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A STUDY OF STRUCTURE OF GRAPH INVARIANT WITH THEIR INDEPENDENCE NUMBERS AND CAYLEY GRAPHS

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ABSTRACT

In this paper the independence numbers and algebraic properties of graph invariant has been discussed and studied. The graph invariant is generalization of Petersen graphs whose independence numbers have been previously known. The authors studied the bounds for the independence number of different graph invariant and sub-classes of graph invariant, and exactly determine the independence number for other graph invariant and sub-classes of graph invariant. Authors also analysed the automorphism groups of the graph invariant.

1. INTRODUCTION:

Graph Theory is a relatively different branch of mathematics, but the study of objects and their relationships to each other is perhaps universal. It is therefore no surprise that the mathematical laws and theorems that have flowered from the study of graphs have wide applications. Indeed, the subject is so large that it is quite difficult to generate broad sweeping theorems and axioms as can often be found in other disciplines. It is in this sense that focusing on specific classes of graphs and their properties becomes essential to the progression of our knowledge in graph theory. In this spirit, this paper focuses on the graph invariant, and to a lesser extent, other generalizations of the Petersen graph. The object of this paper is to study the graphs collectively known as graph invariant, both in terms of the independence number of these graphs, and in terms of the Cayley graphs that represent their automorphism groups. The graph invariant themselves are in fact a generalization of a very famous graph in graph theory, the Petersen graph. In many cases, the Petersen graph provides an important counter example for graph theoretical conjectures. This has given the Petersen graph an important place over time as a tool by which new graph theoretical conjectures may be tested. The graph itself is relatively simple, it consists of ten vertices arranged as two five-cycles attached to one another at each of their vertices. In its normal embedding in the plane there is an inner cycle and an outer cycle, with the inner cycle forming the shape of a five-sided star. In this fashion we observe that in the "outer cycle" every consecutive vertex is connected by an edge and in the "inner cycle" every second vertex is connected by an edge. This notion of an "inner graph" and an "outer graph" along with associated edge patterns for each is important to generalizing the Petersen graph into the class of graphs we study in this paper. The Generalized Petersen graph preserves the basic structure of the Petersen graph in that it consists of two n-cycles attached to each other at every vertex. These graphs are no longer contained to just ten vertices however, and the vertices of each cycle need not be connected according to the pattern observed in the Petersen graph itself. This means that if we embedded a Generalized Petersen graph on the plane as before with an "outer cycle" where every consecutive vertex is attached by an edge, then the vertices of the "inner cycle" do not need to have a skip of two, but can in fact have a skip of any number. This is usually taken to be less than half the number of vertices in a cycle by convention. Yet the class of graph invariant which we study in this paper are an even further generalization of the Generalized Petersen graphs. These graphs viewed in the conventional embedding on the plane allow for not only a skip on the "inner cycle," but also a skip on the "outer cycle." If we try to construct a graph, we may realize that the graph we have constructed could be re-arranged to a different embedding on the plane to resemble a Generalized Petersen graph. In fact, many graph invariant are isomorphic to Generalized Petersen graphs. The first goal of this paper is to study and understand independent sets in the graph invariant. We find bounds and expressions for the size of maximum independent sets. We are studying a set of objects that share no commonalities. In the graph this is simply a set of vertices where no two vertices have an edge between them. Thus, giving parameters for a maximum size independent set in a graph or a class of graphs gives us a notion of the maximum sized set of objects with no redundancies for some system. Graphs admit a great deal of algebraic structure. One such structure on graphs of interest in this thesis is the automorphism group of a graph. Notice that if we start from the conventional drawing of the Petersen graph, certain rotations or reflections of the Petersen graph are indistinguishable from our original starting point. In more mathematical terms, this means that there are isomorphisms between the Petersen graph and itself. These isomorphisms, in fact called automorphisms, form an algebraic structure called a group when paired with the operation of functional composition. The I-graphs being a generalization of the Petersen graphs admit both automorphisms of rotation and reflection, but also other more complex automorphisms described later in this thesis that work on the "inner" and "outer" cycles of the graph. Why are these automorphism groups important to mathematicians? Well in some sense the automorphism group of any object encodes all the various symmetries that object has. This notion of a group of symmetries has many applications and often allows physicists to describe the geometry of different objects and their symmetry properties as they move through space. The automorphism groups of the graph invariant have been established in previous work, which we describe fully in next section, but no further work has been done to characterize properties of these automorphism groups. In this paper, we study the algebraic properties of these automorphism groups via their associated Cayley graphs. These graphs represent the structural properties of a given group. In our case, they succinctly represent the elements of the automorphism groups and the relations between those elements. We provide some important new results concerning the Cayley graphs for the automorphism groups of a graph invariant, both in specific cases and in general. Finding the spectrum of our Cayley graphs gives further insight into the automorphism groups for the graph invariant in this paper. One inadvertent consequence of studying the graph invariant in this paper is that we have gained a small window of insight into how thoroughly the mathematical world and all its various fields are connected. Coxeter in 1950 gave "Self-dual configurations and regular graphs,". Babai in 1979 published "Spectra of Cayley graphs," Again Babai and Rónyai in 1990 studied "Computing irreducible representations of finite groups,". Holton and Sheehan gave in 1993," The Petersen Graph". 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2. Graph Theory Basics:

Some basic graph theory definitions for the benefit of the reader are given below. We start with the definition of a graph itself.

Definition 2.1: A graph is a pair G = (V, E) of sets satisfying $E \subset V^2$. Thus the elements of E called edges of G are two element subsets of V . The elements of V are known as vertices of G. For the purposes of this paper we limit ourselves to the study of simple graphs. This means that our graphs avoid loops and multi-edges. All I-graphs and Generalized Petersen graphs are simple. For convenience, edges of graphs are often denoted by their incident vertices in alphanumeric order and not in set form. In this paper, we concern ourselves with independent sets in graphs. These sets are useful in applications, and give us a graph invariant. We now give the definition of an independent set.

Definition 2.2: Given a graph G, an independent set I(G) is a subset of the vertices of G such that no two vertices in I(G) are adjacent.

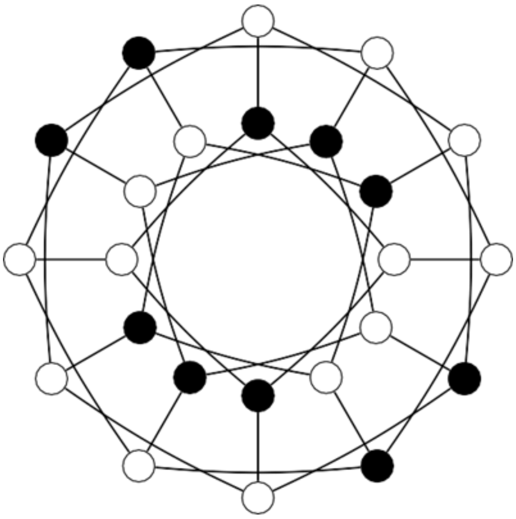

Figure 2.1: The filled in vertices identify an independent set in I(12, 2, 3)

Clearly, these sets have many interesting applications. The independent sets that we are interested in are the maximum independent sets. Note that independent sets are not unique, not even maximum independent sets. The size of the largest independent set in a graph is known as the independence number as we introduce in Definition 2.3. Finding the independence number of I-graphs is what this paper is chiefly concerned with. Finding the independence number is an NP-hard problem, and we therefore rely on finding bounds for the independence number in some cases. We formally define the independence number below.

Definition 2.3: The independence number, a(G) of the graph G, is the cardinality of a largest set of independent vertices. The independent set shown in the previous figure is in fact maximum, and therefore we may say that a(I(12, 2, 3)) = 10.

3. Algebraic Perspectives on Graphs: Another important goal of this paper is to find a well-defined Cayley graph that can represent the automorphism group of an I-graph. To that end, we introduce some basic definitions here such as the notions of isomorphism, automorphism and Cayley graphs. Much work has been done on the connection between group and graph theory.

Definition 3.1: We say $f : G_2 \ldots\ldots\ldots G_1$ is a map from the graph $G_2 = (V_2, E_2)$ to $G_1 = (V_1, E_1)$ if f is associated to the maps $f_v : V_2 \ldots\ldots. V_1$ and f_E: $E_2 \ldots.. E_1$ where $f_E(uv) = f_v(u)f_v(v)$ is in E_1.

We say f is an isomorphism if both f_v and f_E are bijections.

Isomorphisms are important for understanding the characteristics of graphs that are invariant. The notion of isomorphism between graphs motivates the assigning of a group structure to a graph, or likewise, the assigning of a graph to a group structure.

Definition 3.2: An isomorphism, f : G……G, from a graph to itself is called an automorphism. Under the operation of composition, the family of all automorphisms of a graph G forms a group called the automorphism group of G, and is denoted Aut(G).

Note that automorphisms must by definition preserve graph invariants. Only certain vertex/edge re-labelling's are automorphisms of the graph. To emphasize this point we introduce the definitions of vertex-transitivity and edge-transitivity in graphs. These are important notions that further ground our understanding of Cayley graphs and the connections between group and graph theory.

Definition 3.3: A vertex-transitive graph is a graph G such that, given any two vertices v_1 and v_2 of G, there is an automorphism of G that maps v_1 to v_2.

Definition 3.4: Likewise, an edge-transitive graph is a graph G such that, given any two edges e_1 and e_2 of G, there is an automorphism of G that maps e_1 and e_2.

The group of automorphisms of a graph essentially encodes the structure of a graph algebraically. In other words, it is the group of vertex/edge re-labelings that do not change the fundamental structure of the graph. Likewise, we would like to be able to represent groups via a graph that preserves the structure of the group. This is done with Cayley graphs.

Definition 3.5: Let G be a finite multiplicative group. Let $S \subset G$ be a subset which satisfies the condition $S = S^{-1}$ and 1 is not in S. The Cayley graph of (G, S) is the graph G = Cay(G, S) whose vertices are V = G and whose edges E are defined by pairs (g_1, g_2) such that $g_2g^{-1}{}_1$ is in S. Note that G = Cay(G, S) is S-regular, and is connected if and only if S generates G, to explain the connection of Cayley graphs to group theory by linking maps between Cayley graphs to group homomorphisms. This is done by assigning both colors and directions to the Cayley graph and thus making a Cayley color graph. To get the Cayley color graph we say the positive direction of an edge is from g_1 to g_2, and a coloring is assigned. This link between group theory and graph theory is important because it allows us to shift seamlessly between algebraic and graph theoretical observations of the same structure.

Theorem 3.1: Let f: Cay(G, S)……Cay(G^1,S^1) be a colour consistent, direction-preserving, identity preserving map. Then its vertex function coincides with a group homomorphism G……G^1. Conversely, any group homomorphism h: G……G1 such that h(S) is a subset of S^1 coincides with the vertex function of a color-consistent, direction-preserving, identity preserving map Cay(G, S)…….Cay(G^1, S^1).

If our map is a graph isomorphism then the theorem implies coincidence with a group isomorphism. Gross and Tucker go on to the following result which is helpful to our work in that it sheds light on a connection to I-graphs.

Theorem 3.2: Any Cayley graph is vertex transitive. Gross and Tucker move on to note that the converse of this does not hold, and that a notable exception is the Petersen graph itself. This is important to our work here.

4. Graphs Invariant: By now it is surely irritating to realize that none of the definitions and preliminary results we have discussed have been related to our desired class of graphs, that is, graph Invariant. Graph Invariant area generalization of Generalized Petersen graphs. In fact, the Generalized Petersen graphs are themselves special sub-class of graphs Invariant. We further generalize the Generalized Petersen graphs by also allowing a skip in the outer sub-graph.

Definition 4.1: The graph Invariant I(n, j, k) is a graph with vertex set V(I(n, j, k)) = (a_0, a_1, a_2………a_{n-1}, b_0, b_1, b_2……b_{n-1}) and edge set E(I(n, j, k)) = ($a_i a_{i+j}$, $a_i b_i$, $b_i b_{i+k}$: 0<i<n-1)

Since I(n, j, k) is equivalent to I(n, k, j) we assume that j<k . Note that we define the outer and inner sub-graphs as we did before with the Generalized Petersen graphs. It should also be obvious that P(n, k) is equivalent to I(n, 1, k).

The vertices a_i of the outer sub-graph A(n, j, k) the vertices on the outer rim. Likewise, the vertices b_i of B(n, j, k) are the vertices on the inner rim. The edges running between A(n, j, k) and B(n, j, k) are termed spokes.

As with the Generalized Petersen graphs we start with some basic properties of the graphs Invariant. These properties are crucial for this paper, and distinguish when an Invariant graph falls into the sub-class of the Generalized Petersen graphs.

Theorem 4.1: The following are true for graphs Invariant I(n, j, k)

1. The graph I(n, j, k) is connected if and only if gcd(n, j, k) = 1. If gcd(n, j, k) = d > 1, then the graph I(n, j, k) consists of d copies of I(n/d, j/d, k/d).

2. A connected graph I(n, j, k) is bipartite if and only if n is even and j ; k are odd.

3. If j is not equal to +k and -k then I(n, j,k) has a cycle of length 8. If j = k then in I(n, j, k) there exists a cycle of length 4.

4. Let n, j, k and a be positive integers such that gcd(n, j, k) = 1 and gcd(n, a) = 1. Then the graph I(n, aj, ak) is equivalent to I(n, j, k)

5. Let n, j and k be positive integers such that gcd(n, j, k) = 1,gcd(n,j) is not equal to 1 and gcd(n, k) is not equal to 1 then the graph I(n, j, k) is neither vertex-transitive nor edge-transitive.

6. A corollary of (5). A graph I(n, j, k) is a Generalized Petersen graph if and only if gcd(n, j) = 1 or gcd(n, k) = 1. If gcd(n, j) = 1 then I(n, j, k) is equivalent to P(n, s), where s is the solution of the equation k = s.j(mod n)

7. Let n, j, k, j^1 and k^1 be positive integers such that gcd(j, k) = gcd(j^1, k^1)=1 and gcd(n, j) = gcd(n, j^1) is not equal to 1 and gcd(n, k) = gcd(n, k^1) is not equal to 1.

Then the graph I(n, j, k) is equivalent to I(n, j^1, k^1) iff k.j^1=k^1.j(modn)

Understanding this overlap and its implications is important to the work of this paper. Note that a class of the graphs invariant that are not isomorphic to Generalized Petersen graphs themselves.

Definition 4.2: An graph invariant is called proper if it is connected and not isomorphic to a Generalized Petersen graph.

These are the graphs that we are chiefly concerned with in this work and most of our results are tailored for application to this sub-class of the graphs invariant. The class is narrower than might be thought. The two smallest proper graphs invariant, are I(12, 2, 3) and I(12, 3, 4)

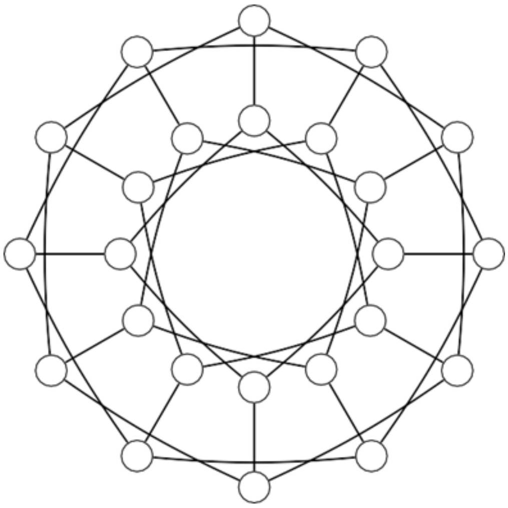

Figure 4.1: I(12, 2, 3)

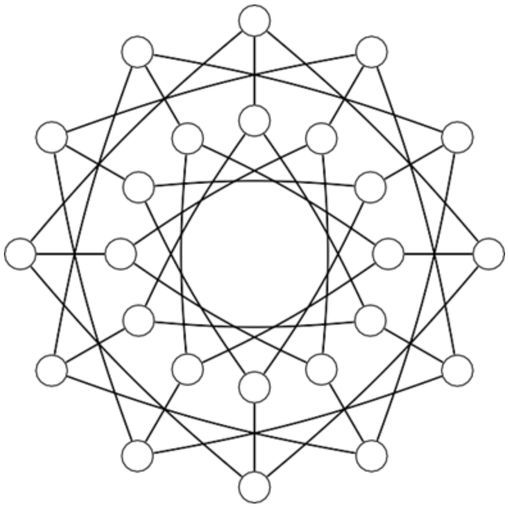

Figure 4.2: I(12, 3, 4)

Figure 4.3: The smallest proper Graph invariant

5. Conclusion: Authors carried out potential areas of future work in regards to the independence number of graphs Invariant, and the algebraic properties of the graphs Invariant. The graphs invariant, especially in their proper forms, have a more complicated edge set then the Generalized Petersen graphs. This means that finding the independence number of graphs Invariant with equality is quite challenging in the general case. We therefore relied on applying bounds to the independence number of graphs Invariant. The sets are started by taking the maximum possible number of consecutively indexed vertices from the inner subgraph of a graph invariant to include in the independent set and filling in as many remaining vertices from the outer subgraph as possible. As indeed is clear from the proofs, these constructed independent sets are not necessarily always maximum and do not necessarily have cardinality equal to the independence number of an graph invariant. In some more specific graph invariant classes, the independence number was able to be determined exactly. This was done by examining subgraphs and determining all the different locally maximal independent sets possible.

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