} \approx 0.5 * E^i_{BY} + RB^i$$

or

$$U^i \approx \frac{2 * RB^i + E^i_{BY} * (BY + 1.5) + E^i_{min} * (BY - 4201.5)}{E^i_{BY} - E^i_{min}}.$$

□

Theorem 15 (boundary condition for U^i)

A useful approximation of U^i is obtained if the inequality condition

$$2 * RB^i < (2099.5 - BY) * \left(E^i_{BY} + E^i_{min}\right) + E^i_{min}$$

is fulfilled.

Proof:

First, we note that $U^i > BY$:

$$\frac{2 * RB^i + E^i_{BY} * (BY + 1.5) + E^i_{min} * (BY - 4201.5)}{E^i_{BY} - E^i_{min}} > BY$$

$$2 * RB^i + E^i_{BY} * (BY + 1.5) + E^i_{min} * (BY - 4201.5) > BY * \left(E^i_{BY} - E^i_{min}\right)$$

$$2 * RB^i + 1.5 * E^i_{BY} + E^i_{min} * (2 * BY - 4201.5) > 0.$$

Then we show that $U^i < 2101$:

$$\frac{2 * RB^i + E^i_{BY} * (BY + 1.5) + E^i_{min} * (BY - 4201.5)}{E^i_{BY} - E^i_{min}} < 2101$$

$$2 * RB^i + E^i_{BY} * (BY + 1.5) + E^i_{min} * (BY - 4201.5) < 2101 * \left(E^i_{BY} - E^i_{min}\right)$$

$$2 * RB^i < (2099.5 - BY) * E^i_{BY} + (2101.5 - BY) * E^i_{min}$$

$$2 * RB^i < (2099.5 - BY) * E^i_{BY} + (2099.5 - BY) * E^i_{min} + E^i_{min}$$

$$2 * RB^i < (2099.5 - BY) * \left(E^i_{BY} + E^i_{min}\right) + E^i_{min}$$

□

Theorem 16 (calculation of T^i)

The point of emission neutrality is given by

$$T^i = \frac{E^i_{BY} * U^i - E^i_{min} * (BY + 0.5)}{E^i_{BY} - E^i_{min}}.$$

Proof:

We insert T^i in equation (11) and obtain the equation

$$\dot{E}^i(T^i) = E^i_{BY} - \frac{E^i_{BY} - E^i_{min}}{U^i - BY - 0.5} * (T^i - BY - 0.5)$$

which we solve for T^i:

$$E^i_{BY} * (U^i - BY - 0.5) = (E^i_{BY} - E^i_{min}) * (T^i - BY - 0.5)$$

$$(E^i_{BY} - E^i_{min}) * T^i = E^i_{BY} * (U^i - BY - 0.5) + (E^i_{BY} - E^i_{min}) * (BY + 0.5)$$

$$T^i = \frac{E^i_{BY} * U^i - E^i_{min} * (BY + 0.5)}{E^i_{BY} - E^i_{min}}.$$

□

Remark 20 (year of emission neutrality)

The year when the emission power function is zero is given by

$$\lfloor T^i \rfloor,$$

the year of emission neutrality, i. e. the year, when the emissions are zero or negative, is given by

$$\lfloor T^i \rfloor, \text{ if } \int_{\lfloor T^i \rfloor}^{\lfloor T^i \rfloor + 1} \dot{E}^i(z)\, dz < 0 \text{ and } \lceil T^i \rceil, \text{ if } \int_{\lfloor T^i \rfloor}^{\lfloor T^i \rfloor + 1} \dot{E}^i(z)\, dz > 0.$$

3.3 Smooth Pathway Models

3.3.1 Smooth Pathway Model (SPM)

Raupach et al. propose a formula for deriving a national pathway from a national budget where the slope at the end of the base year can be specified (Raupach, et al., 2014). This allows a smooth transition from the previous emissions trajectory up to the end of the base year to the emissions trajectory after the base year.

The so-called Smooth Pathway Formula for the emission power $\dot{E}^i$, i. e. the derivative of emissions with respect to time (the emissions per unit of time), of the country *i* at a point of time $z \geq BJ + 1$ is:

$$\dot{E}^i(z) \;=\; \dot{E}^i_{BY+1}\big(1 + \;(r^i + m^i)(z - BY - 1)\big)e^{-m^i(z-BY-1)}, \tag{12}$$

where

$\dot{E}^i_{BY+1}$ emission power of the country *i* at the end of the base year,

r^i change rate of the emission power of the country *i* at the end of the base year $\left(\frac{d\dot{E}^i}{dz}(BY + 1)/\dot{E}^i(BY + 1) = r^i\right)$ and

m^i the mitigation rate (or the decay parameter) of the country *i*.

The mitigation rate m^i is determined such that the allocated remaining budget of the country *i* (RB^i) is met:

$$\int_{BY+1}^{\infty} \dot{E}^i(z)\, dz \;=\; RB^i.$$

Thus, we obtain

$$\int_{BY+1}^{\infty} \dot{E}^i(z)\, dz \;=$$

$$= \int_{BY+1}^{\infty} \dot{E}^i_{BY+1}\big(1 + \;(r^i + m^i)(z - BY - 1)\big)e^{-m^i(z-BY-1)}\, dz \;=$$

$$= \dot{E}^i_{BY+1} \int_{BY+1}^{\infty} e^{-m^i(z-BY-1)}\, dz \;+ \dot{E}^i_{BY+1}(r^i + m^i) \int_{BY+1}^{\infty} (z - BY - 1)e^{-m^i(z-BY-1)}\, dz \;=$$

$$= \dot{E}^i_{BY+1}\left[\frac{-1}{m^i} e^{-m^i(z-BY-1)}\right]_{z=BY+1}^{z=\infty}$$

$$+\dot{E}^i_{BY+1}(r^i + m^i)\left[\frac{-(z - BY - 1)}{m^i} e^{-m^i(z-BY-1)} - \frac{1}{(m^i)^2} e^{-m^i(z-BY-1)}\right]_{z=BY+1}^{z=\infty}$$

$$= \dot{E}^i_{BY+1}\left[\frac{1}{m^i}\right] + \dot{E}^i_{BY+1}(r^i + m^i)\ \left[\frac{1}{(m^i)^2}\right] = RB^i.$$

With the time $T^i = \frac{RB^i}{\dot{E}^i_{BY+1}}$ defined by the remaining budget of the country *i* and the emission power of the country *i* at the end of the base year we obtain

$$T^i\left(m^i\right)^2 - 2m^i - r^i = 0.$$

Thus, if $r^i > -1/T^i$, the mitigation rate m^i is given by

$$m^i \ = \frac{1 + \sqrt{1 + r^i T^i}}{T^i},$$

There is otherwise no solution for the mitigation rate m^i. In this rare case a simple exponential decay function is used:

$\dot{E}^i(z) \ = \ \dot{E}^i_{BY+1} e^{-m^i(z-BY-1)}.$

Since we are more interested in the emissions of the country *i* in the year *t* (E^i_t) than in the emission power at a point of time *z*, we integrate equation (4) and obtain:

$$E^i_t = \int_t^{t+1} \dot{E}^i(z)\, dz =$$

$$-\dot{E}^i_{BY+1}\frac{e^{-m^i(t-BY)}}{(m^i)^2}\left[\left(r^i m^i + \left(m^i\right)^2\right)(t - BY) + 2m^i + r^i\right]$$

$$+\dot{E}^i_{BY+1}\frac{e^{-m^i(t-BY-1)}}{(m^i)^2}\left[\left(r^i m^i + \left(m^i\right)^2\right)(t - BY - 1) + 2m^i + r^i\right].$$

Supplementary information containing mathematical details on the properties of the formula in equation (12) can be retrieved from the “Supplementary Text“ at https://static-content.springer.com/esm/art%3A10.1038%2Fnclimate2384/MediaObjects/41558_2014_BFnclimate2384_MOESM461_ESM.pdf.

3.3.2 Generalised Smooth Pathway Model (GSPM)

In order to allow for net negative emissions, we generalise equation (12) using the following function for the emission power, i. e. the derivative of emissions with respect to time or the emissions per unit of time, of the country *i* at a point of time $z \geq BJ + 1$:

$$\dot{E}^i(z) = p_\infty + \left(p_0 + p_1(z - BY - 1)\right) e^{-p_2(z-BY-1)}, \tag{13}$$

where

the parameter p_∞ is the emission power at infinity and the parameters p_0, p_1 and p_2 are determined in a way that the following constraints hold

(1) $\dot{E}^i(BY + 1) = \dot{E}^i_{BY+1}$,

(2) $\frac{d\dot{E}^i}{dz}(BY + 1)/\dot{E}^i(BY + 1) = r^i$

(3) $\int_{BY+1}^{2101} \dot{E}^i(z)\, dz = RB^i$

with

$\dot{E}^i_{BY+1}$ emission power of the country *i* at the end of the base year,

r^i change rate of the emission power of the country *i* at the end of the base year,

RB^i remaining budget of the country *i* in the period starting at the end of the base year and ending in at the end of the year 2100.

The first constraint leads to $p_0 = \dot{E}^i_{BY+1} - p_\infty$.

The second constraint leads to $p_1 = \dot{E}^i_{BY+1} r + \left(\dot{E}^i_{BY+1} - p_\infty\right) p_2$.

The emissions of the country *i* in the year *t* (E^i_t) are obtained by integrating equation (13):

$$E^i_t = \int_t^{t+1} \dot{E}^i(z)\, dz =$$

$$\left[p_\infty(z - BY - 1) - \frac{p_0}{p_2} e^{-p_2(z-BY-1)} - \frac{p_1(z-BY-1)}{p_2} e^{-p_2(z-BY-1)} - \frac{p_1}{p_2^2} e^{-p_2(z-BY-1)}\right]_{z=t}^{z=t+1}.$$

Thus the parameters p_1 and p_2 are defined implicitly by one linear and one non-linear equation. In general, this system of equations can only be solved iteratively.

3.4 Smart Pathways Models based on Annual Change Rates (SPMCR)[11]

In the SPMCR approaches national emissions pathways meeting a national budget are derived indirectly via an assumption about the property of the annual emissions change rates of the country $\left(RR_t^i\right)$. In contrast to the models presented above there is no need to find out and discuss the properties of the trajectory of the emissions obtained from the model, because the trajectory of emissions is determined beforehand.

This leads to the following approaches for the emissions of the country *i* in the year *t*:

$$E_t^i = E_{t-1}^i * \left(1 + RR_t^i\right) \tag{14}$$

A large variety of functions for RR_t^i are imaginable. In practice, these functions should map a meaningful course that can be justified, for example, economically, technologically or politically.

The functions for RR_t^i can also be defined in more than one section. This can be useful when taking into account net negative emissions. For example, a constant reduction amount can be used when the emissions fall below a threshold until the emissions reach a predefined minimum value.

If you want to indicate a concrete continuous function in one section such, that the resulting emissions meet a given budget, you can start with a family of curves with a free parameter and then determine this parameter iteratively.

We use the scenario types RM 1 – 5 based on annual change rates [cf. (Sargl, et al., 2023b), (Sargl, et al., 2023a), (Sargl, et al., 2021) and (Wiegand, et al., 2021)]. For a comprehensive mathematical description of the RM Scenario Types, we refer to the following paper: (Wolfsteiner & Wittmann, 2023a).

[11] Here is an overview of the tools we offer for this approach: https://climate-calculator.info.

scenario type	course of the annual change rates		basic function type of the annual reduction rates	course of the annual reduction amounts	course of the emission pathways
RM-1-const	linear, no curvature		$y = const$	concave	convex
RM-2-exp	concave, curved to the right		$y = e^x$	u-shaped	s-shaped (first concave then convex)
RM-3-lin	linear, no curvature		$y = ax + b$		
RM-4-quadr	concave, curved to the right		$y = ax^2 + b$		
RM-5-rad	convex, curved to the left		$y = a\sqrt{x} + b$		
RM-6-abs	concave, curved to the right		-	constant	linear

Table 1: Overview of scenario types RM 1 – 6

RM-1, RM-2, …, RM-5 represent a concretisation of the approach in formula (14) and RM-6 is equivalent to the GLPM.

4 List of abbreviations

$\lfloor\ \rfloor$ floor function. The floor function takes as an input a real number T and gives as an output $\lfloor T \rfloor$ the greatest integer less than or equal to T

$\lceil\ \rceil$ ceiling function. The ceiling function takes as an input a real number T and gives as an output $\lceil \mathrm{T} \rceil$ the least integer greater than or equal to T

a^i parameter of the straight line $a^i\, z + b^i$

b^i parameter of the straight line $a^i\, z + b^i$

B global emissions in the convergence period (global budget in the convergence period)

B^i emissions of the country *i* in the convergence period (national budget of the country *i* in the convergence period)

B_t $(= \sum_{l=BY+1}^{t} E_l)$ global emissions until the year *t* (global budget until the year *t*),

B_t^i $\left(= \sum_{l=BY+1}^{t} E_l^i\right)$ emissions of the country *i* until the year *t* (national budget of the country *i* until the year *t*)

BY base year (space of time)

C weighting of the population

C_t weighting of the population in the year *t*

$\widehat{C}_t$ weighting of the population in the year *t* in C&C

$\widetilde{C}_t$ weighting of the population in the year *t* in LIMITS

$\overline{C}_t$ weighting of the population in the year *t* in the RF

$\check{C}_t^i$ weighting of the population of the country *i* in the year *t* used to obtain the national budget of the country *i*

C&C Contraction and Convergence Model

CA_t cap in the year *t*

CDC Common but Differentiated Convergence Model

CY convergence year

E_{BY} global emissions in the base year

E_{BY}^i emissions of the country *i* in the base year

E_{CY} global emissions in the convergence year

E_{CY}^{i} emissions of the country *i* in the convergence year

E_t global emissions in the year *t*

E_t^i emissions of the country *i* in the year *t*

$\widehat{E_t^i}$ emissions of the country *i* in the year *t* in C&C

$\widetilde{E_t^i}$ emissions of the country *i* in the year *t* in LIMITS

$\overline{E_t^i}$ emissions of the country *i* in the year *t* in the RF

$E_t^{i\_bau}$ emissions of the country *i* in the year *t* in a business-as-usual scenario

E_t^{oTH} remaining global emissions in the year *t* for the countries over the threshold in the year *t*

$E_{t-1}^{oTH\_t}$ emissions in the year *t* – 1 of the countries over the threshold in the year *t*

$\dot{E}^i(z)$ emission power emission power (the derivative of emissions with respect to time, emissions per unit of time) of the country *i* at a point of time *z*

$\dot{E}_{BY+1}^{i}$ emission power of the country *i* at the end of the base year

EPM Emission Probability Model

ESPM Extended Smooth Pathway Model

f^i income PDF of the country *i*

$\tilde{f}^i$ emission PDF of the country *i*, scaled PDF

F cumulative distribution function, i. e. the probability of an income equal to z or less is $F(z) = \int_{-\infty}^{z} f(t)\, dt$

F^{-1} inverse function of the cumulative distribution function *F*

$f^i(z; p^i)$ assumed income PDF of the country *i* with parameters p^i to be estimated

$\tilde{f}^i(z; p^i)$ estimated emission PDF of the country *i* with parameters p^i

G-C&C Generalised C&C

G-Limits Generalised LIMITS

GSPF General Smooth Pathway Formula

i country

IC_t^i	implicit weighting of the population of the country *i* in the year *t*
j	country
m^i	mitigation rate (or the decay parameter) of the country *i*
L	explicit representation of the Lorenz curve
$\bar{L}$	parametric representation of the Lorenz curve
$\check{L}^i$	Lorenz curve of the country *i*
LIMITS	LIMITS Model
P	(frozen) global population
P^i	(frozen) population of the country *i*
P_{BY}	global population in the base year
P_{BY}^i	population of the country *i* in the base year
P_{CY}	global population in the convergence year
P_{CY}^i	population of the country *i* in the convergence year
P_t	global population in the year *t*
P_t^i	population of the country *i* in the year *t*
P_t^{oTH}	population in the year *t* of the countries over the threshold in the year *t*
PDF	probability density function
PT	percentage
r^i	change rate of the emission power of the country *i* at the end of the base year $\left(\frac{d\dot{E}^i}{dz}(BY+1)/\dot{E}^i(BY+1)=r^i\right)$
RB	global remaining budget
RB^i	remaining budget of the country *i* (remaining national budget of the country *i*)
RB_{cor}^i	corrected remaining budget of the country *i* or corrected national remaining budget of the country *i* $\left(RB_{cor}^i = RB^i + 0.5 * E_{BY}^i\right)$
RF	Regensburg Formula
RR_t^i	emission reduction rate of the country *i* $\left(\frac{E_t^i}{E_{t-1}^i} - 1\right)$

s scaling factor

s^i scaling factor of the country i $\left(\frac{\text{average emissions in country } i}{\text{average income in country } i}\right)$

SPF Smooth Pathway Formulae

SPFR Smooth Pathway Formula from Raupach et al.

t, T year

T^i point of time defined by the remaining budget of the country *i* and the emission power of the country *i* at the end of the base year $\left(T^i = \frac{RB^i}{\dot{E}^i_{BY+1}}\right)$ (SPM, GSPM),

point of time when the emission power is zero, point of time of emission neutrality $\left(\dot{E}^i(T^i) = 0\right)$ (LPM)

$\tilde{T}^i$ an approximation of T^i (SLPM)

TH_t threshold in the year *t*

U^i point of time when the emission power reaches its minimum $\left(\dot{E}^i(U^i) = E^i_{min}\right)$

$\left(x^i_j, y^i_j\right)$ points of the Lorenz curve $\check{L}^i$ of the country i, i. e. $y^i_j = \check{L}^i\ (x^i_j)$

z point of time (LPM, GLPM, SPM, GSPM),

income (EPM)

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