

HRUSHOVSKI'S AMALGAMATION CONSTRUCTION

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ABSTRACT. An overview is given of the various structures obtained by means of Hrushovski's amalgamation method and variants thereof.

1. INTRODUCTION

In 1986, Ehud Hrushovski modified Fraïssé's construction of a universal homogeneous countable relational structure from the class of its finite substructures, in order to obtain *stable* structures with particular properties. In particular, he constructed

- (i) an \aleph_0 -categorical stable pseudoplane (4.1.1),
- (ii) a strongly minimal set with an exotic geometry which is not disintegrated, but does not interpret any group (4.2.3),
- (iii) the fusion of two strongly minimal sets in disjoint languages in a third one,

obtaining counter-examples to conjectures by Lachlan and Zilber. In particular, a stable \aleph_0 -categorical structure need not be ω -stable (i), and the trichotomy suggested by Zilber for strongly minimal sets (disintegrated / vector-space-like / field-like) does not hold in general (ii); moreover, every strongly minimal set (with an additional condition, the definable multiplicity property DMP) has an essential strongly minimal extension, so there is no maximal one (iii). In particular, an algebraically closed field has proper strongly minimal extensions, even interpreting another algebraically closed field of possibly different characteristic, answering a question of Zilber.

His method was taken up by Baldwin, Baudisch, Evans, Hasson, Hils, Holland, Martín Pizarro, Poizat, Ziegler, Zilber and others, who constructed a variety of strange objects, most notably

- (iv) a disintegrated stable structure with a reduct which is not one-based (3.1.1),
- (v) a new \aleph_1 -categorical nilpotent group of class 2 and exponent p ,
- (vi) a field of Morley rank 2 with an additive subgroup of Morley rank 1 in characteristic $p > 0$ (6.2),
- (vii) a field of Morley rank 2 with a multiplicative subgroup of Morley rank 1 in characteristic 0 (7.2),
- (viii) the fusion of two strongly minimal sets over a common \aleph_0 -categorical reduct.

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(iv) answers negatively the question of Poizat whether a reduct of a disintegrated structure must be disintegrated, or a reduct of a one-based structure must be one-based. (v) yields another counterexample to the trichotomy conjecture, which is not only combinatorial, but carries algebraic structure. (vi) and (vii) answer questions by Zilber and Poizat about the existence of such structures. The problem arose in the context of the analysis of simple groups of finite Morley rank (which were conjectured by Zilber and Cherlin to be algebraic); a soluble non-nilpotent subgroup can be reduced to the form $K^+ \rtimes T$ for some non-trivial $T \leq K^\times$, and in order to prove the existence of elements of finite order, one would have liked to show $T = K^\times$ (but the current approach to the conjecture manages to circumvent this difficulty). Finally, Hrushovski remarks in [16] that it should be possible to do the fusion of (iii) not in disjoint languages, but over a common vector space over a finite field; this is the essential case of (viii).

We shall sketch how these structures are obtained by variations of the original construction.

2. THE ORIGINAL CONSTRUCTION OF FRAÏSSÉ

Let \mathcal{C} be a class of finite structures in a finite relational language, closed under substructures, and with the *amalgamation property* (where we allow $A = \emptyset$)

AP: For all injective $\sigma_i : A \rightarrow B_i$ in \mathcal{C} for $i = 1, 2$ there are injective $\rho_i : B_i \rightarrow D \in \mathcal{C}$ with $\rho_1\sigma_1 = \rho_2\sigma_2$.

Then there is a unique countable structure \mathfrak{M} satisfying

Richness: For all finite $A \subset \mathfrak{M}$ and $A \subset B \in \mathcal{C}$ there is an embedding $B \rightarrow \mathfrak{M}$ which is the identity on A .

The proof is by successive amalgamation over all possible situations, using AP.

We call \mathfrak{M} the *generic model*; it is ultrahomogeneous, and hence \aleph_0 -categorical (since the language is finite). It is axiomatized by the universal axioms for \mathcal{C} (which proscribe the sets not in \mathcal{C}) and the inductive axioms for richness.

3. MODIFICATION I

Rather than considering all inclusions $A \subset B \in \mathcal{C}$, we only consider *certain* inclusions $A \leq B$, which we call *closed*. We require \leq to be transitive and preserved under intersections: If $A \leq B$ and $C \leq B$, then $A \cap C \leq C$. We only demand AP for closed inclusions, and obtain a generic structure \mathfrak{M} satisfying

Closed Richness: For any finite $A \leq \mathfrak{M}$ and $A \leq B \in \mathcal{C}$ there is a *closed* embedding $B \hookrightarrow \mathfrak{M}$ which is the identity on A .

Moreover, \mathfrak{M} is ultrahomogeneous for finite *closed* subsets.

Let $\bar{\mathcal{C}}$ be the class of structures whose finite substructures are in \mathcal{C} . For infinite embeddings $A \subseteq B \in \bar{\mathcal{C}}$ we put $A \leq B$ if $A \cap C \leq C$ for all finite $C \subseteq B$. For $A \subset \mathfrak{M} \in \bar{\mathcal{C}}$ we define the *closure* $\text{cl}_{\mathfrak{M}}(A)$ to be the smallest $B \leq \mathfrak{M}$ containing A . It is easily seen to be unique, but it can be infinite (for instance \mathfrak{M} itself).

In order to axiomatize, we need to express $A \leq \mathfrak{M}$. If this can be done by a first-order formula, and if there is a finite bound on the size of the closure of a finite set in terms of size of the set itself (and hence a finite bound on the number of n -types for all $n < \omega$, since the language is finite), the generic model is \aleph_0 -categorical.

However, even in a finite relational language, closedness need not be a definable property. If $A \leq \mathfrak{M}$ is only *type*-definable, or if closures can be infinite, we need definability of

Approximate Richness: If $A \leq B \in \mathcal{C}$ and A is sufficiently closed in \mathfrak{M} , then there is an embedding of B into \mathfrak{M} over A whose image has a pre-described level of closedness.

This yields homogeneity for closures of finite subsets in an \aleph_0 -saturated model, or for countable closed subsets in an \aleph_1 -saturated model.

3.1. Example. [10] Consider the class \mathcal{C} of finite directed graphs without directed cycles and out-valency 2. Define $A \leq B$ if all descendants of A in B are already in A . Note that the closure is *disintegrated*:

$$\text{cl}_{\mathfrak{M}}(A) = \bigcup_{a \in A} \text{cl}_{\mathfrak{M}}(a).$$

This class has AP (namely the free amalgam, i.e. the disjoint union over the intersection, with no edges added). Clearly, richness (for finite sets) is definable; let T be the theory of (finitely) rich directed graphs without directed cycles and out-valency 2. By compactness, an \aleph_0 -saturated model of T is rich even for closures of finite sets, so T is complete, and isomorphic closed sets have the same type.

3.1.1. A non-disintegrated reduct. Now consider the undirected reduct of models of T . By induction, one sees that a finite undirected graph has an orientation without directed cycles and of out-valency 2 iff every subgraph has a vertex of valency at most 2. For these graphs we define $A \leq B$ if B has such an orientation in which A is closed in the directed sense. This class is closed under free amalgamation; its generic model is the reduct of a generic model of the directed class.

However, the closure of A in \mathfrak{M} is the intersection of all directed closures of A in \mathfrak{M} , for all possible orientations. This closure is no longer disintegrated; since no group is definable, it is not even one-based. Counting types, one can show that both theories are stable.

4. MODIFICATION II

For every relation $R \in \mathcal{L}$ we choose a weight $\alpha_R > 0$ and define a *predimension* on finite \mathcal{L} -structures:

$$\delta(A) = |A| - \sum_{R \in \mathcal{L}} \alpha_R |R(A)|, \quad \text{as well as}$$

$$\delta(A/B) = \delta(AB) - \delta(B) = |A \setminus B| - \text{weights of the new relations};$$

the latter makes sense even if B is infinite. Define

$$A \leq B \iff \delta(B'/A) \geq 0 \text{ for all } B' \subseteq B.$$

Let \mathcal{C} be the universal class of all finite \mathcal{L} -structures whose substructures have non-negative predimension (*ab initio*).

It is closed under free amalgamation, and thus has AP. Closedness is type-definable (this uses finiteness of the language), richness is approximately definable, and \aleph_1 -saturated models are rich. The generic model is ω -stable if all the α_R are rational, and stable otherwise.

Since we have free amalgamation, cl is equal to algebraic closure (in the model-theoretic sense).

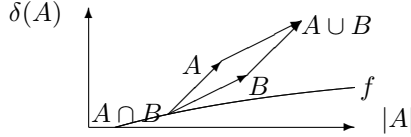
If A is a finite subset of the generic model, then $\text{cl}_{\mathfrak{M}}(A)$ is the unique smallest superset of A with minimal predimension (for irrational α_R this may be infinite, and we interpret its predimension as a suitable limit). Define the *dimension* of A in \mathfrak{M} to be $d_{\mathfrak{M}}(A) = \delta(\text{cl}_{\mathfrak{M}}(A))$.

Then two closed sets are independent in the forking sense iff they are freely amalgamated over their intersection, and the amalgam is closed in \mathfrak{M} . It is now easy to see that the generic model has weak elimination of imaginaries.

Finally, for rational α_R Evans [11] has defined a notion of *directed hypergraph* such that $A \leq B$ in the predimension sense iff there is an orientation of B in which A is closed in the directed sense. So 3.1.1 above generalizes, and all ω -stable ab initio constructions arise as reducts of disintegrated geometries.

4.1. \aleph_0 -categoricity. If we want the resulting structure to be \aleph_0 -categorical, there must be a bound on the size of the closure of a finite set (as the closure is always contained in the algebraic closure).

Note that the free amalgam of A and B over $A \cap B$ forms the fourth point of a parallelogram:



We choose an unbounded increasing convex cut-off function f and consider the subclass $\mathcal{C}_f = \{A \in \mathcal{C} : \delta(A) \geq f(|A|)\}$.

If the slope of f at $|B|$ is at most the minimal slope of an arrow at B , then \mathcal{C}_f has free amalgamation. This requires in particular $\delta(B/A)$ to be strictly positive, and the α_R to be linearly independent over \mathbb{Q} . Axiomatizability and \aleph_0 -categoricity follow.

4.1.1. Pseudoplanes. An incidence system $I \subset P \times L$ is a *pseudoplane* if

- every point $p \in P$ lies on infinitely many lines $\ell \in L$,
- every line contains infinitely many points,
- every two points are incident with finitely many lines,
- every two lines intersect in finitely many points.

The pseudoplane is *complete* if I is a complete type.

Hrushovski has shown that for a single binary relation the set of (irrational) α which allow an unbounded increasing convex cut-off function is co-meagre. Putting $P = L = \mathfrak{M}$ and $I = R$, this yields an \aleph_0 -categorical stable complete pseudoplane (answering a question of Lachlan).

It is easy to see that a disintegrated stable theory cannot type-define a complete pseudoplane. So if an ω -stable complete pseudoplane has a disintegrated expansion à la Evans, the pseudoplane obviously still exists in the expansion, but is no longer complete.

4.2. Strong minimality.

4.2.1. *Geometry.* Suppose all the α_R are integers. For a single point a and a set B (which we may assume to be closed) of parameters in a generic structure \mathfrak{M} , there are two possibilities:

- $d_{\mathfrak{M}}(a/B) = \delta(a/B) = 1$. Then $aB \leq \mathfrak{M}$, so this determines a unique type, the *generic* type.
- $d_{\mathfrak{M}}(a/B) = 0$. So a is in the *geometric* closure $\text{gcl}(B)$.

Clearly gcl is increasing and idempotent, hence a closure operator, which in addition satisfies the exchange rule:

If $a \in \text{gcl}(Bc) \setminus \text{gcl}(B)$, then $c \in \text{gcl}(Ba)$.

We should like to restrict the class \mathcal{C} so that gcl becomes algebraic closure, thus yielding a *strongly minimal* set (every definable subset is uniformly finite or cofinite). For that, we have to bound the number of possible realisations of any $a \in \text{gcl}(B)$ uniformly and definably.

4.2.2. *Minimal and pre-algebraic extensions, codes.* A proper closed extension $A \leq B \in \mathcal{C}$ is *minimal* if $A \leq A' \leq B$ implies $A' = A$ or $A' = B$. Equivalently, $\delta(B/A') < 0$ for all $A \subset A' \subset B$.

It is *pre-algebraic* if $\delta(B/A) = 0$. In this case, minimality is equivalent to $\delta(B'/A) > 0$ for all $A \subset B' \subset B$.

For a minimal pre-algebraic extension $A \leq B$ let $A_0 \leq A$ be the closure of the points in A related to some points in $B \setminus A$. This is the unique minimal closed subset of A over which $B \setminus A$ is pre-algebraic (and in fact minimal).

We call $A_0 \leq B$ *bi-minimal pre-algebraic*; its *code* $\varphi(\bar{x}, \bar{y})$ is the quantifier-free diagram of $(B \setminus A, A_0)$.

Clearly it is sufficient to bound the number of realisations for each bi-minimal pre-algebraic code.

4.2.3. *New strongly minimal sets.* [17] Let μ be a function from the set of codes to the integers, satisfying $\mu(\varphi(\bar{x}, \bar{y})) \geq \delta(\bar{y})$. Let \mathcal{C}^μ be the class of $A \in \mathcal{C}$ such that for any code φ and any A_0 there are at most $\mu(\varphi)$ disjoint realizations of $\varphi(\bar{x}, A_0)$. Again, this is a universal class.

Hrushovski has shown that \mathcal{C}^μ has *thrifty amalgamation*:

If $A \leq B \in \mathcal{C}^\mu$ is minimal and $A \leq M \in \mathcal{C}^\mu$, then either the free amalgam of A and M over B is still in \mathcal{C}^μ , or B embeds closedly into M over A .

So a generic model exists; since approximate richness is still definable, its theory is strongly minimal.

5. EXTENSION TO GENERAL LANGUAGES

If the language is infinite or non-relational (but still countable), one considers a class \mathcal{C} of finitely generated structures with AP, and a predimension δ on \mathcal{C} satisfying

- (1) $\delta(\emptyset) = 0$, and
- (2) *Submodularity*: $\delta(A) + \delta(B) \geq \delta(A \cup B) + \delta(A \cap B)$ for closed A, B .

Two main difficulties arise:

- Axiomatization of the class \mathcal{C} (the excluded structures need not be finite).
- Axiomatization of the richness condition, or even of approximate richness.

In practice, they are usually overcome by requiring definability of rank (and possibly also degree), and *modularity* of the negative part of the predimension (i.e. the inequality above is strengthened to equality).

5.1. Poizat's black field of rank $\omega \cdot 2$. [19] A *black field* is an ω -stable algebraically closed field K with a predicate N for a subset of comparable rank. According to Berline and Lascar [18] the Lascar rank of an infinite superstable field must be of the form $\omega^\alpha \cdot n$ for some ordinal α and some positive integer n ; they conjectured that $n = 1$. Since $U(K) > U(N)$, black fields yield a counter-example to this conjecture.

Let \mathcal{C} be the class of finitely generated fields k of fixed characteristic with a unary predicate N , such that for all finitely generated subfields k'

$$\delta(k') = 2 \operatorname{tr.deg}(k') - |N(k')| \geq 0,$$

and for $k' \in \bar{\mathcal{C}}$ finitely generated over $k \in \bar{\mathcal{C}}$ put

$$\delta(k'/k) = 2 \operatorname{tr.deg}(k'/k) - |N(k') \setminus N(k)|,$$

where again $\bar{\mathcal{C}}$ is the class of structures whose finitely generated substructures are in \mathcal{C} .

Since \mathcal{C} has free amalgamation, a generic model \mathfrak{M} exists. The class \mathcal{C} is universally axiomatized by formulas of the form $\forall \bar{x} \neg[\varphi(\bar{x}) \wedge \nu(\bar{x})]$, where φ is a quantifier-free formula in the field language explicitly making strictly more than half of \bar{x} algebraic over the rest, and $\nu(\bar{x})$ states that all elements of \bar{x} are distinct and black. In a similar way, modulo the axioms for algebraically closed fields, richness is approximately definable by inductive formulas

$$\forall \bar{y} [\vartheta(\bar{y}) \rightarrow \{\exists \bar{x} [\varphi(\bar{x}, \bar{y}) \wedge \nu(\bar{x})]\}],$$

where φ and ϑ are quantifier-free formulas in the pure language of fields, $\varphi(\bar{x}, \bar{y})$ implies that $\operatorname{tr.deg}(\bar{x}'/\bar{x}''\bar{y}) < 2|\bar{x}'|$ for every non-trivial partition $\bar{x} = \bar{x}'\bar{x}''$, and $\vartheta(\bar{y})$ states that $\varphi(\cdot, \bar{y})$ has Morley rank $|\bar{x}|/2$ and Morley degree one (which are definable properties in pure algebraically closed fields). So \aleph_0 -saturated models of $\operatorname{Th}(\mathfrak{M})$ are rich.

For a point a and a set B in \mathfrak{M} there are three possibilities:

- (1) $d_{\mathfrak{M}}(a/B) = 2$. Then a is not black, and aB is closed (*white generic*).
- (2) $d_{\mathfrak{M}}(a/B) = 1$. There is a black point a' interalgebraic with a over B , and $a'B$ is closed (*black generic*).
- (3) $d_{\mathfrak{M}}(a/B) = 0$. Then either a is algebraic over B , or pre-algebraic.

One can show that minimal pre-algebraic extensions have rank 1, that the black generic is the limit of pre-algebraic types of arbitrarily big finite rank (and hence of rank ω), and that the white generic has rank $\omega \cdot 2$ (all other types have rank strictly less than $\omega \cdot 2$).

5.2. The collaps. [19, 1, 2] If $\varphi(\bar{x}, \bar{y})$ and $\vartheta(\bar{y})$ are as in one of the inductive axioms, a *code* is given by a formula $\varphi \wedge \vartheta$ (where we assume in addition \bar{y} to be the canonical parameter). For a code φ we fix a quantifier-free definable function $f_\varphi(\bar{x}_1, \dots, \bar{x}_{n_\varphi}, \bar{y})$ which explicitly defines \bar{y} over n_φ (independent) generic

realizations \bar{x}_i of $\varphi(., \bar{y})$. Let μ be a function from the set of codes to ω with $\mu(\varphi) \geq (2n+2)n_\varphi$, and \mathcal{C}^μ the class of structures in \mathcal{C} which for any code φ contain at most $\mu(\varphi)$ disjoint black tuples such that any n_φ of them have the same image \bar{y} under f , and they all satisfy $\varphi(., \bar{y})$. Then \mathcal{C}^μ has thrifty amalgamation; if μ is finite-to-one, a generic model is ω -saturated, and hence of Morley rank 2. A black generic has Morley rank 1; a white (field) generic has Morley rank 2, and is the sum of two independent generic black points.

Since the axiomatization is inductive and any complete theory of fields of finite Morley rank is \aleph_1 -categorical, Lindström's theorem implies model-completeness of the theory of the generic model.

6. FIELDS WITH AN ADDITIVE SUBGROUP

6.1. Poizat's red field of rank $\omega \cdot 2$. [20] A *red field* is an ω -stable algebraically closed field K with a predicate R for a connected additive subgroup of comparable rank. Note that in characteristic 0 this gives rise to an infinite definable subfield $\{a \in K : aR \leq R\}$, so the structure has rank at least ω .

Let \mathcal{C} be the class of finitely generated fields k of characteristic $p > 0$ with a predicate R for an additive subgroup, such that for all finitely generated subfields k'

$$\delta(k') = 2 \operatorname{tr.deg}(k') - \operatorname{lin.dim}_{\mathbb{F}_p}(R(k')) \geq 0.$$

This condition is universal, since we have to say that $2n$ linearly independent red points do not lie in any variety of dimension $< n$.

For $k \subseteq k' \in \mathcal{C}$ with k' finitely generated over k put

$$\delta(k'/k) = 2 \operatorname{tr.deg}(k'/k) - \operatorname{lin.dim}_{\mathbb{F}_p}(R(k')/R(k)).$$

Since \mathcal{C} has free amalgamation, a generic model \mathfrak{M} exists; as \mathcal{C} is (universally) axiomatizable and richness is approximately definable, \aleph_0 -saturated models of $\operatorname{Th}(\mathfrak{M})$ are rich.

For a point a and a set B in \mathfrak{M} there are three possibilities:

- (1) $d_{\mathfrak{M}}(a/B) = 2$. Then a is not red, and aB is closed. $RM(a/B) = \omega \cdot 2$.
- (2) $d_{\mathfrak{M}}(a/B) = 1$. There is a red point a' interalgebraic with a over B , and $a'B$ is closed. $\omega \cdot 2 > RM(a/B) \geq RM(a'/B) = \omega$.
- (3) $d_{\mathfrak{M}}(a/B) = 0$. Then a is algebraic or pre-algebraic over B , of rank $< \omega$.

6.2. The collaps. [8] We have to restrict the number of bi-minimal pre-algebraic extensions. A *code* is a formula $\varphi(\bar{x}, \bar{y})$ in the field language with $n = |\bar{x}|$ such that

- (1) For all \bar{b} either $\varphi(\bar{x}, \bar{b})$ is empty, or has Morley degree 1.
- (2) $\operatorname{tr.deg}(\bar{a}/\bar{b}) = n/2$ and $\operatorname{lin.dim}_{\mathbb{F}_p}(\bar{a}/\bar{b}) = n$ for generic $\bar{a} \models \varphi(\bar{x}, \bar{b})$, and $2 \operatorname{tr.deg}(\bar{a}/U\bar{b}) < n - \operatorname{lin.dim}_{\mathbb{F}_p}(U)$ for all non-trivial subspaces U of $\langle \bar{a} \rangle$.
- (3) If $\operatorname{tr.deg}(\varphi(\bar{x}, \bar{b}) \cap \varphi(\bar{x}, \bar{b}')) = n/2$, then $\bar{b} = \bar{b}'$.
- (4) If $\varphi(\bar{x}, \bar{b})$ is disintegrated for some \bar{b} , it is disintegrated (or empty) for all \bar{b} .
- (5) For any $H \in \operatorname{GL}_n(\mathbb{F}_p)$, \bar{m} and \bar{b} there is \bar{b}' with $\varphi(H\bar{x} + \bar{m}, \bar{b}) \equiv \varphi(\bar{x}, \bar{b}')$.

Remark:

- (1) says that $\varphi(\bar{x}, \bar{b})$ determines a unique generic type $p_{\varphi(\bar{x}, \bar{b})}$ (or is empty).

- (2) says that $\bar{b} \leq \bar{a}\bar{b}$ with \bar{a} red is minimally pre-algebraic. Moreover, $\delta(\bar{a}'/B) < 0$ for any $B \ni \bar{b}$ and non-generic red $\bar{a}' \notin \text{acl}(B)$ realizing $\varphi(\bar{x}, \bar{b})$.
- (3) says that \bar{b} is the canonical base for $p_{\varphi(\bar{x}, \bar{b})}$, so the extension $\bar{b} \leq \bar{a}\bar{b}$ is bi-minimal.
- (4) says that φ fixes the type of the extension: disintegrated, or generic in a group coset (minimal pre-algebraic types are locally modular).
- (5) says that affine transformations preserve the code.

Inductively one constructs a set \mathcal{S} of codes such that every minimal pre-algebraic extension is coded by a unique $\varphi \in \mathcal{S}$.

6.2.1. *Difference sequences.* For a code φ and some \bar{b} consider a Morley sequence $(\bar{a}_0, \bar{a}_1, \dots, \bar{a}_k, f)$ for $p_{\varphi(\bar{x}, \bar{b})}$, and put $\bar{e}_i = \bar{a}_i - \bar{f}$.

We can then find a formula $\psi_\varphi \in \text{tp}(\bar{e}_0, \dots, \bar{e}_k)$ in the field language such that

- Any realization $(\bar{e}'_0, \dots, \bar{e}'_k)$ of ψ_φ is \mathbb{F}_p -linearly independent, and $\models \varphi(\bar{e}'_i, \bar{b}')$ for some unique \bar{b}' definable over sufficiently large finite subsets of the \bar{e}'_i .
- ψ_φ is invariant under the finite group of *derivations* generated by

$$\partial_i : \bar{x}_j \mapsto \begin{cases} \bar{x}_j - \bar{x}_i & \text{if } j \neq i \\ -\bar{x}_i & \text{if } j = i \end{cases}$$

for $0 \leq i \leq k$.

- Some condition ensuring dependence of affine combinations, and invariance under the stabiliser of the group for coset codes.

A *difference sequence* for a code φ is any realization of ψ_φ .

6.2.2. *A counting Lemma.* Given a code φ and natural numbers m, n , there is some λ such that for every $M \leq N \in \mathcal{C}$ and red difference sequence $(\bar{e}_0, \dots, \bar{e}_\lambda)$ in N with canonical parameter \bar{b} , either

- the canonical parameter for some derived sequence lies in M , or
- for every $A \subset N$ of size m the sequence $(\bar{e}_0, \dots, \bar{e}_\lambda)$ contains a Morley subsequence in $p_{\varphi(\bar{x}, \bar{b})}$ over MA of length n .

Let μ be a sufficiently fast-growing finite-to-one function from \mathcal{S} to ω , and \mathcal{C}^μ the class of $A \in \mathcal{C}$ which do not contain a red difference sequence for φ of length $\mu(\varphi)$ for any $\varphi \in \mathcal{S}$.

The above lemma allows us to characterize when a minimal pre-algebraic extension of some $M \in \mathcal{C}^\mu$ is no longer in \mathcal{C}^μ , and to prove thrifty amalgamation for \mathcal{C}^μ .

6.2.3. *Axiomatization.* It follows that there is a generic model \mathfrak{M} ; since approximate richness remains definable, \aleph_0 -saturated models of $\text{Th}(\mathfrak{M})$ are rich, \mathfrak{M} has Morley rank 2, and $R^\mathfrak{M}$ has Morley rank 1.

Alternatively, the field can be inductively axiomatized as follows:

- Finitely generated subfields are in \mathcal{C}^μ .
- ACF_p .
- The extension of the model generated by a red generic realization of some code instance $\varphi(\bar{x}, \bar{b})$ is not in \mathcal{C}^μ .

So Lindström's theorem again implies model-completeness of $\text{Th}(\mathfrak{M})$.

7. FIELDS WITH A MULTIPLICATIVE SUBGROUP

7.1. Poizat's green field of rank $\omega \cdot 2$. [20] A *green field* is an ω -stable algebraically closed field K with a predicate \dot{U} for a connected multiplicative subgroup of comparable rank. Note that in characteristic $p > 0$ a green field of finite rank implies that there are only finitely many *p-Mersenne primes* $\frac{p^n-1}{p-1}$, and \mathbb{F}_p is a prime model. Its existence is thus improbable; in any case it cannot be constructed as generic model by amalgamation methods.

Let \mathcal{C} be the class of finitely generated fields k of characteristic 0 with a predicate \dot{U} for a torsion-free multiplicative subgroup, such that for all finitely generated subfields k'

$$\delta(k') = 2 \operatorname{tr.deg}(k') - \operatorname{lin.dim}_{\mathbb{Q}}(\dot{U}(k')) \geq 0,$$

where the linear dimension is taken multiplicatively. For $k' \in \bar{\mathcal{C}}$ finitely generated over $k \subseteq k'$ put

$$\delta(k'/k) = 2 \operatorname{tr.deg}(k'/k) - \operatorname{lin.dim}_{\mathbb{Q}}(\dot{U}(k')/\dot{U}(k)).$$

7.1.1. The weak CIT. While linear dimension over a finite field is definable, this is no longer true for dimension over \mathbb{Q} , as there are infinitely many scalars (exponents). Poizat used Zilber's weak CIT, a consequence of Ax' differential Schanuel conjecture:

For any uniform family $V_{\bar{z}}$ of varieties there is a finite set T_0, \dots, T_r of tori, such that for any torus T , any $V_{\bar{b}}$ and any irreducible component $W \ni \bar{a}$ of $V_{\bar{b}} \cap \bar{a} \cdot T$, for some $i \in [0, r]$ $W \subseteq \bar{a} \cdot T_i$ and $\dim(T_i) - \dim(V \cap \bar{a} \cdot T_i) = \dim T - \dim W$, or $\dim T - \dim W \geq n - \dim V$, where $V \subseteq (K^\times)^n$.

This specifies finitely many possibilities for \mathbb{Q} -linear relations on a family of varieties which could render δ negative. Hence \mathcal{C} is again universal, approximate richness axiomatizable, the generic model exists, and \aleph_0 -saturated models of its theory are rich. It has Morley rank $\omega \cdot 2$, and a generic green point has rank ω .

7.2. The collaps. [5] A *code* is a formula $\varphi(\bar{x}, \bar{y})$ in the field language with $n = |\bar{x}|$ such that

- (1) For all \bar{b} either $\varphi(\bar{x}, \bar{b})$ is empty, or has Morley degree 1.
- (2) $\operatorname{tr.deg}'(\bar{a}/\bar{b}) = n/2$ and $\operatorname{lin.dim}_{\mathbb{Q}}(\bar{a}/\bar{b}) = n$ for generic $\bar{a} \models \varphi(\bar{x}, \bar{b})$, and for $i = 2, \dots, r$ and any W irreducible component of $V \cap \bar{a}T_i$ of maximal dimension, $\dim(T_i) > 2 \cdot \dim(W)$ if $V \cap \bar{a}T_i$ is infinite.
- (3) If $\operatorname{tr.deg}(\varphi(\bar{x}, \bar{b}) \cap \varphi(\bar{x}, \bar{b}')) = n/2$, then $b = b'$.
- (4) For any multiplicatively invertible \bar{m} and \bar{b} there is \bar{b}' with $\varphi(\bar{x} \cdot \bar{m}, \bar{b}) \equiv \varphi(\bar{x}, \bar{b}')$.

7.2.1. Toric correspondences. This time $\operatorname{GL}_n(\mathbb{Q})$ acts on the codes, which is infinite. Hence we cannot put invariance under $\operatorname{GL}_n(\mathbb{Q})$ into the axioms, but have to deal with it outside the codes. Using weak CIT we obtain:

There exists a collection \mathcal{S} of codes such that for every minimal prealgebraic definable set X there is a unique code $\varphi \in \mathcal{S}$ and finitely many tori T such that $T \cap (X \times \varphi(\bar{x}, \bar{b}))$ projects generically onto X and $\varphi(\bar{x}, \bar{b})$ for some \bar{b} . We call such a T a *toric correspondence*. In particular, for any code φ only finitely many tori can induce a toric correspondence between instances of φ .

7.2.2. *Difference sequences.* For every code φ there is some formula $\psi(\bar{x}_0, \dots, \bar{x}_k) \in \text{tp}(\bar{e}_0 \cdot \bar{f}^{-1}, \dots, \bar{e}_k \cdot \bar{f}^{-1})$ for some Morley sequence $(\bar{e}_0, \dots, \bar{e}_k, \bar{f})$ in $p_{\varphi(\bar{x}, \bar{b})}$ such that:

- (1) Any realization $(\bar{e}'_0, \dots, \bar{e}'_k)$ of ψ is disjoint, and $\models \varphi(\bar{e}'_i, \bar{b}')$ for some unique \bar{b}' definable over sufficiently large finite subsets of the \bar{e}'_i .
- (2) If $\models \psi(\bar{e}_0, \dots, \bar{e}_k)$, then $\models \psi(\bar{e}_0, \dots, \bar{e}_{k'})$ for each $k' \leq k$, and ψ is invariant under derivations.
- (3) Let $i \neq j$ and $(\bar{e}_0, \dots, \bar{e}_k)$ realize ψ with canonical parameter \bar{b} . If there is some toric correspondence T on φ and \bar{e}'_j with $(\bar{e}_j, \bar{e}'_j) \in T$, then $\bar{e}_i \not\perp_{\bar{b}} \bar{e}'_j \cdot \bar{e}_i^{-1}$ in case \bar{e}_i is a generic realization of $\varphi(\bar{x}, \bar{b})$.

7.2.3. *Counting.* Miraculously, this is enough to obtain the same counting lemma as before.

Since \mathbb{Q} -linear dependence need not be uniform, we have to use the weak CIT in order to uniformize dependencies in a non-generic difference sequence, and then the finite Ramsey theorem to obtain a long derived sequence inside the original model.

Again, we choose a fast-growing finite-to-one function μ from \mathcal{S} to ω , and define \mathcal{C}^μ to be the class of all $A \in \mathcal{C}$ who do not have a difference sequence for φ of length $\mu(\varphi)$. We obtain a characterisation when a structure in \mathcal{C}^μ has a minimal pre-algebraic extension not in \mathcal{C}^μ similar to the red case, and can prove thrifty amalgamation. Hence there is a generic model \mathfrak{M} .

7.2.4. *Axiomatization.* When we want to axiomatize approximate richness, we have to say that for all $\bar{a} \in A \leq \mathfrak{M}$, a code instance $\varphi(\bar{x}, \bar{a})$ has an A -generic realization in \mathfrak{M} , *unless* for a generic realization B we would have $AB \notin \mathcal{C}^\mu$. The weak CIT allows us to limit the possible \mathbb{Q} -linear dependencies we have to consider, with an extra twist: We may first have to extend by finitely many green generic points.

It follows that \aleph_0 -saturated models of $\text{Th}(\mathfrak{M})$ are rich, \mathfrak{M} has Morley rank 2, and $\ddot{U}^{\mathfrak{M}}$ has Morley rank 1.

Moreover, there also is an alternative inductive axiomatization analogous to the red case, which yields model-completeness by Lindström's theorem.

8. FUSION

Using similar techniques as for the expansions of fields, one also obtains:

- (1) Two strongly minimal sets with definable multiplicity property (DMP) in disjoint languages can be amalgamated freely with predimension $\delta(A/B) = RM_1(A/B) + RM_2(A/B) - |A \setminus B|$, and collapsed to a strongly minimal set [16].
- (2) Two strongly minimal sets with DMP with a common \aleph_0 -categorical reduct, one expansion preserving multiplicities, can be amalgamated freely with predimension $\delta(A/B) = RM_1(A/B) + RM_2(A/B) - RM_0(A/B)$, and collapsed to a strongly minimal set. See [14] for the one-based case, [7] for the fusion over vector spaces, which implies the full result according to [15].

- (3) Two theories of finite and definable Morley rank with DMP and of the same Morley degree can be amalgamated freely with predimension $\delta(A/B) = n_1 \cdot RM_1(A/B) + n_2 \cdot RM_2(A/B) - n \cdot |A \setminus B|$, where $n_1 \cdot RM(T_1) = n_2 \cdot RM(T_2) = n$, and collapsed to a structure of Morley rank n [21]. (It may be necessary to extend the language.)

In all cases except for the one-based collapse in (2), the language has to be countable in order to collapse.

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