



## GIPERGEOMETRIK FUNKSIYALARNING CHEKSIZ YIG'INDILARI

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### Annotatsiya

Ushbu maqolada ikki o'zgaruvchili gipergeometrik funksiyalarning cheksiz yig'indilari formulalari isbotini tadqiq qilingan

### Абстрактный

В данной статье изучается доказательство формул бесконечных сумм гипергеометрических функций двух переменных.

### Abstract

In this article, the proof of formulas of infinite sums of two-variable hypergeometric functions is studied

Ushbu maqolada

p surat va q maxraj parametrlari  ${}_pF_q$  umumlashgan gipergeometrik funksiyani quyidagicha aniqlash mumkin.

$${}_pF_q \left[ \begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix} \middle| z \right] := {}_pF_q \left[ \alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q; z \right] := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_p)_n z^n}{(\beta_1)_n \dots (\beta_q)_n n!}$$

Bu yerda  $(a)_n$  Pochhammer belgisi (siljish faktorialini bildiradi)



$$(1)_n = 1 \cdot 2 \cdot \dots \cdot n = n!$$

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)\dots(a+n-1), & n \in N = 1, 2, 3, \dots \\ 1, & n = 0 \end{cases}$$

$\Gamma$ -funksiya quyidagicha

$$\Gamma(x) = \int_0^{\infty} u^{x-1} e^{-u} du, \quad \Re(x) > 0$$

Srivastava va Panda tomonidan quyidagi umumlashgan gipergeometrik funksiyalar kiritilgan

$$F_{g;c;d}^{h;a;b} \left[ \begin{matrix} (H_h):(A_a):(B_b); \\ (G_g):(C_c):(D_d); \end{matrix} x, y \right] = \sum_{m,n \geq 0} \frac{((H_h))_{m+n} ((A_a))_m ((B_b))_n x^m y^n}{((G_g))_{m+n} ((C_c))_m ((D_d))_n m! n!}$$

Bu yerda  $(H_h)$   $(H_1, H_2, \dots, H_h)$  parametrlar ketma-ketligini bildiradi.

$$((H_h))_n = (H_1)_n \cdot (H_2)_n \cdot \dots \cdot (H_h)_n$$

Gipergeometrik funksiyalar nazariyasida ikkita o'zgaruvchili gipergeometrik funksiyalar muhim ahamiyat kasb etadi. Masalan quyidagi Appelning to'rtta gipergeometrik funksiyasini misol qilib keltirishimiz mumkin.

$$F_1[a; b_1, b_2; c; x, y] = \sum_{m,n \geq 0} \frac{(a)_{m+n} (b_1)_m (b_2)_n x^m y^n}{(c)_{m+n} m! n!}, \quad |x|, |y| < 1$$

$$F_2[a; b_1, b_2; c_1, c_2; x, y] = \sum_{m,n \geq 0} \frac{(a)_{m+n} (b_1)_m (b_2)_n x^m y^n}{(c_1)_m (c_2)_n m! n!}, \quad |x| + |y| < 1$$

$$F_3[a_1, a_2; b_1, b_2; c; x, y] = \sum_{m,n \geq 0} \frac{(a_1)_m (a_2)_n (b_1)_m (b_2)_n x^m y^n}{(c)_{m+n} m! n!}, \quad |x|, |y| < 1$$

$$F_4[a; b; c_1, c_2; x, y] = \sum_{m,n \geq 0} \frac{(a)_{m+n} (b)_{m+n} x^m y^n}{(c_1)_m (c_2)_n m! n!}, \quad \sqrt{|x|} + \sqrt{|y|} < 1$$



Shuningdek Erdelyi va Hornlar 34 ta turli ikki o'zgaruvchili yaqinlashuvchi gipergeometrik funksiyalarida manashu ishlarida keltirishgan. Shulardan beshtasi quyidagi ko'rinishlarda aniqlanadi.

$$\Xi_1(\alpha_1, \alpha_2; \beta; \gamma; x, y) = \sum_{m, n \geq 0} \frac{(\alpha_1)_m (\alpha_2)_n (\beta)_m}{(\gamma)_{m+n} m! n!} x^m y^n, \quad |x|, |y| < 1$$

$$\Xi_2(\alpha; \beta; \gamma; x, y) = \sum_{m, n \geq 0} \frac{(\alpha)_m (\beta)_n}{(\gamma)_{m+n} m! n!} x^m y^n, \quad |x|, |y| < 1$$

$$\Phi_1(\alpha; \beta; \gamma; x, y) = \sum_{m, n \geq 0} \frac{(\alpha)_{m+n} (\beta)_n}{(\gamma)_{m+n} m! n!} x^m y^n, \quad |x|, |y| < 1$$

$$\Phi_2(\beta_1, \beta_2; \gamma; x, y) = \sum_{m, n \geq 0} \frac{(\beta_1)_m (\beta_2)_n}{(\gamma)_{m+n} m! n!} x^m y^n, \quad x, y \in C$$

$$\Psi_1(\alpha; \beta; \gamma_1, \gamma_2; x, y) = \sum_{m, n \geq 0} \frac{(\alpha)_{m+n} (\beta)_n}{(\gamma_1)_m (\gamma_2)_n m! n!} x^m y^n, \quad |x|, |y| < 1$$

Appel va Horn funksiyalari uchun bazi cheksiz yig'indi formulalarini keltirib chiqamiz.

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(A_1)_k}{k!} t^k F_{g;c;d}^{h;a;b} \left[ \begin{matrix} (H_h): A_1 + k, A_2, \dots, A_a; (B_b); \\ (G_g): (C_c); (D_d); \end{matrix} \middle| w, z \right] = \\ = (1-t)^{-A_1} F_{g;c;d}^{h;a;b} \left[ \begin{matrix} (H_h): (A_a); (B_b); \\ (G_g): (C_c); (D_d); \end{matrix} \middle| \frac{w}{1-t}, z \right]; \end{aligned}$$

(2.1)

$$\sum_{k=0}^{\infty} \frac{(H_1)_k}{k!} t^k F_{g;c;d}^{h;a;b} \left[ \begin{matrix} H_1 + k, H_2, \dots, H_h; (A_a); (B_b); \\ (G_g): (C_c); (D_d); \end{matrix} \middle| w, z \right] =$$



$$= (1-t)^{-H_1} F_{g;c;d}^{h;a;b} \left[ \begin{matrix} (H_h):(A_a):(B_b); \\ (G_g):(C_c):(D_d); \end{matrix} \frac{w}{1-t}, \frac{z}{1-t} \right];$$

(2.2)

Isbot.  $F_{g;c;d}^{h;a;b}$  gipergeometrik funksiya tarifiga asosan quyidagi holatga o'tamiz.

$$\sum_{k,m,n=0}^{\infty} \frac{(A_1)_k ((H_h))_{m+n} (A_1+k)_m (A_2)_m \dots (A_a)_m ((B_b))_n}{((G_g))_{m+n} ((C_c))_m ((D_d))_n m!n!k!} w^m z^n t^k$$

Oddiy o'zgartirishni  $(A_1)_k (A_1+k)_m = (A_1)_m (A_1+m)_k$  qo'llab quyidagi yig'indisini hisoblab chiqamiz.

$$\sum_{m,n=0}^{\infty} \frac{((H_h))_{m+n} ((A_a))_m ((B_b))_n}{((G_g))_{m+n} ((C_c))_m ((D_d))_n m!n!} w^m z^n {}_1F_0 \left[ \begin{matrix} A_1+m \\ - \end{matrix} \middle| t \right]$$

$$\sum_{m,n=0}^{\infty} \frac{((H_h))_{m+n} ((A_a))_m ((B_b))_n}{((G_g))_{m+n} ((C_c))_m ((D_d))_n m!n!} w^m z^n (1-t)^{-A_1-m}$$

bu yerda  ${}_1F_0(t)$  Nyutonning umumlashgan binomial teoremasi bilan baholanadi.

$${}_1F_0 \left[ \begin{matrix} a \\ - \end{matrix} \middle| z \right] = (1-z)^{-a} \quad (2.3)$$

Biroz soddalashtirilgandan so'ng, biz formulaning o'ng tomonini biriktiramiz.

Ushbu teoremadagi parametrlarni xususiyl holda, biz Appell funksiyalarining cheksiz yig'indisi formulalarini quyidagicha olishimiz mumkin.

$$\sum_{k=0}^{\infty} \frac{(b_1)_k}{k!} t^k F_1 \left[ a; b_1+k, b_2; c; w, z \right] = (1-t)^{-b_1} F_1 \left[ a; b_1, b_2; c; \frac{w}{1-t}, z \right]$$



$$\sum_{k=0}^{\infty} \frac{(a)_k}{k!} t^k F_4[a+k; b, c_1, c_2; w, z] = (1-t)^{-a} F_4\left[a; b, c_1, c_2; \frac{w}{1-t}, \frac{z}{1-t}\right]$$

Horn funksiyalarining cheksiz yig'indisi formulalarini quyidagicha yozish mumkin. Bu yerda biz faqat uchta formulani misol qilib keltiramiz.

$$\sum_{k=0}^{\infty} \frac{(\alpha_1)_k}{k!} t^k \Xi_1[\alpha_1+k, \alpha_2; \beta; \gamma; w, z] = (1-t)^{-\alpha_1} \Xi_1\left[\alpha_1, \alpha_2; \beta; \gamma; \frac{w}{1-t}, z\right]$$

$$\sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} t^k \Phi_1[\alpha+k; \beta; \gamma; w, z] = (1-t)^{-\alpha} \Phi_1\left[\alpha; \beta; \gamma; \frac{w}{1-t}, \frac{z}{1-t}\right]$$

$$\sum_{k=0}^{\infty} \frac{(\beta)_k}{k!} t^k \Psi_1[\alpha; \beta+k; \gamma_1, \gamma_2; w, z] = (1-t)^{-\beta} \Psi_1\left[\alpha; \beta; \gamma_1, \gamma_2; \frac{w}{1-t}, z\right]$$

Biz quyidagi yangi cheksiz yig'indi formulasini olamiz.

Ikki o'zgaruvchili gipergeometrik funksiyaning cheksiz yig'indisi formulasi quyidagicha

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{((H_h))_k ((A_a))_k}{((G_g))_k ((C_c))_k k!} t^k F_{g;c;d}^{h;a;b} \left[ \begin{matrix} (H_h+k):(A_a+k);(B_b); \\ (G_g+k):(C_c+k);(D_d) \end{matrix} \right] w, z = \\ = F_{g;c;d}^{h;a;b} \left[ \begin{matrix} (H_h):(A_a);(B_b); \\ (G_g):(C_c);(D_d) \end{matrix} \right] w+t, z \end{aligned} \quad (2.4)$$

Isbot:  $(a)_k (a+k)_m = (a)_{k+m}$  o'zgarish kiritib chap tomonini quyidagicha ifodalash mumkin.

$$\sum_{k,m,n=0}^{\infty} \frac{((H_h))_{k+m+n} ((A_a))_{k+m} ((B_b))_n}{((G_g))_{k+m+n} ((C_c))_{k+m} ((D_d))_n m! n! k!} w^m z^n t^k$$

Yuqoridagi natijada  $k+m=p$  almashtirishni bajarib va biroz soddalashtirilgandan so'ng quyidagi natijani olamiz.





$$\sum_{p,n=0}^{\infty} \frac{((H_h))_{p+n} ((A_a))_p ((B_b))_n z^n}{((G_g))_{p+n} ((C_c))_p ((D_d))_n p! n!} \sum_{k=0}^p \binom{p}{k} w^{p-k} t^k$$

$$\sum_{p,n=0}^{\infty} \frac{((H_h))_{p+n} ((A_a))_p ((B_b))_n}{((G_g))_{p+n} ((C_c))_p ((D_d))_n p! n!} (w+t)^p z^n$$

bu yerda binomial teoremaning maxsus holatini qo'lladik

$$\sum_{k=0}^n \binom{n}{k} x^k y^{n-k} = (x+y)^n \quad (2.5)$$

Ushbu teoremadagi parametrlarni xususiy holatlarini ko'rib chiqamiz, biz Appell va Horn funksiylarining cheksiz yig'indisi formulalarini quyidagicha olamiz:

$$\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c_1)_k k!} t^k F_4[a+k; b+k; c_1+k, c_2; w, z] = F_4[a; b; c_1, c_2; w+t, z]$$

$$\sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\beta)_k}{(\gamma)_k k!} t^k \Xi_1[\alpha_1+k, \alpha_2; \beta+k; \gamma+k; w, z] = \Xi_1[\alpha_1, \alpha_2; \beta; \gamma; w+t, z]$$

$$\sum_{k=0}^{\infty} \frac{(\beta_1)_k (\beta_2)_k}{(\gamma)_k k!} t^k \Phi_2[\beta_1+k, \beta_2+k; \gamma+k; w, z] = \Phi_2[\beta_1, \beta_2; \gamma; w+t, z]$$

Ikki gipergeometrik funksiyaning quyidagi cheksiz yig'indisi formulalari to'g'ri keladi:

$$\sum_{k=0}^{\infty} \frac{(A_2)_k \dots (A_a)_k ((H_h))_k (r)_k}{((G_g))_k ((C_c))_k k!} w^k F_{g;c;d}^{h;a;b} \left[ \begin{matrix} (H_h+k): A_1, A_2+k, \dots, A_a+k; (B_b); \\ (G_g+k): (C_c+k); (D_d) \end{matrix} \middle| w, z \right] =$$

$$= F_{g;c;d}^{h;a;b} \left[ \begin{matrix} (H_h): A_1+r, A_2, \dots, A_a; (B_b); \\ (G_g): (C_c); (D_d) \end{matrix} \middle| w, z \right] \quad (2.6)$$



$$\sum_{k=0}^{\infty} \frac{(H_2)_k \dots (H_h)_k ((A_a))_k (r)_k}{((G_g))_k ((C_c))_k k!} w^k F_{g:c;d}^{h;a;b} \left[ \begin{matrix} H_1, H_2 + k, \dots, H_h + k : (A_a)(B_b); \\ (G_g + k) : (C_c + k); (D_d) \end{matrix} \middle| w, z \right] =$$

$$= F_{g:c;d+1}^{h;a;b+1} \left[ \begin{matrix} H_1 + r, H_2, \dots, H_h; (A_a); H_1, (B_b); \\ (G_g) : (C_c); H_1 + r(D_d) \end{matrix} \middle| w, z \right] \quad (2.7)$$

Isbot:  $F_{g:c;d}^{h;a;b}$  ta'rifni qo'llab  $(a)_k (a+k)_m = (a)_{k+m}$  o'zgartirish kiritsak chap tomonini quyidagicha ifodalash mumkin.

$$\sum_{k,m,n=0}^{\infty} \frac{(H_h)_{m+n+k} (A_1)_m (A_2)_{m+k} \dots (A_a)_{m+k} ((B_b)_n (r)_k)}{((G_g))_{m+n+k} ((C_c))_{m+k} ((D_d))_n k! m! n!} w^{m+k} z^n$$

$$\sum_{p,n=0}^{\infty} \frac{(H_h)_{p+n} ((A_a))_p ((B_b))_n (r)_k}{((G_g))_{p+n} ((C_c))_p ((D_d))_n p! n!} w^p z^n {}_2F_1 \left[ \begin{matrix} -p, r \\ 1 - A_1 - p \end{matrix} \middle| 1 \right]$$

bu yerda biz ikkinchi tenglamada  $m+k=p$  almashtirishni amalga oshirdik.  ${}_2F_1(1)$  ichki yig'indini quyidagi Vandermonde teoremasi orqali baholash mumkin.

$${}_2F_1 \left[ \begin{matrix} -n, a \\ c \end{matrix} \middle| 1 \right] = \frac{(c-a)_n}{(c)_n} \quad (2.8)$$

Biroz soddalashtirilgandan so'ng, biz (2.6) ning o'ng tomonini hosil qilamiz. Cheksiz yig'indi formulasi (2.7) xuddi shunday usul bilan isbotlanishi mumkin. Biz misol sifatida Appell va Horn funksiyalarining quyidagi cheksiz yig'indi formulalarini hosil qilamiz.

$$\sum_{k=0}^{\infty} \frac{(a)_k (r)_k}{(c)_k k!} w^k F_1[a+k; b_1, b_2; c+k; w, z] = F_1[a; b_1+r, b_2; c; w, z]$$

$$\sum_{k=0}^{\infty} \frac{(a)_k (r)_k}{(c_1)_k k!} w^k F_2[a+k; b_1, b_2; c_1+k; w, z] = F_2[a; b_1+r, b_2; c_1, c_2; w, z]$$



$$\sum_{k=0}^{\infty} \frac{(a)_k (r)_k}{(c_1)_k k!} w^k F_4[a; b+k; c_1+k, c_2; w, z] = F \begin{matrix} 2:0;1 \\ 0:1;2 \end{matrix} \left[ \begin{matrix} a+r, b:-; a; \\ -:c_1; a+r, c_2; \end{matrix} w, z \right]$$

$$\sum_{k=0}^{\infty} \frac{(\beta)_k (r)_k}{(\gamma)_k k!} w^k \Xi[\alpha_1, \alpha_2; \beta+k; \gamma+k; w, z] = \Xi[\alpha_1+r, \alpha_2; \beta; \gamma; w, z]$$

$$\sum_{k=0}^{\infty} \frac{(\beta_2)_k (r)_k}{(\gamma)_k k!} w^k \Phi_2[\beta_1, \beta_2+k; \gamma+k; w, z] = \Phi[\beta_1+r, \beta_2; \gamma; w, z]$$

$$\sum_{k=0}^{\infty} \frac{(\alpha)_k (r)_k}{(\gamma_1)_k k!} w^k \Psi_1[\alpha+k, \beta; \gamma_1+k, \gamma_2; w, z] = \Psi_1[\alpha, \beta+r, \beta_2; \gamma_1, \gamma_2; w, z]$$

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