Properties of Gauss Hypergeometric Functions of Specific Parameters: ${}_2F_1\left(\frac{1}{2n}, b, \frac{1}{2n}+1; -t^{2n}\right)$

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Abstract

This paper showed some properties of Gauss Hypergeometric function ${}_2F_1(a, b, c; -t^{2n})$ of specific parameters $a = 1/2n, b \geq 0, c = 1/2n + 1, n \in \mathbb{N}^+$. Generating equation is presented and basic properties, monotonicity, bounded range, and inequality are discussed. With these parameters, ${}_2F_1$ is monotonic decreasing function for $|t|$ value, bounded on $(0, 1]$, and ${}_2F_1(, b_1,) > {}_2F_1(, b_2,), \forall 0 \leq b_1 < b_2, b_1, b_2 \in \mathbb{R}$. In addition, inequality for two functions of $n_1 < n_2 \in \mathbb{N}^+$ and same b value have threshold point that reverses the inequality if $b < 1 + \frac{1}{2}(\frac{1}{n_1} + \frac{1}{n_2})$. Visual representation of the cases are presented.

***K*eywords** Hypergeometric function · Monotonic function · Inequality

1 Introduction

This paper discuss about some properties of the Gauss hypergeometric function of form, ${}_2F_1(1/2n, b, 1/2n+1; -t^{2n})$, where n is a positive integer and $b \geq 0, b \in \mathbb{R}$. Generating integral of such form function was suggested and with the integral relationship, some properties are investigated. Visual representation of the results are also presented.

The motivation of the subject is determining a positive solution linear system of $(1 + x^{2n})^{-b}$ kernel matrix. With Kaykobad's Theorem of positive solution, the integration of kernel function can be used to determined the existence of positive solution of the system by the dimension size[1]. However, the precious value and properties of the definite integration of the kernel is required.

2 Notations, Definitions and Properties

In this paper, **GHF** is an abbreviation of Gauss hypergeometric function, and without special notation, upper case F denotes special form of GHF.

$$F(z, b, n) := {}_2F_1\left(\frac{1}{2n}, b, \frac{1}{2n} + 1, -z^{2n}\right) \tag{1}$$

The superscript $+, -$ on some number set $\mathbb{X} = \mathbb{N}, \mathbb{Q}, \mathbb{R}$ indicate positive and negative elements of set respectively and zero is not contained without additional notation. For example, $\mathbb{X}^+ = \{x | x \in \mathbb{X}, x > 0\}$.

Definition 2.1 (Gauss hypergeometric function)**.**

$${}_2F_1(a, b, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n, (b)_n}{(c)_n} \frac{z^n}{n!} \tag{2}$$

where, $|z| < 1, c \notin \mathbb{N}^- \cup \{0\}$ and $(x)_n$ is a Pochhammer symbol which is

$$(a)_n = \frac{\Gamma(a + n)}{\Gamma(a)} \tag{3}$$

$_2F_1(1/2n, b, 1/2n+1, -t^{2n})$

The function is analytic everywhere on complex plane, $\mathbb{C}$, with principal branch $(1, \infty)$ [2].

With analytic continuation and Euler integral representation, next equation can be used for expanded complex version of the function if the integral is well defined[3].

For $\mathbb{R}(c) > \mathbb{R}(b) > 0$,

$$_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a} dt \tag{4}$$

where, $\Gamma(x)$ is a gamma function.

By the definition of GHF, GHF have next properties.

- Symmetric for a, b, $_2F_1(a, b, c; z) = {_2F_1}(b, a, c; z)$
- $_2F_1(a, b, c; z) = 1$ if $a = 0$ or $b = 0$ or $z = 0$.
- $_2F_1(a, b, b; z) = (1-z)^{-a}$

Pfaff's tranform of GHF,

$$_2F_1(a, b, c, ; z) = (1-z)^{-a} {_2F_1}(a, c-b, c; \frac{z}{z-1}) \tag{5}$$

and it is valid for analytic continuation of GHF[4].

Derivative of GHF can be represented with contiguous relations of GHF[5]. One example is

$$z \frac{d}{dz} {_2F_1}(a, b, c, z) = a[{_2F_1}(a+1, b, c, z) - {_2F_1}(a, b, c, z)] \tag{6}$$

3 2F1 function of special form

3.1 Generating Integral

Gauss hypergeometric function $_2F_1(a, b, c; z)$ of $a = 1/2n, b \geq 0, c = 1/2n+1, z \leq 0$, is derived with scale value t from the next integral, for $z = -t^{2n}$, $t \geq 0$ and $n \in \mathbb{N}^+$,

$$\int_0^t (1+x^{2n})^{-b} dx = t \cdot {_2F_1}\left(\frac{1}{2n}, b, \frac{1}{2n}+1, -t^{2n}\right) \tag{7}$$

Proof. Let, $x = ts$, then $dx = tds$,

$$\int_0^t (1+x^{2n})^{-b} dx = t \int_0^1 (1+t^{2n}s^{2n})^{-b} ds \tag{8}$$

and again substituting parameter, $s^{2n} = z$, $2ns^{2n-1}ds = dz$,

$$ds = \frac{1}{2n} z^{1/2n-1} dz \tag{9}$$

Thus,

$$\int_0^t (1+x^{2n})^{-b} dx = t \frac{1}{2n} \int_0^1 \frac{z^{1/2n-1}}{(1-(-t^{2n})z)^b} dz \tag{10}$$

The right term of the above equation is an Euler integral representation of Gauss hypergeometric function Eq(4) with $a = 1/2n, c = 1/2n+1$,

$$_2F_1\left(\frac{1}{2n}, b, \frac{1}{2n}+1; z\right) = \frac{1}{2n} \int_0^1 \frac{t^{1/2n-1}}{(1-zt)^b} dt \tag{11}$$

□

$_2F_1(1/2n, b, 1/2n+1, -t^{2n})$

This relationship also hold true for generalized hypergeometric function, $_pF_q, |z| < 1$ with series representation and $\alpha \neq 0$,

$$ {}_pF_q\left(\begin{array}{c} a_1, a_2, \dots a_p \\ b_1, b_2, \dots b_q \end{array}; z^\alpha\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{z^{\alpha k}}{k!} $$

$$ \begin{aligned} &\int {}_pF_q\left(\begin{array}{c} a_1, a_2, \dots a_p \\ b_1, b_2, \dots b_q \end{array}; z^\alpha\right) dz \\ &= \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{1}{\alpha k + 1} \frac{z^{\alpha k+1}}{k!} \\ &\because \frac{a}{a+k} = \frac{(a)_k}{(a+1)_k} \\ &= z \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{(\frac{1}{\alpha})_k}{(\frac{1}{\alpha}+1)_k} \frac{z^{\alpha k}}{k!} \\ &= z \cdot {}_{p+1}F_{q+1}\left(\begin{array}{c} a_1, a_2, \dots, a_p, \frac{1}{\alpha} \\ b_1, b_2, \dots, b_q, \frac{1}{\alpha}+1 \end{array}; z^\alpha\right) \end{aligned} $$

3.2 Properties

Theorem 1 (Inequality 1). *$\forall t$ and $b \in \mathbb{R}^+ \cup \{0\}, n \in \mathbb{N}^+$, GHFs of form $_2F_1(1/2n, b, 1/2n, -t^{2n})$ has next inequalities for $0 \leq b_1 < b_2$,*

$$ F(t, b_1, n) \geq F(t, b_2, n) $$

and equality holds true when $t = 0$.

Proof. From Eq(7), because $(1 + x^{2n}) > 0, \forall |x| \geq 0$,

$$ (1 + x^{2n})^{-b_1} > (1 + x^{2n})^{-b_2} \tag{12} $$

this inequality is conserved through integral operation. Thus,

$$ \begin{aligned} \int_0^t (1 + x^{2n})^{-b_1} dx &> \int_0^t (1 + x^{2n})^{-b_2} dx \\ &\to tF(t, b_1, n) > tF(t, b_2, n). \end{aligned} $$

Without loss of generality we can assume $t > 0$, if $t < 0$, then $\int_0^t = -\int_0^{-t}$. Therefore, we get next

$$ F(t, b_1, n) > F(t, b_2, n). \tag{13} $$

□

Theorem 2 (Monotonic property). *$F(t, b, n)$ is a strictly monotonic decreasing function for $|t| > 0$ value. That is, $\forall t > 0$*

$$ \frac{d}{dt} F(t, b, n) < 0 \tag{14} $$

and the inequality is reversed for $t < 0$.

${}_2F_1(1/2n, b, 1/2n+1, -t^{2n})$

Proof. From Eq(6),

$$\frac{d}{dt}F(t,b,n) = \frac{1}{t}[(1+t^{2n})^{-b} - {}_2F_1\left(\frac{1}{2n}, b, \frac{1}{2n}+1, -t^{2n}\right)] \tag{15}$$

Applying Pfaff's transform,

$$= \frac{1}{t(1+t^{2n})^b}[1 - {}_2F_1\left(1, b, \frac{1}{2n}+1, \frac{t^{2n}}{1+t^{2n}}\right)] \tag{16}$$

Let, $z(t) = t^{2n}/(1+t^{2n})$, then $0 \le z < 1$ and with Eq(4), the above GHF term can be represented as

$${}_2F_1\left(1, b, \frac{1}{2n}+1, z\right) = \frac{1}{2n}\int_0^1 \frac{(1-s)^{\frac{1}{2n}-1}}{(1-zs)^b} ds \tag{17}$$

from the regions $0 < z < 1$ and $0 < s < 1$, the numerator and denominator of the RHS integral are always positive and $(1-zs)^{-b} > 1 \; \forall b > 0$. Therefore, $(1-s)^{\frac{1}{2n}-1}/(1-zs)^b > (1-s)^{\frac{1}{2n}-1}$ and consequently, we get

$${}_2F_1\left(1, b, \frac{1}{2n}+1, z\right) > \frac{1}{2n}\int_0^1 (1-s)^{\frac{1}{2n}-1} ds = 1. \tag{18}$$

Thus, for $t > 0$,

$$\frac{d}{dt}F(t,b,n) < 0 \tag{19}$$

and $F(t,b,n)$ is an even function, it is enough.

□

Theorem 3. *$F(t,b,n)$ is bounded on $(0,1], \forall n \in \mathbb{N}^+, b > 0$.*

Proof. By the definition, Eq (2), $\lim_{t\to 0} {}_2F_1(\dots, -t^{2n}) = {}_2F_1(\dots, 0) = 1$. It monotonically strictly decreases $\forall |t|$ by Theorem 2. Therefore, maximum value 1 exists at point $t = 0$, and with Eq (7), $F(t,b,n) > 0, \forall t \in \mathbb{R}$, it is bounded below by value 0. □

Theorem 4. *It is a non-integrable function, such as*

$$\lim_{s\to\infty}\int_0^s F(t,b,n)dt = \infty$$

Proof. With substitution integration, $t^{2n} = x$,

$$\int_0^s F(t,b,n)dt$$

$$= \int_0^{s^{2n}} \frac{1}{2n} x^{1/2n-1} {}_2F_1\left(\frac{1}{2n}, b, \frac{1}{2n}+1, -x\right) dx$$

with $s^{2n} \to \infty$, we have Mellin transform of Gauss hypergeometric function. Let $0 < \epsilon < \epsilon_0 = \min(\frac{1}{2n}, b)$, we have next [2]15.14.1

$$\lim_{\epsilon\to\epsilon_0} \frac{1}{2n}\int_0^\infty x^{\epsilon-1} {}_2F_1\left(\frac{1}{2n}, b, \frac{1}{2n}+1, -x\right) dx = \lim_{\epsilon\to\epsilon_0} \frac{1}{2n} \frac{\Gamma(\epsilon)\Gamma(\frac{1}{2n}-\epsilon)\Gamma(b-\epsilon)}{\Gamma(\frac{1}{2n})\Gamma(b)\Gamma(\frac{1}{2n}+1-\epsilon)} \tag{20}$$

The terms on the numerator, $\Gamma(\frac{1}{2n}-\epsilon)\Gamma(b-\epsilon)$ diverge to ∞ always as $\epsilon \to \epsilon_0$ for $b > \frac{1}{2n}$. If $b < \frac{1}{2n}$ then, next identity guarantees the same result.

${}_2F_1(1/2n, b, 1/2n+1, -t^{2n})$

$$\lim_{\epsilon\to\epsilon_0}\int_1^\infty x^{\epsilon-1}{}_2F_1\left(\frac{1}{2n}, b, \frac{1}{2n}+1, -x\right)dx = \infty < \int_1^\infty x^{\frac{1}{2n}-1}{}_2F_1\left(\frac{1}{2n}, b, \frac{1}{2n}+1, -x\right)dx \tag{21}$$

□

Theorem 5 (Inequality 2). *$\forall n_1 < n_2 \in \mathbb{N}^+$ and $b > 0$, there exists threshold value b_{th} such as*

$$b_{th} = 1 + \frac{1}{2}\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \tag{22}$$

and let

1. *$b \geq b_{th}$: $\forall t,\ F(t, b, n_1) < F(t, b, n_2)$*
2. *$\frac{1}{2n_1} < b < b_{th}$: $\exists! t_{th},$ s. t. $1 < t_{th}\ \forall t > t_{th}\ F(t, b, n_1) > F(t, b, n_2)$*

Proof. Define two function f, g for the given $n_1 < n_2$ and $b > 0$

$$f(t) :={}_2F_1\left(\frac{1}{2n_2}, b, \frac{1}{2n_2}+1, -t^{2n_2}\right) - {}_2F_1\left(\frac{1}{2n_1}, b, \frac{1}{2n_1}+1, -t^{2n_1}\right) \tag{23}$$

$$g(t) :=(1+t^{2n_2})^{-b} - (1+t^{2n_1})^{-b} \tag{24}$$

The proof is enough to show that $f(t)$ has unique maximum point in $t < 1$ and only if $b < b_{th}$ there is unique minimum point in $1 < t$ and the minimum value is negative.

Since, $tf(t) = \int_0^t g(x)dx$,

$$t(g(t) - f(t)) = 0. \tag{25}$$

is enough to determine the roots of $f'(t)$, because, $t^2 f'(t) = t(g(t) - f(t))$ and we can construct only $g(t)$ and operations as,

$$\int_0^t xg'(x)dt = t^2 f'(t) = 0 \tag{26}$$

To show properties of $f(t)$ function, we need some precious properties of $g(t)$.

Lemma 1. *$\forall b > 0, n_1 < n_2$, $g(t)$ has two extreme values at point $0 < t_0 < 1 < t_1$, and they are maximum and minimum point respectively for t_0 and t_1.*

Proof.

$$g'(t) = -\frac{2b}{t}l(t) \tag{27}$$

$$l(t) = [n_2\frac{t^{2n_2}}{(1+t^{2n_2})^{b+1}} - n_1\frac{t^{2n_1}}{(1+t^{2n_1})^{b+1}}] \tag{28}$$

$l(t) = 0$ yields next equality.

$$\frac{1}{b+1}\ln(\frac{n_2}{n_1}) = \ln\left(\frac{1+t^{2n_2}}{1+t^{2n_1}}\right) - \frac{2(n_2-n_1)}{b+1}\ln(t) \tag{29}$$

The RHS diverse to $+\infty$ as $x \to 0+$ and $t \to \infty$, and has unique minimum point, t_m. The location of minimum point is $t_m > 1$ and $t_m < 1$ respectively for $b < 1$ and $b > 1$. In addition, $\ln(n_2/n_1)/(b+1)$ is positive for all $b > 0$ and $n_1 < n_2$. Therefore, in any case of b, n_1, n_2, $g'(t)$ has two roots, t_0, and t_1. In addition, $0 < t_0 < 1 < t_1$. □

Corollary 1. *$f'(t)$ has an root at $t_0 < t_{e_1} < 1$ and if $b < b_0$ than there exists an another root, t_{e_2}, and $t_1 < t_{e_2}$.*

Proof. For $0 < t < t_0$, $tg'(t) > 0 \to t^2 f'(t) > 0$, and by the definition, $g(1) = 0 = f(1) + f'(1)$ and $f(1) > 0$. Therefore, $f'(1) = -f(1) < 0$. Since, $tg'(t) < 0$ in $t_0 < t < 1$, $t^2 f'(t)$ monotonically decreases in the region. Consequently, the root of $t^2 f'(t)$, $\exists! t_e, t_0 < t_e < 1$.

${}_2F_1(1/2n, b, 1/2n+1, -t^{2n})$

For $t > 1$, limit of $t^2 f'(t)$ as $t \to \infty$ is helpful to determine the other roots.

$$\lim_{x\to\infty} t^2 f'(t) = \lim_{t\to\infty} tg(t) - tf(t) \tag{30}$$

$$= \lim_{t\to\infty} t \left\{ \frac{1 - {}_2F_1(1, b, \frac{1}{2n_2}+1, t^{2n_2}/(1+t^{2n_2}))}{(1+t^{2n_2})^b} - \frac{1 - {}_2F_1(1, b, \frac{1}{2n_1}+1, t^{2n_1}/(1+t^{2n_1}))}{(1+t^{2n_1})^b} \right\} \tag{31}$$

The asymptotic behavior of $(1 - {}_2F_1(1, b, \frac{1}{2n}+1, t^{2n_2}/(1+t^{2n})))/(1+t^{2n})^b$ at $t = \infty$ depends on the range of b. Using argument unity of Hypergeometric function http://dlmf.nist.gov/15.4.ii [2], next asymptotic equations are achieved by the cases.

1. $0 < b < \frac{1}{2n}$: $\frac{2nb}{2nb-1} t^{-2nb}$
2. $b = \frac{1}{2n}$: $t^{-1}[1 - \ln(t)]$
3. $\frac{1}{2n} < b$: $t^{-2nb} - t^{-1}\Gamma(b)^{-1}[\Gamma(\frac{1}{2n}+1)\Gamma(b - \frac{1}{2n})]$

Thus, the limit, Eq (31), have 5 cases of asymptotic equations of form, $p(t)t^{1-2n_2b} - q(t)t^{1-2n_1b} + c$. See Table 1.

Table 1: Asymptotic equation at $t = \infty$ [2]

b	$p(t)$	$q(t)$	c	Limit at $x = \infty$
$0 < b < \frac{1}{2n_2}$	$\frac{2n_2b}{2n_2b-1}$	$\frac{2n_1b}{2n_1b-1}$	0	∞
$b = \frac{1}{2n_2}$	$1 - \ln(t)$	$\frac{2n_1b}{2n_1b-1}$	0	∞
$\frac{1}{2n_2} < b < \frac{1}{2n_1}$	1	$\frac{2n_1b}{2n_1b-1}$	$-\Gamma(b)^{-1}[\Gamma(\frac{1}{2n_2}+1)\Gamma(b-\frac{1}{2n_2})]$	∞
$b = \frac{1}{2n_1}$	1	$1 - \ln(t)$	$-\Gamma(b)^{-1}[\Gamma(\frac{1}{2n_2}+1)\Gamma(b-\frac{1}{2n_2})]$	∞
$\frac{1}{2n_1} < b$	1	1	$\Gamma(b)^{-1}[\Gamma(\frac{1}{2n_1}+1)\Gamma(b-\frac{1}{2n_1})$ $-\Gamma(\frac{1}{2n_2}+1)\Gamma(b-\frac{1}{2n_2})]$	$\Gamma(b)^{-1}[\Gamma(\frac{1}{2n_1}+1)\Gamma(b-\frac{1}{2n_1})$ $-\Gamma(\frac{1}{2n_2}+1)\Gamma(b-\frac{1}{2n_2})]$

With the asymptotic equations in Table 1, we can simply express the limit of $x^2 f'(x)$ with $b_{th} = 1 + \frac{1}{2}(\frac{1}{n_1} + \frac{1}{n_2})$, as next,

$$\lim_{t\to\infty} t^2 f'(t) = \begin{cases} \infty & b \le \frac{1}{2n_1} \\ c > 0 & \frac{1}{2n_1} < b < b_{th} \\ 0 & b = b_{th} \\ c < 0 & b > b_{th} \end{cases}. \tag{32}$$

where c is an arbitrary constant. $t_1^2 f'(t_1)$ <0, because for $t_0 < t < t_1$, $t^2 f'(t)$ monotonically decrease so that there exists a root, $t_{r'}$. In addition, for $t_1 < t$, $t^2 f'(t)$ monotonically increases and its limits are well shown in Eq (32) by the values of b. Therefore, if $b < b_0$ there exists an another root, t_{e_2} where $t_1 < t_{e2}$, of $t^2 f'(t)$ and it is unique for $1 < t$.

As $f'(t)$ and $t^2 f'(t)$ are sharing roots and sign at each point, t_e, and t_{e_2} are roots of $f'(t)$ too. □

$_2F_1(1/2n, b, 1/2n+1, -t^{2n})$

The above asymptotic representation result for $\frac{1}{2n_1} < b$ can be achieved with Mellin transform of Gauss Hypergeometric function.

$$\lim_{t\to\infty} tf(t) \tag{33}$$

$$= \lim_{t\to\infty} \int_0^t (1+x^{2n_2})^{-b} - (1+x^{2n_1})^{-b} dx \tag{34}$$

$$= \lim_{t\to\infty} \int_0^t {}_2F_1(b,1,1,-x^{2n_2}) - {}_2F_1(b,1,1,-x^{2n_1}) dt \tag{35}$$

$$\because \int_0^t {}_2F_1(b,1,1,-x^{2n}) dx = \frac{1}{2n} \int_0^{t^{2n}} x^{1/2n-1} {}_2F_1(b,1,1,-x) dx \tag{36}$$

$$= \frac{\Gamma(\frac{1}{2n_2})\Gamma(b-\frac{1}{2n_2})}{2n_2\Gamma(b)\Gamma(1)} - \frac{\Gamma(\frac{1}{2n_1})\Gamma(b-\frac{1}{2n_1})}{2n_1\Gamma(b)\Gamma(1)} \tag{37}$$

$$= \Gamma(b)^{-1} \left[\Gamma(\frac{1}{2n_2}+1)\Gamma(b-\frac{1}{2n_2}) - \Gamma(\frac{1}{2n_1}+1)\Gamma(b-\frac{1}{2n_1})\right] \tag{38}$$

Now, with the above results, $f(t)$ has a maximum point at $t = t_e$, $t_e < t_0 < 1$ and if $b < b_{th}$, there exists t_{e_2} and minimum point which is bigger than 1 is unique. As, the limit of $f(t)$ for $\frac{1}{2n_1} < b$ is

$$\lim_{t\to\infty} f(t) = \begin{cases} 0+ & b \geq b_{th} \\ 0- & b < b_{th} \end{cases} \tag{39}$$

$f(t_{e_2}) < 0$ and there exists a root of $f(t)$ in $t_e < t < t_{e_2}$.

□

Observation 1. *$\forall \alpha > 1$ and $b > \frac{1}{\alpha}$, $t \to \infty$*

$$F(t, b, \frac{\alpha}{2}) \sim \Gamma(b)^{-1}\Gamma(\frac{1}{\alpha}+1)\Gamma(b-\frac{1}{\alpha})\frac{1}{t} \tag{40}$$

That is,

$$\lim_{t\to\infty} tF(t, b, \frac{\alpha}{2}) = \Gamma(b)^{-1}\Gamma(\frac{1}{\alpha}+1)\Gamma(b-\frac{1}{\alpha}) \tag{41}$$

Proof. For $\alpha > 1$ and $b > \frac{1}{\alpha}$, Mellin transform of GHF is possible as

$$\lim_{t\to\infty} tF(t, b, \frac{\alpha}{2}) = \int_0^\infty {}_2F_1(b,1,1,-x^\alpha) dx = \Gamma(b)^{-1}\Gamma(\frac{1}{\alpha}+1)\Gamma(b-\frac{1}{\alpha}) \tag{42}$$

Thus,

$$\lim_{t\to\infty} \frac{F(t,b,\frac{\alpha}{2})}{\Gamma(b)^{-1}\Gamma(\frac{1}{\alpha}+1)\Gamma(b-\frac{1}{\alpha})\frac{1}{t}} = 1 \tag{43}$$

□

In addition, we can derive well known identity, $\forall \alpha > 1$

$$\int_0^\infty \frac{1}{1+x^\alpha} dx = \frac{\pi}{\alpha}\csc(\frac{\pi}{\alpha}) \tag{44}$$

with $b = 1$ and Gamma identity.

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{45}$$

See exercise 11 of [6], Eq (44) identity can be derived with residual theorem.

${}_2F_1(1/2n, b, 1/2n+1, -t^{2n})$

Table 2: Special cases of $F(t, b, n)$

Parameters		
n	b	${}_2F_1(1/2n, b, 1/2n+1; -t^{2n})$
1	1/2	$t^{-1}\ln(t+\sqrt{1+t^2})$
1	1	$t^{-1}\arctan(t)$
-	$1/2n+1$	$(1+t^{2n})^{-1/2n}$

3.3 Special case b =1

$b = 1$ and $\alpha \in \mathbb{R}$ case of the generating integral,

$$\int_0^t \frac{1}{1+x^\alpha}dx = t\,F(t, 1, \frac{\alpha}{2}) \tag{46}$$

has been well studied by many researchers. Those results are useful to calculate $F(t, 1, \alpha/2)$ values.

The evaluation of $\alpha = 2^m, m \in \mathbb{N}$ cases were introduced by Gopalan and Ravichandran using partial fraction expansion[7]. For example, if $n = 2$ and With $\log$ and $\arctan(p(t)/q(t))$ identity,

$$\int_0^t \frac{1}{1+x^4}dt = \frac{1}{2\sqrt{2}}\left[\arctan(1+\sqrt{2}t) - \arctan(1-\sqrt{2}t) + \text{arctanh}\left(\frac{\sqrt{2}t}{1+t^2}\right)\right].$$

Palagallo and Price showed $\alpha = 2m, 2m+1, m \in \mathbb{N}^+$ and expanded the result to rational number[8].

With $\theta_j = \pi\frac{2j-1}{\alpha}$, define a function $k_\alpha(t)$ as,

$$k_\alpha(t) := -\frac{1}{\alpha}\sum_{j=1}^{m}\left[\cos(\theta_j)\log(t^2+1-2\cos(\theta_j)t) - 2\sin(\theta_j)\arctan(\frac{t-\cos(\theta_j)}{\sin(\theta_j)})\right] \tag{47}$$

$$F(t, 1, m) = \frac{1}{t}k_\alpha(t) \tag{48}$$

$$F(t, 1, m+\frac{1}{2}) = \frac{1}{t}(k_\alpha(t) + \frac{1}{\alpha}\log(t+1)) \tag{49}$$

4 Visual representation

There are some special cases of $F(t, b, n)$ function. See DLMF 15. Hypergeometric function[2].

$_2F_1(1/2n, b, 1/2n+1, -t^{2n})$

Fig 1 and 2 are plotting graphs of $_2F_1(1/2n, b, 1/2n+1; -t^{2n})$ and special case of Table 2. Fig 3 is an example plot of Thm 5 and its proof process. The implementation routine of GHF function is imported from *mpmath* library[9].

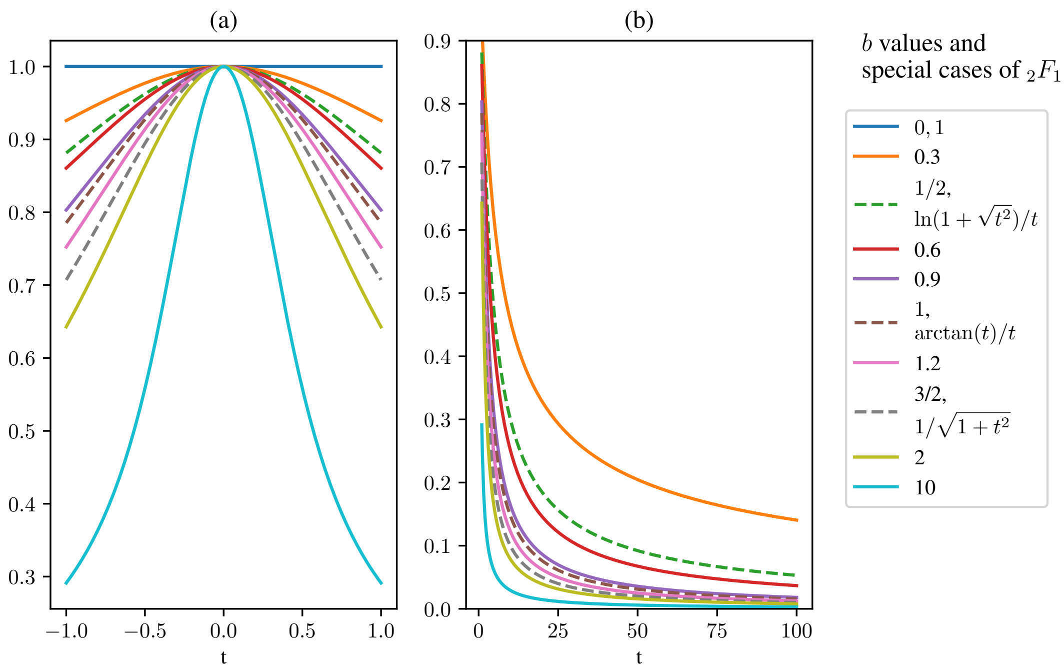

Figure 1: Plot graphs of $F(t, b, n)$ function for $b \geq 0$, $n = 1$. (a): From $t = -1$ to $t = 1$. (b): From $t = 0$ to $t = 100$. b values of the each graphs are noted in legend. After number and formula after comma in the legend are special cases of $F(t, b, n)$.

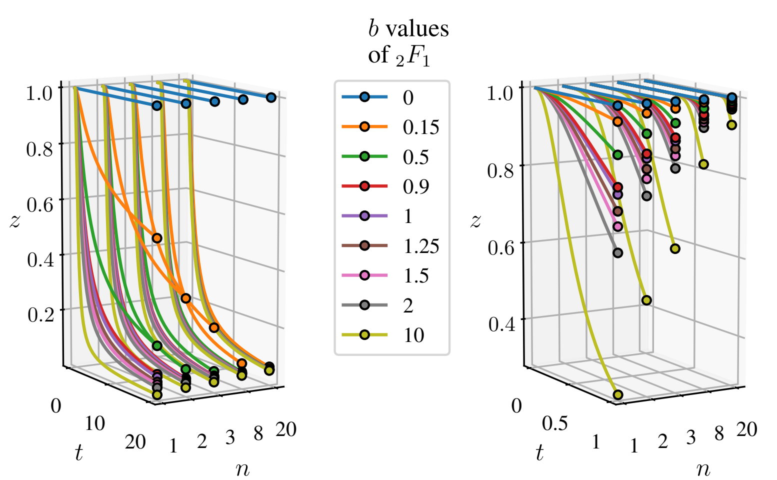

Figure 2: Plot graphs of $F(t, b, n)$ function from $t = 0$ to $t = 20$, for several $b \geq 0$ and $n = 1, 2, 3, 8, 20$.

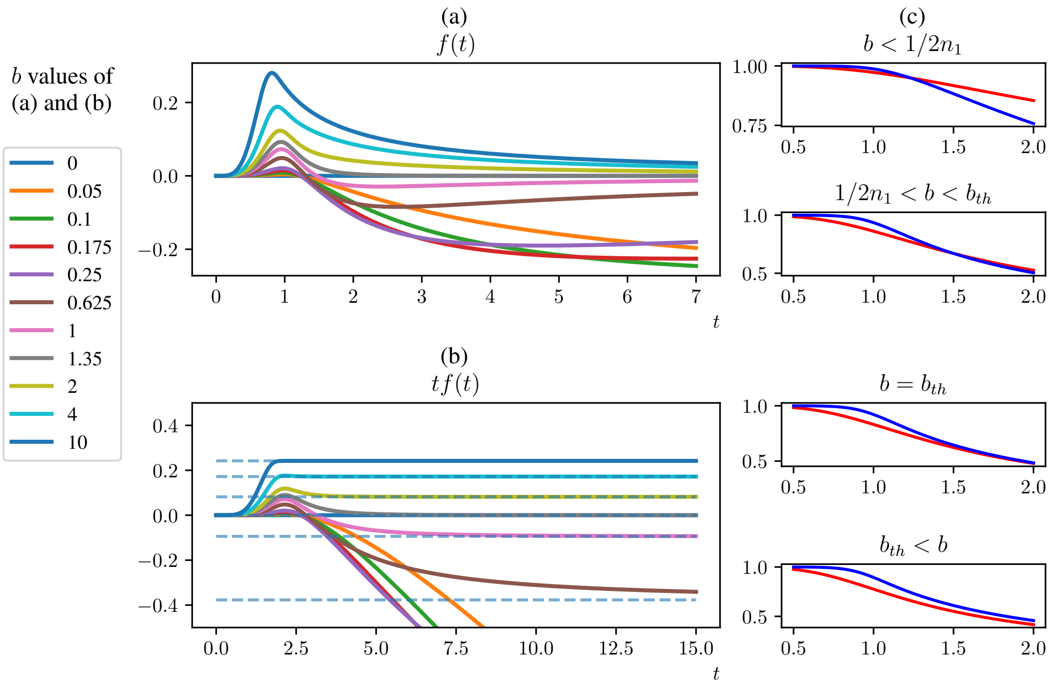

Figure 3: $n_1 = 2, n_2 = 5$ case plot. $b_{th} = 1.35$. (a):$f(t)$ graphs by b values in legends. (b): $tf(t)$ graphs and its converged values calculated with Eq (38), denoted as transparented dashed line. (c): $F(t, b, n)$ behavior by b values of each title for $n_1 < n_2$. The red line is n_1 and the blue line is n_2 graph.

5 Further Works

$n \in \mathbb{N}$ cases were studied but those works can be expanded to rational number, p/q cases, but it is a cumbersome work now.

6 Conclusion

$_2F_1(1/2n, b, 1/2n+1, -t^{2n})$ has next properties for $n \in \mathbb{N}^+, b \in \mathbb{R}^+ \cup \{0\}$,

- Monotonically decreasing function as $|t|$ increasing.
- Bounded on range $(0, 1]$.
- For the given real values, $0 \leq b_1 < b_2$, $_2F_1(1/2n, b_1, 1/2n+1; -t^{2n}) \geq {}_2F_1(1/2n, b_2, 1/2n+1; -t^{2n})$, and the equality holds true when $t = 0$.
- It is a non-integrable function.
- For $n_1 < n_2$, a function $f(t) = {}_2F_1(1/2n_2, b, 1/2n_2+1, -t^{2n_2}) - {}_2F_1(1/2n_1, b, 1/2n_1+1, -t^{2n_1})$ have maximum point on < 1. If $b < 1 + \frac{1}{2}(\frac{1}{n_1} + \frac{1}{n_2})$, there is a root and a minimum point on > 1.

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${}_2F_1(1/2n, b, 1/2n+1, -t^{2n})$

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