Order in the particle zoo V3

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Abstract

The standard model of physics classifies particles into elementary leptons and hadrons composed of quarks. In this article the existence of an alternate ordering principle will be demonstrated giving particle energies to be quantized as a function of the fine-structure constant, α. The quantization can be derived from the relationship of a point charge and a photon representation of energy. Necessary input parameters are square of the elementary charge divided by electric constant and one model specific constant. The value of α itself can be approximated numerically by the gamma functions of the integrals involved.

1 Introduction

Particle zoo is the informal though fairly common nickname to describe what was formerly known as "elementary particles" [1]. The standard model of physics [2] divides these particles into leptons, considered to be the fundamental "elementary particles" and the hadrons, composed of two (mesons) or three (baryons) quarks. Well hidden in the data of particle energies lies another ordering principle which can be derived by interpreting particles as electromagnetic objects. There exist numerous attempts to calculate particle energy with electromagnetic models going back as far as 1881 with the work of J.J.Thomson [3 4]. In the work presented here, to obtain quantifiable results, the electromagnetic field will be modified with an appropriate exponential function, $\Psi(r, e^2/\varepsilon, \rho, \alpha)$, serving as probability amplitude of the field, with r = distance from origin, e = elementary charge, ε = electric constant and ρ = model specific constant. The two integrals needed to calculate energy in point charge and photon representation exhibit the following two relations:

1) Their product - resulting from energy conservation - is characterized by containing the product of the two gamma functions $\Gamma(1/3)|\Gamma(-1/3)| \approx \alpha^{-1}/(4\pi)$,

2) their ratio features a quantization of energy states with powers of $1/3^n$ over some base α_0, a relation that can be found in the particle data with $\alpha_0 = \alpha$ as:

$$W_n/W_e = 1.509\left(y_l^m\right)^{-1/3}\Pi_{k=0}^{n}\alpha\text{^}(-1/3^k) \qquad (1)$$

with n = {0;1;2;..}, W_e = energy of electron, W_n= energy of particle n and y_l^m being a function of the spherical harmonics. For spherical symmetry $y_0^0 = 1$ holds, corresponding particles are e, μ, η, p/n, Λ, Σ and Δ. The factor 1.509 is related to angular momentum |J| = 1/2.

The particles are interpreted as some kind of standing electromagnetic wave originating from a rotating electromagnetic field with the E-vector pointing towards the origin. Neutral particles are supposed to exhibit nodal planes and corresponding equal volume elements of opposite polarity. The terms for calculating energy do not distinguish between charged and neutral particles and have to be considered a first approximation, accurate only within order of magnitude of the spread of energies of particle families. Typical relative error of calculated parameters compared to experimental values is in a range of ± 0.01, within the same range the approximations made below are valid.

Many details of the model still have to be worked out, yet the basic equations presented below are considered sufficient proof for relation (1) to be more than mere coincidence.

2 Results

2.1 Calculation of energy - point charge

To calculate particle energies the integral over the electrical field E of a point charge is used as a first approximation. However, it can not be expected that the expression derived from Coulomb's law for two interacting particles can be used unaltered and it will be demonstrated below that a factor 4π is needed as modification to yield a half integral angular momentum, giving:

$$W_{Coul,n} = 4\pi \int_0^\infty \varepsilon_0 E(r)^2 d^3 r = 4\pi \int_0^\infty \frac{e^2}{4\pi\varepsilon_0 r^2} dr = 4\pi b_0 \int_0^\infty r^{-2} dr \tag{2}$$

with $b_o = e^2/(4\pi\varepsilon_0)$ used for brevity.
The field E is modified with a function

$$\Psi(r) = \exp(-\{(\sigma\,\tau\,b_0^2\,r^{-3}) + [(\sigma\,\tau\,b_0^2\,r^{-3})^2 - 4\,\tau\,b_0^2\,r^{-3}]^{0,5}\}/2) \tag{3}$$

The first term, $\exp(-\sigma\,\tau\,b_0^2\,r^{-3})$, avoids divergence of the E-field for r −> 0, the part in square brackets provides an integration limit, r_l, where the root term equals zero. r_l of particle n can be given by:

$$r_{l,n} = (\sigma^2 \tau_n b_0^2/4)^{1/3} \tag{4}$$

providing a boundary condition for the problem.
Coefficient σ is a constant (σ = 1.76E+8[-]) related to constant angular momentum J (see below), τ is a parameter representing particle energy, $\tau_n \sim W_n^{-3}$. The coefficient τ_{n+1} of a particle can always be expressed by a term multiplying the coefficient of its predecessor n (defined in this work by $W_n < W_{n+1}$) with a parameter $\alpha_{\tau,n+1}$: $\tau_{n+1} = \tau_n\,\alpha_{\tau,n+1}$. In general for the coefficient of particle n a partial product is formed relative to a reference particle, chosen here to be the electron, τ_e (electron coefficient τ_e = 1.68E+6 [m/J^2]):

$$\tau_n = \tau_e \Pi_{k=0}^{n} \alpha_{\tau,k} = \tau_e \Pi_{\tau,n} \tag{5}$$

In all integrals over Ψ(r) given below equ. (6) may be used as approximation for (3) up to $r = r_l$ with relative error << 0.01:

$$\Psi_n(r < r_l) \approx \exp(-\sigma\,\tau_n\,b_0^2\,r^{-3}) = \exp(-\beta_n/2\,r^{-3}) \tag{6}$$

where $\beta_n = 2\,\sigma\,\tau_n\,b_0^2$ is used for brevity. The factor 2 takes into account, that Ψ(r) appears squared in the integrals below.
There are four closely related integrals over the approximation of Ψ(r) according to equ. (6) that are of interest to the problem:

$$\int^{r_l} \Psi(r)^2 r^{-(m+1)} dr = \Gamma(m/3,\ \beta/r_L^3)\ \beta^{-m/3}/3 \tag{7}$$

with m = {-1;0;1;2;}. The term $\Gamma(m/3,\ \beta/r_l^3)$ denotes the upper incomplete gamma function, given by the Euler integral of the second kind with s = m/3 and $x = \beta/r_l^3$ as lower integration limit:

$$\Gamma(s,x) = \int_x^\infty t^{s-1} e^{-t}\, dt \tag{8}$$

It follows from the boundary condition (4) that the integration limit $x = \beta/r_l^3$ has to be a constant for all particles:

$$\beta_n/r_{l,n}^3 = 2\sigma\tau_n b_0^2/\,r_{l,n}^3 = 8/\sigma \tag{9}$$

For m = {1;2} $\Gamma(m/3,\ \beta/r_l^3))$ −> Γ(m/3) gives a sufficient approximation for the equations of interest here and will be used below. For m = {-1;0} the integrals (7), (8) depend critically on the integration limit and have to be integrated numerically.
The integral for m = 1 is needed to calculate $W_{Coul,n}$. Inserting (6) and (7) in equ. (2) will turn out:

$$W_{Coul,n} = 4\pi \int_0^\infty \varepsilon_0 E(r)^2 \Psi_n(r)^2 d^3r = 4\pi b_0 \int_0^{r_{l,n}} \Psi_n(r)^2 r^{-2} dr = 4\pi\, b_0\, \Gamma(1/3)\, \beta_n^{-1/3}/3 \quad (10)$$

Equation (10) is the source of $\tau_n \sim W_n^{-3}$. From (5) and (10) follows:

$$\tau_n/\tau_e = \Pi_{k=0}^{n} \alpha_{\tau,k} = \Pi_{\tau,n} = \Pi_{k=0}^{n} \alpha_{W,k}^{-3} = \Pi_{W,n}^{-3} \quad (11)$$

with $\alpha_{W,k}$ being coefficients of a general partial product $\Pi_{W,n}$ for particle energies. Through equ. (4) the relations $\tau_n \sim r_{l,n}^3$ and $W_n \sim r_{l,n}^{-1}$ hold.

The factor 4π added in equ. (2) may be derived by applying a semi-classical approach for angular momentum J, using J = r x p(r) = r $W_n(r)/c_0$ (assuming $W_{kin,n} = 1/2\, W_n$):

$$|J| = \int_0^{r_{l,n}} J_n(r) dr = 4\pi \frac{b_0}{c_0} \int_0^{r_{l,n}} \Psi_n(r)^2 r^{-1} dr \quad (12)$$

From (7), (8) follows for m = 0:

$$\int_0^{r_{l,n}} \Psi(r)^2 r^{-1} dr = 1/3 \int_{8/\sigma}^{\infty} t^{-1} e^{-t} dt = 5.447 \approx \alpha^{-1}/8\pi \quad (13)$$

yielding the constant $\alpha^{-1}/8\pi$ for all particles. Inserting (13) in (12) provides a half integer angular momentum, |J| = 1/2:

$$|J| = 4\pi \frac{b_0}{c_0} \frac{\alpha^{-1}}{8\pi} = 1/2\ [\hbar] \quad (14)$$

Analogous to the postulate for neutral particles to be composed of volume elements of opposite charge, integer spin particles as well as particles with J = 3/2, etc. are supposed to be composed of a combination of half integer contributions of angular momentum J = ± 1/2, adding up accordingly.

2.2 Calculation of energy - photon

For m = -1 equations (7), (8) give a relation between radii and Euler-integral:

$$r_{x,n} = \int_0^{r_{x,n}} \Psi_n(r)^2 dr = \beta_n^{1/3}/3 \int_{\beta/r_{x,n}^3}^{\infty} t^{-4/3} e^{-t} dt \quad (15)$$

Using the value of the Compton wavelength, λ_C, in the term for the energy of a photon gives hc_0/λ_C. With equ. (15) λ_C can be given by:

$$\lambda_{C,n} = \int_0^{\lambda_{C,n}} \Psi_n(r)^2 dr = \beta_n^{1/3}/3 \int_{\beta/\lambda_{C,n}^3}^{\infty} t^{-4/3} e^{-t} dt \approx \beta_n^{1/3}/3\ \ 18\pi \left| \Gamma(-1/3) \right| \quad (16)$$

According to (10) particle energy is proportional to $\beta_n^{-1/3}$ and $\lambda_{C,n} \sim \beta_n^{1/3}$ has to hold, requiring the lower integration limit of the Euler integral and the factor ≈ 18π to be a constant for all particles. Energy of a photon can be expressed by:

$$W_{Phot,n} = hc_0/\lambda_{C,n} = \frac{hc_0}{\int^{\lambda_{C,n}} \Psi_n(r)^2 dr} = \frac{3hc_0}{18\pi |\Gamma(-1/3)| \beta_n^{1/3}} \quad (17)$$

2.3 Relation of integrals for $W_{Coul,n}$ and $W_{Phot,n}$ with α

The energy of a particle has to be the same in both photon and point charge description. From (10) and (17) follows:

$$W_{Coul,n} = W_{Phot,n} = 4\pi b_0\, \Gamma(1/3)\, \beta_n^{-1/3}/3 = \frac{3\,hc_0}{18\,\pi|\Gamma(-1/3)|\beta_n^{1/3}} \quad (18)$$

which my be rearranged to emphasize the relationship $\Gamma(1/3)\,|\Gamma(-1/3)| = 2.679 \cdot 4.062 = 0.998\, \alpha^{-1}/(4\pi) \approx \alpha^{-1}/(4\pi)$ and expanded by 2π to transform h into ħ, giving:

$$\Gamma(1/3)\,|\Gamma(-1/3)| \approx \alpha^{-1}/4\pi = \frac{2\pi\, 9\, hc_0}{2\pi\, 4\pi\, 18\pi\, b_0} = \frac{\hbar c_0}{4\pi\, b_0} \quad (19)$$

2.4 Coefficient 1.509 and related parameters

It is unclear if equation (19) can be used to directly link α with the quantization condition given in (1). However, the first term in (1), $W_\mu/W_e = 206.8 = 1.509\, \alpha^{-1}$ is within the accuracy of the calculations identically to the factor determing the integration limit, $1.501\, \alpha^{-1} \approx 1.5\, \alpha^{-1}$, being a key factor related to |J| = 1/2.

According to equation (15) $r_{l,n}$ may be given by :

$$r_{l,n} = \int_0^{r_{l,n}} \Psi(r)^2 dr = \beta_n^{1/3}/3 \int_{8/\sigma}^{\infty} t^{-4/3} e^{-t}\, dt \approx 1.501\, \alpha^{-1}\,|\Gamma(-1/3)|\, \beta_n^{1/3}/3 \quad (20)$$

Consequently the equivalent term from (1) will cancel in the expression for $r_{L,\mu}$ (note: $W_n \sim 1/r_{l,n}$) :

$$r_{l,e} \approx 1.5\, \alpha^{-1}\,|\Gamma(-1/3)|\, \beta_e^{1/3}/3 \quad (21)$$

$$r_{l,\mu} \approx 1.5^{-1}\, \alpha^{+1}\,[1.5\, \alpha^{-1}\,|\Gamma(-1/3)|\, \beta_e^{1/3}/3\,] = |\Gamma(-1/3)|\, \beta_e^{1/3}/3 = 1.5\, \alpha^{-1}\,|\Gamma(-1/3)|\, \beta_\mu^{1/3}/3 \quad (22)$$

Assuming an identity of both terms, the value for $W_\mu/W_e = 1.509\, \alpha^{-1}$ will be used in all calculations as least biased value for $\approx 1.5\, \alpha^{-1}$, see discussion section. The coefficient σ is related to factor $1.509\, \alpha^{-1}$ by equ. (9) and (20) to be:

$$\sigma = 8\, r_{l,n}^3 / \beta_n = 8\,(1.509\, \alpha^{-1}\,|\Gamma(-1/3)|\,/3)^3 = 1.76\text{E+8}[-] = 68.3\, \alpha^{-3}\,[-] \quad (23)$$

Coefficients $1.5\, \alpha^{-1}$ and σ are part of the terms setting the integration limits in equ. (13), determining the value of J=1/2.

In analogy to σ the coefficient τ_e will be defined as

$$\tau_e = \rho\, 1.509^3\, \alpha^{-3} = \rho\, 3.4\, \alpha^{-3} \quad , \quad (24)$$

the coefficient ρ being the dimension bearing remainder. The actual value used in this work is obtained from a least square fit of energies of particles of the y_0^0 group, $\rho = 0.193\ [m/J^2]$.

2.5 Quantization with powers of $1/3^n$ over α

To find a source for the quantization with powers of $1/3^n$ over α the ratio of the integrals used in (10) and (17) for the point charge and photon representation of energy may be examined.

$$Q(\psi_n) = \frac{\int_{\lambda_{C,n}}^{r_{l,n}} \Psi_n(r)^2 r^{-2} dr}{\int \Psi_n(r)^2 dr} = \frac{\Gamma(1/3)}{18\pi\ \Gamma(-1/3)\beta_n^{2/3}} \sim \frac{\Gamma(1/3)}{\Gamma(-1/3)} \frac{\alpha_{\tau,0}^{1/3}\alpha_{\tau,1}^{1/3}.....\alpha_{\tau,n}^{1/3}}{\alpha_{\tau,0}\alpha_{\tau,1}....\alpha_{\tau,n}} \tau,0 \tag{25}$$

with n = {0;1;2;..}. The term given by (25) is related to the boundary condition (4) (see discussion) and via (10) and (17) to the square of particle energy $W_n^2 \sim \tau_n^{-2/3}$. The last expression of (25) is obtained by expanding $\Pi_{\tau,n}^{-2/3}$ of $\beta_n^{-2/3}$ with $\Pi_{\tau,n}^{1/3}$ From this term it is obvious that a relation $\alpha_{n+1} = \alpha_n^{1/3}$ such as in equation (1) yields a distinct solution for $Q(\psi_n)$, $Q(\psi_n)$ being a function of coefficient α_n and α_0 only. By comparison with experimental data $\alpha_{\tau,0}$ can be identified as $\alpha_{\tau,0} = \alpha^3$ and $Q(\Psi_n)$ can in general be given by (n = {0;1;2;..}):

$$Q(\psi_n) \sim \frac{\Gamma(1/3)}{\Gamma(-1/3)} \frac{\alpha^1\alpha^{1/3}\alpha^{1/9}....\alpha\wedge(1/3^n)}{\alpha^3\alpha^1\alpha^{1/3}....\alpha\wedge(3/3^n)} = \frac{\Gamma(1/3)}{\Gamma(-1/3)}\alpha\wedge(1/3^n)/\alpha^3 \tag{26}$$

where all intermediate particle coefficients cancel out.
All other particle parameters of successive particles have to be related by a corresponding 1/3rd power relationship as well. Equation (5) turns into (n = {0;1;2;..}):

$$\tau_n = \tau_e\, 0.291 \quad \Pi_{k=0}^{n}\alpha\wedge(3/3^k) = \rho\alpha^{-3}\Pi_{k=0}^{n}\ \alpha\wedge(3/3^k) = \rho\,\alpha^{-3}\,\Pi_n \tag{27}$$

The factor $0.291 = 1.509^{-3}$ has to be taken from the experimental W_μ / W_e ratio.

2.6 Extension to non-spherical symmetry

Up to here only spherical symmetry is considered, introduced through equ. (2), (10). For a simple test if the model might be extendible to other symmetries equ. (26) is used. The integral over r^{-2} in $Q(\Psi_n)$ actually represents a volume integral, the factor 4π being included in equ. (2), (10) and thus implicitly in all related terms and coefficients. For non-spherically symmetric states an appropriate spherical harmonic factor, y_l^m, should be added to equ. (26), given by the integral over non-normalized spherical harmonics i.e. the inverse of the square of the normalization factor N_l^m, corrected by 4π:

$$y_l^m = \frac{1}{4\pi}\int P_l^m\cos(\vartheta)e^{im\varphi}P_l^m\cos(\vartheta)e^{-im\varphi}\sin(\vartheta)\,d\vartheta d\varphi = \frac{1}{4\pi(N_l^m)^2} \tag{28}$$

turning relation (26) into

$$Q(\Psi_n) \sim \quad y_l^m\,\alpha\wedge(1/3^n)/\alpha^3 \tag{29}$$

For the second spherical harmonic this gives y_1^0 = 4π /(4π3) = 1/3, providing a second set of particle coefficients which is given by the coefficients according to (27) divided by 3. Table 1 shows results for y_0^0, y_1^0 relative to experimental values in col. 5. These are calculated according to equ. (10) using the coefficients of col. 4 in β_n. Relative energy values calculated by equ. (1) with the coefficients of col. 3 would be shifted by + 0.003 due to the electron becoming a reference particle.
For the transition from y_0^0 to y_1^0 the factor 1/3 in the coefficients τ (col. 4) appears as $3^{-1/3}$ =1.44 in the coefficients for energy ratio (col. 3). A change in angular momentum is expected for this transition which is actually observed with ΔJ = ± 1 except for the pair μ/π with Δ J = 1/2.
Included is a particle energy derived by expanding the model to energies below the electron with a coefficient of α^3 in equ. (1): $W_v / W_e = 1.509\,\alpha^3$. This gives a state with energy 0.3eV (for y_0^0) which is in a range expected for a neutrino [6].

	$W_{n,Lit}$ [MeV]	$\Pi_{k=0}^{n}$ α^(-1/3^k) equ (1)	$\tau/\rho = \alpha^{-3} * \Pi_{\tau,n}$ equ (27)	W_{calc}/W_{Lit} equ(10, 27)	J
ν	**3E-7 ***	$\boldsymbol{\alpha^{+3}}$	$\boldsymbol{\alpha^{-3}\alpha^{-9}}$	**-**	**1/2**
e+-	**0.51**	Reference	$\boldsymbol{\alpha^{-3}}$	**0.997**	**1/2**
μ+-	**105.66**	$\boldsymbol{\alpha^{-1}}$	$\boldsymbol{\alpha^{-3}\alpha^{3}}$	**0.997**	**1/2**
π+-	*139.57*	*1.44* α^{-1}	$\alpha^{-3}\alpha^{3}/3$	*1.089*	*0*
K	495				0
η 0	**547.86**	$\boldsymbol{\alpha^{-1}\alpha^{-1/3}}$	$\boldsymbol{\alpha^{-3}\alpha^{3}\alpha^{1}}$	**0.992**	**0**
ρ0	*775.26*	*1.44* $(\alpha^{-1}\alpha^{-1/3})$	$(\alpha^{-3}\alpha^{3}\alpha^{1})/3$	*1.011*	*1*
ω0	*782.65*	*1.44* $(\alpha^{-1}\alpha^{-1/3})$	$(\alpha^{-3}\alpha^{3}\alpha^{1})/3$	*1.001*	*1*
K*	894				1
p+-	**938.27**	$\boldsymbol{\alpha^{-1}\alpha^{-1/3}\alpha^{-1/9}}$	$\boldsymbol{\alpha^{-3}\alpha^{3}\alpha^{1}\alpha^{1/3}}$	**1.000**	**1/2**
n	**939.57**	$\boldsymbol{\alpha^{-1}\alpha^{-1/3}\alpha^{-1/9}}$	$\boldsymbol{\alpha^{-3}\alpha^{3}\alpha^{1}\alpha^{1/3}}$	**0.999**	**1/2**
η'0	958				0
Φ'0	1019				1
Λ0	**1115.68**	$\boldsymbol{\alpha^{-1}\alpha^{-1/3}\alpha^{-1/9}\alpha^{-1/27}}$	$\boldsymbol{\alpha^{-3}\alpha^{3}\alpha^{1}\alpha^{1/3}\alpha^{1/9}}$	**1.009**	**1/2**
Σ0	**1192.62**	$\boldsymbol{\alpha^{-1}\alpha^{-1/3}\alpha^{-1/9}\alpha^{-1/27}\alpha^{-1/81}}$	$\boldsymbol{\alpha^{-3}\alpha^{3}\alpha^{1}\alpha^{1/3}\alpha^{1/9}\alpha^{1/27}}$	**1.003**	**1/2**
Δ	**1232.00**	$\boldsymbol{\alpha^{-3/2}}$	$\boldsymbol{\alpha^{-3}\alpha^{9/2}}$	**1.001**	**3/2**
Ξ	1318				1/2
*Σ*0*	*1382.80*	*1.44* $(\alpha^{-1}\alpha^{-1/3}\alpha^{-1/9})$	$(\alpha^{-3}\alpha^{3}\alpha^{1}\alpha^{1/3})/3$	*0.978*	*3/2*
Ω-	*1672.45*	*1.44* $(\alpha^{-1}\alpha^{-1/3}\alpha^{-1/9}\alpha^{-1/27})$	$(\alpha^{-3}\alpha^{3}\alpha^{1}\alpha^{1/3}\alpha^{1/9})/3$	*0.971*	*3/2*
tau+-	*1776.82*	*1.44* $(\alpha^{-3/2})$	$(\alpha^{-3}\alpha^{9/2})/3$	*1.001*	*1/2*

Table 1: Particles up to tau energy; calculated values for $\mathbf{y_0^0}$ **(bold)**, y_1^0 *(italic)* ; col. 2: energy values from literature [5] except *: calculated from model; Exponent of -3/2, 9/2 for Δ and tau is equal to the limit of the partial products in (1) and (27);

The wave function character of Ψ(r) in the model has potential for quantitative description of other particle properties. Calculation for angular momentum has been demonstrated above. Using $m = e \pi r^2 / T$ (period $T = 2 \pi r/c_0$) with $r = r_{l,n}$ as simple approximation for the absolute value of the magnetic moment, m_n

$$|m_n| = 1/2 \; e \; c_0 \; r_{l,n} \tag{30}$$

gives the values in tab 2.

	|m| Calc [Am²]	m Lit [Am²]	|m| Calc/|m| Lit
e+-	2.11E-22	-9.28E-24	22.74
μ+-	1.04E-24	-4.49E-26	23.17
p+-	1.16E-25	1.41E-26	8.22
n	1.16E-25	-9.66E-27	11.99

Table 2: Absolute values calculated for magnetic moment [5]

3 Discussion

3.1 Relation to standard model and classical quantum mechanics

The model presented derives its inspiration more from electrodynamics and quantum mechanics than from the quarks of the standard model and does not directly reproduce the classification into leptons, mesons and baryons of the latter. Mesons constitute a separate group of particles due to their integer angular momentum

which is considered to be a combination of half-integer contributions in both models. However, there is no obvious difference in the group of particles identified as leptons and baryons to be found on the present level of understanding of this model and the tentatively assigned y_0^0 and y_1^0 groups each include all three particle types of the standard model. Any rigorous analysis of symmetry properties and an in-depth comparison with the standard model requires more detailed information about the wave function used here, which in turn requires a differential equation providing an exact solution for Ψ(r, ϑ, φ). This subject is still under research.
The relation of this model to classical quantum mechanics may be given by interpreting Ψ(r) as probability amplitude applied to a field instead of a particle. This implies that concepts such as orthonormalization and calculation of eigenvalues may not be applicable on the level of the differential equation. Properties have to be calculated by integration over the spatial extent of the field.
The quantization condition itself is not exclusive. The special solution of (26) coincides with the rest mass of particles of sufficiently high mean lifetime to be experimentally observable but does not prohibit the existence of particles with any other mass.
As for the number of parameters needed to calculate energy states the model resembles the simplicity of basic quantum mechanical models, relying essentially on $4\pi\, b_0 = e^2/\varepsilon$ and J = 1/2 to yield the expression (1). The second parameter ρ is needed to transform the relative energy scale of (1) into an absolute one.

3.2 Boundary condition

Equation (26) features not only the 1/3rd power relationship characteristic for all particle parameters but also the inverse relation of coefficients with the first/reference coefficient as well as a coefficient ≈ 1.5 in form of the ratio $|\Gamma(-1/3)| / \Gamma(1/3) = 1.516$. A relation with the boundary condition (4) is given by replacing $r_{l,n}$ in equation (9) by r, multiplying with $\Psi_n(r)^2$ and integrating, yielding the following term (left side):

$$\beta_n \int_0^\infty \Psi_n(r)^2 r^{-3} dr \quad = \quad \frac{\Gamma(2/3)\beta_n}{3(\beta_n)^{2/3}} \quad = \quad \frac{\Gamma(2/3)\beta_n^{1/3}}{3} \tag{31}$$

where the integral $\int\Psi(r)^2 r^{-3}\, dr$ of (31) is directly proportional to $Q(\Psi_n)$, equ. (25), via the term $\beta_n^{-2/3}$. Since $Q(\Psi_n) \sim \alpha_{\tau,n+1}$ equ. (31) is proportional to $\Pi_{\tau,n}\, \alpha_{n+1} = \Pi_{\tau,n+1}$ and may be used to calculate particle coefficients τ_{n+1}.
The integral over the right side of (9) gives:

$$\frac{8}{\sigma}\int_0^{r_x} \Psi(r)^2 dr \quad = \quad \frac{8}{\sigma}\beta_n^{1/3}/3 \int_x^\infty t^{-4/3} e^{-t}\, dt \quad = \quad \beta_n^{1/3}/3 \int_y^\infty t^{-4/3} e^{-t}\, dt \quad = \quad \frac{1}{3}\Gamma(-1/3)\beta_n^{1/3}/3 \tag{32}$$

To match (31) the integration limit has to be adapted accordingly by either replacing the limit 8/σ of equ. (20) with the limit $x \approx 1/\sigma^{3}$ or y ≈ 1. The term on the right results from comparison with the right term of equ. (31) using the relation $|\Gamma(-1/3)| = 3\,\Gamma(2/3)$. Setting $\beta_n = \beta_e$ basically reproduces the inverse relation of equ. (22), i.e. for any given particle parameter τ_n equation (31) produces the particle radius $r_{l,n+1}$ of the next particle.
The various relationships between the terms given above as well as their significance are not completely understood and subject of further research. A particular simple interpretation may be given, considering that the ratio $r_{l,n} / r_{l,n+1}^3$ is constant, which gives using (4):

$$r_{l,n} / r_{l,n+1}^3 = (\sigma\, \beta_e\, \Pi_{\tau,n} /8)^{1/3}) / (\sigma\, \beta_e\, \Pi_{\tau,n+1} /8) = \text{const} \tag{33}$$

To be valid for all n this implies $\Pi_{\tau,n} \in \Pi_{\tau,n+1} \vee \Pi_{\tau,n}^{1/3} \in \Pi_{\tau,n+1}$. Since $W_{n+1}^3 / W_n \sim \lambda_{C,n} / \lambda_{C,n+1}^3 \sim r_{l,n} / r_{l,n+1}^3$ this result is a restatement of the relations given above though suggesting that some geometrical interpretation in r- or k-space might be conceivable.

3.3 Accuracy

The values calculated for y_0^0 agree within ± 0.01 with experimental data. There are two major causes preventing a significant improvement of accuracy.
1) Especially in the case of particle families effects on top of the relations given in this work have to play a

role to explain different energy levels for differently charged particles. This limits accuracy and the possibility to precisely identify candidates for calculated energies (e.g. both ρ^0 and ω^0 are given for 1.44 $\alpha^{-1}\alpha^{-1/3}$ in tab. 1).
If possible, particles chosen for y_0^0 in table 1 are of charge ± 1. In cases such as Σ with three energy levels, the intermediate energy level is chosen. For y_1^0 particles of the same charge as their y_0^0 equivalent are preferred in table 1.
Remaining particles in table 1 may be explained by higher excitation or linear combinations of lower states. At the present level of understanding and accuracy of the model it is considered too speculative to attempt to assign additional particle states.
Conversely, energy states belonging to higher terms of the y_0^0, y_l^0 partial products may be missing an identifyable experimental counterpart. The next y_0^0 particle following Σ^0 is expected at 1217 MeV, the next y_l^0 particle following Ω^- is expected at 1726 MeV with J = 3/2. At least for the latter there exists a resonance at 1720 MeV with J = 3/2 [7] as possible candidate.
2) The second effect is due to ambiguity in fitting model parameters to experimental values. The results presented in this article are calculated using 1.509 α^{-1} as value for ≈ 1.5 α^{-1} originating from direct experimental data of the energy ratio of μ and e. This value is used to calculate σ via equ. (23). Parameter ρ is calculated using a least square fit of energies of y_0^0 particles using equ. (10). Replacing the approximation (7) with the exact term (3) in equation (10) or choosing other sets of fitting particles may change results by roughly ± 0.01.
All procedures of this kind, i.e. fitting only energies with the parameter ρ seem to give systematically low values of $|J| \approx 0.998/2$ [ħ] (calculated numerically with appropriate parameter set). To obtain exact values for both energy and momentum requires a fit of both σ and ρ yielding a slightly higher value for r_l. Relative errors of W_e and J significantly lower than ± 0.001 may be achieved with a parameter set of σ ≈ 1.83E+8[-] and ρ ≈ 0.181 [m/J^2]. As a consequence equ. (9) does not hold exactly, integration limits and values of the Euler integrals change slightly, see below.

3.4 Approximation for the value of α

Equation (19) uses three approximations, calculated below with the standard parameter set and the values from the σ, ρ fit as given in 3.3 in brackets:
1) Γ(1/3) is used in place of the incomplete Γ-function $\Gamma(1/3, \beta/ r_l^3) = 0.9960\Gamma(1/3)$ (0.9960)
2) the approximation for $\alpha^{-1}/(8\pi)$ in equ. (13) requires a correction factor of 0.9981 (0.9993) for 4π in the equation for $W_{Coul,n}$ if the experimental value of α is used.
3) For the integration limit $\beta_n/r_{x,n}^3 \ll 0$ the result of the Euler integral in (15) is approximately given by

$$\int_{\beta_n/r_{x,n}^3}^{\infty} t^{-4/3} e^{-t}\, dt \approx 3\, (\beta_n/r_{x,n}^3)^{-1/3} \tag{34}$$

Inserting this in equ. (16) gives the identity $\lambda_{C,n} = (\beta_n^{1/3}/3)\,(3\,\lambda_{C,n}/\beta_n^{1/3})$ yielding $3\,\lambda_{C,n}/(\beta_n^{1/3}\,\Gamma(1/3)) = 56.87 = 1.0057\,(18\pi)$ as approximation for 18π.

All three factors add up to change the remaining inequality of (19) from 0.9980 to 0.9978 (0.9990). Calculation errors, approximation residuals as well as possible higher order correction terms of e.g. QED type have to be considered to contribute to the remaining discrepancy.

3.5 Other applications

Apart from calculating properties of elementary particles the model might have some other useful applications.

Using the equations above to calculate energies of Dirac magnetic monopoles[7] is straightforward, replacing e by the magnetic charge e_m

$$e_m = e/(2\alpha) \tag{35}$$

turns b_0 into b_m. The integral (13) yields only minor variations even when changing input parameters by several orders of magnitude. This indicates the product $4\pi b_0 = x b_m$ has to be essentially a constant to provide half integer spin. The proportionality $\lambda_{C,n} \sim \beta_n^{1/3}$ has to be applicable for magnetic monopoles as well, yielding the same factor 18π in (16). As a result equ. (19) holds for both electric and magnetic monopoles. Using the same coefficients τ_n according to equ. (27) as for electric monopoles in equ. (10) would leave $(2\alpha)^{4/3} = 1/280$ as ratio between electric and magnetic particle energies placing the latter approximately in the same energy range as their electric counterparts.

The model should be applicable in describing non-Coulomb particle-particle interaction.

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LibreCalc files:

1) Numerical calculation of particle energies:
Num Calc W.ods
http://doi.org/10.5281/zenodo.570158
2) Results of tables:
Results.ods
http://doi.org/10.5281/zenodo.570159
3) Numerical calculation of Euler integrals:
Euler.ods
http://doi.org/10.5281/zenodo.570160