A Generalized Class of∗-Bisimple Ample$\boldsymbol{\omega}$-Semigroups

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Abstract:- It is clear that most results in ample semigroups are but analogues of inverse semigroups. Unlike bisimple inverse ω-semigroups which $\mathcal{H}$-classes contains regular elements as studied in [28] and later extended in [2] and [3] to a class of ample semigroups called $\ast$ - bisimple Ample ω-semigroup and $\ast$ - simple Ample ω-semigroup, there exists a class of $\ast$ - bisimple Ample ω-semigroups in which certain $\mathcal{H}^*$-classes contains no regular elements. Close look at the internal structure of this class of Ample ω-semigroups reveals that some of the $\mathcal{H}^*$-classes rather contains bisystems of cancellative monoids. However, the presence of these bisystems of cancellative monoids makes this class of semigroups different from the once studied in [28], [2], [3] and [22]. Thus, in this work, we study such a class of $\ast$ - bisimple Ample ω-semigroups as an extension of the binary array of bisystems of cancellative monoids. The array of bisystems were closed and then certain rules are imposed to ensure the closure of multiplication of elements in the binary array of bisystems. Thus, we construct and study few ofits properties and then characterize them as a special extension of binary array of bisystems of sequence of cancellative monoids.

Keywords:- *Cancellative monoids, abundant semigroups, adequate semigroups, ample semigroups, ω-chain, *-bisimple semigroups, binary array of bisystems of cancellative monoids.*

I. INTRODUCTION AND PRELIMINARY RESULTS

[6] extensively studied monoids whose right S -systems were projective. [17] later extended this work by studying monoids in which their principal ideals were projective and termed them right principal projective monoids (RPP monoids). [8] used certain internal characterization of these RPP monoids to obtain a wider class of semigroups. He termed certain RPP monoids whose idempotents formed a semilattice “idempotent cancellable monoids”. With this [8], [9], [10] then obtained a class of semigroups whose structure parallels inverse semigroups (see [15], [20],[23]). This class of semigroups since then attracted many brilliant minds and stimulated deep investigations mostly because of it striking relationship with inverse semigroup. Thus, to study ample semigroups, it is natural to put it side by side with inverse semigroups as most of its results are found to be analogues of the later. In any case, there exists a subclass of ample semigroups whose structures does not duplicate those of inverse semigroups. [2]characterized the structure of $\ast$ - bisimpleAmple ω-semigroup in which each $\mathcal{H}^*$-class contains regular elements as a generalized Bruck-Reilly extension $BR^*(T,\theta)$ of cancellative monoid T,θ being a homomorphism on T. His result were clearly analogues of bisimple inverse ω-semigroups studied in[28]. Unlike the situation in inverse semigroup there is a class of $\ast$ - bisimple type $A\ \omega$ −semigroup in which certain $\mathcal{H}^*$ −classes contain no regular elements. When the internal structure of this class ample ω − semigroup is analyzed, it is found that some of the $\mathcal{H}^*$-classes contains bi systems of cancellative monoids. Thus, this result makes this class of Ample ω-semigroups surprisingly different from the one studied by [2] and also there is no analogue of this result to those studied in [28] under inverse semigroups.

This paper studies this subclass of ample semigroups. Particularly, this class of semigroups would be constructed in this paper, some of its properties would be presented.

The following well known results are presented:

Let $a,b\ \in S$, then Green’s $\ast$ −relations are defined as follows:

$$\mathcal{L}^* = \{(a,b) \in S\times S: \forall\, x,y\ \in S^1\ ax = ay\ \Leftrightarrow bx = by\}$$
$$\mathcal{R}^* = \{(a,b) \in S\times S: \forall\, x,y\ \in S^1\ xa = ya\ \Leftrightarrow xb = yb\}$$
$$\mathcal{H}^* = \{(a,b) \in S\times S: (a,b) \in \mathcal{L}^* \cap \mathcal{R}^*\}$$
$$\mathcal{D}^* = \{\exists\, c\ \in S: (a,c) \in \mathcal{L}^*: (c,b) \in \mathcal{R}^*\ or\ a\mathcal{L}^*c\mathcal{R}^*b\} = \mathcal{L}^* \vee \mathcal{R}^*$$

A. Lemma 1.1 (6)

$\mathcal{R}^*(\mathcal{L}^*)$*is a left(right) congruence.*

Proof

Let $(a,b) \in \mathcal{R}^*$, then $xa = ya\ \Leftrightarrow xb = yb$, let $c \in S$, so that $(xc)a = x(ca)$ and by associativity in S.

$x(ca) = y(ca)\ \Leftrightarrow (xc)b = x(cb) = y(cb)$ and so $(ca,cb) \in \mathcal{R}^*$ and $\mathcal{R}^*$ is a left congruence. Similar arguments hold for $\mathcal{L}^*$ which is a right congruence. □

B. Lemma 1.2

$$\mathcal{R} \subseteq \mathcal{R}^*, \mathcal{L} \subseteq \mathcal{L}^*$$

Proof

Let $(a,b) \in \mathcal{R}, \exists$ then $s,t \in S^1: as = b, bt = a$

$a = bt \Longrightarrow as = bts$, for $x,y \in S^1$, then $xas = xbts \Leftrightarrow xb = xas$ and if $xa = ya$, then

$xb = yas \Longrightarrow xb = yb$. Conversely $as = b \Longrightarrow ast = bt \Longrightarrow xbt = xast$ and if $xb = yb$, then

$xa = yast = ybt = ya$ and $(a,b) \in \mathcal{R}^*$, showing that $\mathcal{R} \subseteq \mathcal{R}^*$.

Similar argument holds for $\mathcal{L} \subseteq \mathcal{L}^*$ □

C. Corollary 1.3
If S is regular then $\mathcal{L} = \mathcal{L}^*$,$\mathcal{R} = \mathcal{R}^*$ □

$(a,b) \in \mathcal{L}^* \circ \mathcal{R}^* \Longrightarrow a\mathcal{L}^*c\mathcal{R}^*b$ and $(a,b) \in \mathcal{R}^* \circ \mathcal{L}^* \Longrightarrow a\mathcal{R}^*c\mathcal{L}^*b$. But clearly

$a\mathcal{L}^*c\mathcal{R}^*b = \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{R}^* \circ \mathcal{L}^* = a\mathcal{R}^*c\mathcal{L}^*b$ only when $xa = ax$, that is $\mathcal{L}^* \circ \mathcal{R}^* = \mathcal{R}^* \circ \mathcal{L}^*$ only when S commutes.

Hence generally $\mathcal{L}^* \circ \mathcal{R}^* \neq \mathcal{R}^* \circ \mathcal{L}^*$. □

A semigroup S is called ***left abundant*** if each $\mathcal{R}^* -$ class of S contains an idempotent and ***right abundant*** if each $\mathcal{L}^* -$class contains an idempotent. S is called ***abundant*** if it is both left and right abundant.

S is called ***left adequate*** if:
- it is left abundant
- idempotents commute
- each $\mathcal{R}^* -class$ contains a unique idempotent.

Similarly, S is called ***right adequate*** if:
- it is right abundant
- idempotents commute
- each $\mathcal{L}^*$-contains a unique idempotent.

S is ***adequate*** if it is *left* and *right* adequate.

Denote the unique idempotent of each $\mathcal{R}^* -$ class containing a as a^+ for left adequate and that of each $\mathcal{L}^* -$ class containing a as a^*.

D. Lemma 1.4
Let $a \in S, e \in E_S$. Then the following are holds.
- $(e,a) \in \mathcal{L}^*$.
- $ae = a$ and for all $x,y \in S^1, ax = ay \Longrightarrow ex = ey$.

Proof
- Recall that $\mathscr{L}\subseteq\mathscr{L}^*$, so suppose $(e,a) \in \mathscr{L}$ then $\exists x,y \in S^1: xe = a, ya = e$. So $xe = a$. Now suppose that $ex = ey$, then $xex = xey \Longrightarrow ax = ay$ and then $(e,a) \in \mathscr{L}^*$.
- Since from (i) above, $xe = a \Longrightarrow xee = xe^2 = xe = ae$ and so $ae = a$
- Similarly
- Similarly,$(a,e) \in \mathscr{R}^*$
- $ea = a$ and for all $x,y \in S^1, xa = ya \Longrightarrow xe = ye$. □

E. Lemma 1.5
Let S be an adequate semigroup with semilattice of idempotents E, then$\forall\ a,b \in S$,
- *$(a,b) \in \mathcal{R}^*$ if and only if $a^+ = b^+$; $(a,b) \in \mathcal{L}^*$ if and only if $a^* = b^*$.*
- *$(ab)^* = (a^*b)^*$ and $(ab)^+ = (ab^+)^+$*
- *$aa^* = a = a^+a$.*

Proof
- $(a,e) \in \mathcal{R}^*, (b,f) \in \mathcal{R}^*$. Where $e(f)$ are the idempotents in the $\mathcal{R}^* -$class containing $a(b)$. But $(a,b) \in \mathcal{R}^* \Longrightarrow xa = ya \Longrightarrow xb = yb$. But since$(a,e) \in \mathcal{R}^*$, then $xa = ya \Longrightarrow xe = ye$. Also $(b,f) \in \mathcal{R}^*$ since a,b are in the same $\mathcal{R}^* -$ classSo $(b,f) \in$

$\mathcal{R}^* \Longrightarrow xb = yb \Longrightarrow xf = yf$.So $(a,b) \in \mathcal{R}^* \Longrightarrow xa = ya \Longrightarrow xb = yb \Longrightarrow xe = ye \Longrightarrow xf = yf$. So $(e,f) \in \mathcal{R}^*$. But since S is adequate then the idempotent in each $\mathcal{R}^*$ $-$classis unique so $e = f$. If $e = a^+$ and $f = b^+$, the result follows. Converse is clear and straight forward.

- this follows immediately from (i) and the fact that $\mathcal{L}^*$ is a right identity on S and $\mathcal{R}^*$ isleft identity on S. So $(ab)^+ = (afb)^+ = (ab^+b)^+ = (ab^+)^+$.
- follows immediately from (i) and (ii).

Let S be left adequate, then S is called ***left ample*** if $\forall\ e \in E_S$, then$ae = (ae)^+a$ and ***right ample*** if $ea = a(ea)^*, \forall\ a \in S$.

In view of the definition above, let S be a semigroup with a semilattice of idempotents E_S. Then S is called an ***ample*** semigroup if and only if:

- S is cancellable
- for every $e \in E_S$ and $a \in S$, then $ae = (ae)^+a,\ ea = a(ea)^*$ □

Let $E_S = \{e_0, e_1, e_2, \dots, e_{n-1}, e_n, \dots\}$ where $e_0 \geq e_1 \geq e_2 \geq \cdots \geq e_{n-1} \geq e_n$.
Let $e_0, e_1, \dots, e_{n-1}, e_n, \dots$ be idempotents in Sthen E_S as defined is called ω $-$*chain*.
Let S be an ample semigroup with E_S as defined above, then S is called ***ample ω −semigroup***.
Observe that $E_S = \{e_n : n \geq 0\}, e_m \geq e_n \Leftrightarrow m \leq n$ and $m, n \in N$.

Let S be an ample semigroup. An ideal I of S is said to be $*$ $-$*ideal* if $\mathcal{L}_a^*, \mathcal{R}_a^* \subseteq I$. The smallest $*$ $-$*ideal* containing a which is the union of $\mathcal{D}^*$ $-$*classes* is denoted by $\mathcal{J}^*$.
Let S be *ample ω −semigroup* then S is called $*$ $-$*bisimple* if $\mathcal{D}^*$ is an identity relation on S. That is S is called $*$ $-$*bisimple* if it has a single $\mathcal{D}^*$-class.

Let S be a semigroup, and let $a, b \in S$, the relation $\widetilde{\mathcal{D}}$ on S is defined by:
$a\widetilde{\mathcal{D}}b \iff a^*\mathcal{D}b^*, a^+\mathcal{D}b^+$.
Observe that if $a, b \in S$ and $a^*, a^+, b^*, b^+, \in E$ then $a\widetilde{\mathcal{D}}a \Rightarrow a^*\mathcal{D}a^*, a^+\mathcal{D}a^+$.
Also, if $b\widetilde{\mathcal{D}}a \Longrightarrow b^*\mathcal{D}a^*, b^+\mathcal{D}a^+$then$a\widetilde{\mathcal{D}}b \Longrightarrow b\widetilde{\mathcal{D}}a$.
If $a\widetilde{\mathcal{D}}b, b\widetilde{\mathcal{D}}c$, then this implies that $a^*\mathcal{D}b^*, a^+\mathcal{D}b^+, b^*\mathcal{D}c^*, b^+\mathcal{D}c^+$ which implies that $a^*\mathcal{D}b^*\mathcal{D}c^*, a^+\mathcal{D}b^+\mathcal{D}c^+ \Longrightarrow a\widetilde{\mathcal{D}}c$ and then $\widetilde{\mathcal{D}}$ is an equivalence relation on S. □

What follows is an introduction of bisystems. Going forward, the terminologies used are as in [2] and [3].

II. BISYSTEMS

Consider a monoid M and let S be a set. If there exists a mapping $S \times M \to S$ such that the following holds:

- $x.1 = x,$
- $(xa)b = x(ab), \forall\ x \in M, a, b \in S$, then S is called a right $S -$ system.

Dually, if

- $1.x = x,$
- $a(bx) = (ab)x, \forall\ x \in M, a, b \in S$, then S is called a left $S -$ system.

Now, suppose that M_1, N_1 are monoids, then S is an $M_1, N_1 -$ bisystem if it is both right and left S $-$system and for $a, b \in N_1, x \in S, (ax)b = a(xb)$.

Let S_1be (M_1, N_1) bisystem while S_2 an (M_2, N_2) bisystem, then a mapping $f: S_1 \to S_2$ is a morphism$f$ from $S_1$to S_2if there exists $\theta: M_1 \to M_2$ and $\varphi: N_1 \to N_2$ such that for all $a \in M_1$ and $b \in N_1$, we have:

$(axb)f = a\theta.xf.b\varphi$. However, if S_1 and S_2 are both (M, N) bisystem and $\theta = \varphi = i$, the identity, then

$f: S_1 \to S_2$ is a morphism if for all $a \in M, b \in N$ and $x \in S, (axb)f = a.xf.b$

III. BINARY ARRAY OF BISYSTEMS

Consider the sequence of cancellative monoids $M_i, 0 \leq i \leq d-1$ with the linking homomorphism $\alpha_{i,j}: M_i \to M_j,\ i \leq j$ between them. By considering (M_i, M_j) bisystems $B_{i,j}, 0 \leq i, j \leq d-1$ such that for $0 \leq i \leq k \leq d-1, 0 \leq j \leq l \leq d-1$, there is a bisystem morphism from $B_{i,j}$ to $B_{k,l}$. That is:

$\gamma_{k,l}: B_{i,j} \to B_{k,l}$ defined by $x\gamma_{k,l} = e_k x e_l$ which then gives a sequence of bisystem mappings:

$B_{i,j} \to B_{i+1,j+1} \to B_{i+2,j+2} \to \cdots B_{k,l}$ where $k = d-1, l = j-i+d-1$ if $i > j$ and $k = i-j+d-1$ if $i < j$. Considering the mapping $\delta: B_{i,j} \to B_{k,l}$ where $x\delta = e_k.x.e_l$is a morphism. This is so because the linking homomorphism $\alpha_{i,k}: M_i \to M_k$ and $\alpha_{j,l}: M_j \to M_l$ are defined respectively by

$a\alpha_{i,k} = e_k.a, b\alpha_{j,l} = be_l$ for $a \in M_i, b \in M_j$. Now we observe the following:

For any $x \in B_{i,j}$, then;

$$(axb)\,\delta = e_k(axb)e_l$$
$$= e_k.axb.e_l$$
$$= e_k.a.xbe_l$$
$$= a\alpha_{i,k}.x.b\alpha_{j,l}$$
$$= (a\alpha_{i,k}).e_kxe_l.(b\alpha_{j,l})$$
$$= (a\alpha_{i,k}).x\delta.(b\alpha_{j,l})$$

Now let $\boldsymbol{B} = \cup \{B_{i,j}: i,j = 0,1,2,\dots,d-1\}$ be a collection of bisystems where we have $M_i = B_{i,i}$,

$$0 \leq i \leq d-1\}.$$

Now if θ and φ are morphisms on $\boldsymbol{B}$ such that

$\theta: B_{m,n} \to B_{m-n,0}$ and $\varphi: B_{m,n} \to B_{0,n-m}$ [2.01]

Where we put $\overline{m-n} = m-n$ if $m > n$, or $m-n+d$ if $m < n$, then we have:

A. Lemma 2.1

- $B_{m,n}\,\theta\varphi \subseteq B_{0,\overline{n-m}}$.
- $B_{m,n}\,\varphi\theta \subseteq B_{\overline{m-n},0}$
- $M_m\,\theta = B_{m,m}\,\theta \subseteq B_{0,0} = M_0$

Proof

- $B_{m,n}\,\theta\varphi = (B_{m,n}\,\theta)\varphi = B_{\overline{m-n},0}\,\varphi = B_{0,0-(\overline{m-n})} = B_{0,(\overline{n-m})} \subseteq B_{0,(\overline{n-m})}$.
- $B_{m,n}\,\varphi\theta = (B_{m,n}\,\varphi)\theta = B_{0,(\overline{n-m})}\theta \subseteq B_{\overline{m-n},0}$.
- off course $M_m\,\theta = B_{m,m}\,\theta \subseteq B_{0,0} = M_0 = M_m\varphi$.

Following lemma 2.1 above, we assume the following conditions:

- $x\theta = x\varphi\theta$ if $m > n$
- $x\theta = x\varphi$ if $m = n$
- $x\theta\varphi = x\varphi$ if $m < n$

Following the construction above, we shall refer to the collection of bisystems$\boldsymbol{B} = \cup B_{i,j}$ as a binary array of bisystems if there is a multiplication " $*$" on $\boldsymbol{B}$ such that if $x, y \in \boldsymbol{B}$, then $x * y \in \boldsymbol{B}$.

Thus, we use an operation "$*$" on the collection $\boldsymbol{B} = \cup B_{i,j}$, where for $x \in B_{m,n}$ and $y \in B_{p,q}$, so that

$x * y \in B_{\overline{m-n+t,q-p+t}}, t = \max(n,p)$ [2.02]

Now we observe that if:

- $t = n$ then $x * y \in B_{m,q-p+n}$
- $t = p$ then $x * y \in B_{\overline{m-n+p},\,q}$
- $t = n = p$ then $x * y \in B_{m,q}$.

Suppose that $t = n$, and $q-p+n > d$, then we put $q-p+n = d+k, 0 \leq k < d$ so that;

$B_{m,\overline{q-p+n}} = B_{m,k}$. Thus, we have that for every $x \in B_{m,n}, y \in B_{p,q}$, $x * y = B_{m,\overline{q-p+n}}$ implies that;

$x * y \in B_{m,k}$ for $q-p+n > d$. But if $q-p+n > d$, then $n > d+p-q$ and then $B_{d+p-q,0}, = B_{\overline{p-q},0}$.

More so, we observe that:

$y\theta \implies \theta: B_{p,q}\theta \rightarrow B_{p-q,0}$, and so:

$x * y\theta \in B_{\overline{m-n+t},\overline{q-p+t}} = B_{m,k}$. But $t = \max(n, q-p)$.

But if $t = n$ and so $x * y\theta \in B_{m,\overline{q-p+n}} = B_{m,k}$

Following these observations, we assume the following conditions:

- $x * y \in B_{\overline{m-n+t},\overline{q-p+t}} = x * y\theta$ if $q - p + n > d$. [2.03ai]
- $(x * y)\theta \in B_{(m-n+t)-(q-p+t),0} = B_{m-n-p-q,0}$ and $x\theta * y\theta = B_{m-n-p-q,0}$ for
- $0 < q - p < d - n$
- But if $0 \leq q - p < d - n$, then $(x * y)\theta = x * y\theta$ [2.03aii]
- Lastly if $q < p$, we then impose the condition $(x * y)\theta = x\theta * y\theta$ [2.03aiii]

One can see that the morphism φ is the dual of θ and we have the following conditions which corresponds to [2.03 ai], [2.03aii] and [2.03aiii]. thus, we have:

$x * y = x\varphi * y$, if $m - n + p \geq d$ [2.03bi]

$(x * y)\varphi = x\varphi * y$, if $0 \leq m - n < d - p$ [2.03bii]

$(x * y)\varphi = x\varphi * y\varphi$ if $m < n$ [2.03biii]

Let $x \in B_{m,n}, y \in B_{p,q}, z \in B_{r,s}$

Considering the binary array of bisystem$\boldsymbol{B}$, we observe the following:

$(x * y) * z \in B_{\overline{m-n+t},\overline{q-p+t}} * B_{r,s}$ and $x * (y * z) = B_{m,n} * B_{p-q+u,s-r+u}$, [2.04]

where $t = \max(n, p)$ and $u = \max(q, r)$. Now see that if $u = r$, then with $p - q + r \geq d$ and $r > q - p + t$, we find that both $(x * y) * z \in B_{m-n+p-q+n,s}$, and $x\theta * (y * z) \in B_{m-n+p-q+n,s}$.

Thus, we assume that:

$(x * y) * z = x\theta * (y * z)$, if $p - q + r \geq d$. [2.05i]

Dually, let$(x * y) * z\varphi = x * (y * z)$ for $t = n$ and $p - q + n \geq d$ [2.05ii]

And $(x * y) * z = x * (y * z)$ if $p - q + n \leq d$ [2.05iii]

Following the observations [2.05i] through [2.05iii], we remark that the binary array of bisystems$(\boldsymbol{B}, d, \theta, \varphi)$ is a collection of bisystems $\boldsymbol{B} = \cup\{B_{i,j}: i, j \in d\}$ such that;

- If $i = j$, then $B_{i,i} = M_i, 0 \leq i, j \leq d - 1$ is a sequence of cancellative monoids with the linking homomorphism $\alpha_{i,j}: M_i \rightarrow M_j, 0 \leq i, j \leq d - 1$ between them.
- Each $B_{i,j}$ for $i \neq j$ is a (M_i, M_j) bisystems where $0 \leq i, j \leq d - 1$ and
- $\theta, \varphi : \boldsymbol{B} = \cup\{B_{i,j}: 0 \leq i, j \leq d - 1\} \rightarrow \boldsymbol{B}$ are mappings as earlier defined in 2.1 satisfying the binary multiplication defined in 2.02 under which the conditions [2.03ai] - [2.03aiii] and [2.05bi] - [2.05biii] are true.

Now suppose that $B_{m,n}, B_{n,m}$ are bisystems in $(\boldsymbol{B}, d, \theta, \varphi)$ then the product in 2.02 implies that:

$B_{m,n} * B_{n,m} = B_{m-n+n,m-n+n} \subseteq B_{m,m}$. Let $B_{m,n} * B_{n,m} = I_m \subseteq B_{m,m}$ [2.06]

where I_m is a subset of M_m not containing a unit of M_m.

Let $x \in B_{m,n}, y \in B_{n,m}$and e_m, e_n respectively the identities in M_m, M_n.

Clearly,$M = \bigcup_{i=0}^{d-1} M_i$,Now we observe that $B_{m,m} * B_{m,m} \subseteq B_{m,m}$ and so let $e_m \subseteq B_{m,m}$

$y * e_m = B_{n,m} * B_{m,m} \subseteq B_{n,m} = y$ and $x * e_n = B_{m,n} * B_{n,n} \subseteq B_{m,n} = x$

Also see that $x * y * x = (B_{m,n} * B_{n,m}) * B_{m,n} \subseteq B_{m,m} * B_{m,n} = B_{m,n} = x$.

Thus if $x \in M_m, y \in M_n$, then $x * y \in M_t$, $t = \max(m,n)$, $M\theta = B_{0,0} = M\varphi$ and then $\theta| M = \varphi| M$.

Thus, we have proved:

B. Lemma 2.2

Let $x \in B_{m,n}, y \in B_{n,m}$ and e_m, e_n the respective identities in M_m, M_n and let $\boldsymbol{M} = \bigcup_{i=0}^{d-1} M_i$, then:

- $y * e_m = y,\ x * e_n = x$
- $x * y * x = x$;
- if $x \in M_m, y \in M_n$, then $x * y \in M_t$, $t = \max(m,n)$, and $M\theta = M\varphi$.

C. Lemma 2.3

Let $x \in B_{m,n}, m, n \in d$. Then for $\alpha, \beta \in N$:

- $x\theta^\alpha\varphi^\beta = x\varphi^{\alpha+\beta}$ if $m \geq n$ and $x\theta^\alpha\varphi^\beta = x\varphi^{\alpha+\beta-1}$ if $m < n$.
- $x\theta^\alpha\varphi^\beta = x\varphi^{\alpha+\beta}$ if $m \leq n$ and $x\theta^\alpha\varphi^\beta = x\varphi^{\alpha+\beta-1}$ if $m > n$.

Proof

If $m = n$ then clearly, $x \in M_m$ and with $\varphi = \theta$ and the lemma holds evidently. However, if we suppose that $m \neq n$, then for $x \in B_{m,n}$, evidently $x\varphi \in B_{0,n-m}, n - m \neq 0$, with $x\theta = x\varphi\theta$ if $m > n$ and $x\varphi = x\theta\varphi$ if $m < n$, this quickly follows that:

$$(x\theta)\varphi = (x\theta\varphi)\varphi = (x\varphi)\varphi = x\varphi^2$$

Thus, by induction principle, we have:

$x\theta\varphi^k = (x\theta\varphi)\varphi^{k-1} = (x\varphi)\varphi^{k-1} = x\varphi^k$, if $m < n$ and $x\theta\varphi^k = (x\varphi\theta\varphi)\varphi^{k-1} = (x\varphi^2)\varphi^{k-1} = x\varphi^{k+1}$, if $m \geq n$, for all-natural numbers. Moreover, we observe that:

$x\theta^2\varphi^k = (x\theta)x\theta\varphi^k = x\varphi^k,\ m < n.$

Therefore, by inductive assumption that the result holds for $h - 1$, it follows that:

$x\theta^h\varphi^k = (x\theta^{h-1})(\theta\varphi^{k+1}) = x\varphi^{h-1}\varphi^{k+1} = \cdots x\varphi^{h+k}, m \geq n$ and $\varphi^k = (x\theta^h)(\theta\varphi^k) = (x\theta)\theta^{h-1}\varphi^k = x\varphi^{k+h-1}, m < n$ proving (i). thus (ii) follows dually.

IV. CONSTRUCTION

In this section, we construct a generalized $* -$ bisimple type ample $\omega -$ semigroup.

Considering a given array of bisystems$(\boldsymbol{B}, d, \theta, \varphi)$ satisfying the conditions already stated above. We denote the set of triples by:

$S(\boldsymbol{B}, d, \theta, \varphi) = \{(m, x, n): m, n \in N, x \in B_{\overline{m},\overline{n}}\}$, for simplicity, let $S(\boldsymbol{B}, d, \theta, \varphi) = S$

Suppose that $(m, x, n), (p, y, q) \in S(\boldsymbol{B}, d, \theta, \varphi)$, define the multiplication on $S(\boldsymbol{B}, d, \theta, \varphi)$ by:

$$(m, x, n)(p, y, q) = (m - n + t, x\theta^{t\prime - n_d}y\varphi^{t\prime - p_d}, q - p + t), \qquad [3.01]$$

where $t = \max(n, p)$, $t' = \max(n_d, p_d)$.

It should be noted that $\theta^{t\prime - n_d}\varphi^{t\prime - p_d}$ is an appropriate endomorphisms while $\theta^0\varphi^0$ is an appropriate identity endomorphism on the array of bisystems in $\boldsymbol{B}$.

Now observe that under the multiplication as in 3.01, if $t = n, t' = n_d$, then 3.01 becomes:

$(m, x, n)(p, y, q) = (m, x\theta^0y\varphi^{n_d - p_d}, q - p + n) = (m, x * y\varphi^{n_d - p_d}, q - p + n)$, where it is clear that $x * y\varphi^{n_d - p_d} \in B_{\overline{m},\overline{q-p+n}}$ and then $(m, x * y\varphi^{n_d - p_d}, q - p + n) \in S$

If $t = p, t' = p_d$, then;

$(m, x, n)(p, y, q) = (m, x\theta^{p_d - n_d}y\varphi^o, q - p + n) = (m - n + p, x\theta^{p_d - n_d} * y, q)$, and so $x\theta^{p_d - n_d} * y \in B_{\overline{m-n+p},q}$, and then again $(m - n + p, x\theta^{p_d - n_d} * y, q) \in S$.

Lastly, if $n = p$, then $x\theta^{t\prime - n_d}y\varphi^{t\prime - p_d} = x * y \in B_{m,q} \in S$. Thus S is closed under the multiplication.

Now suppose that $a, b, c \in S$, where $a = (m, x, n), b = (p, y, q), c = (r, z, s)$ then:

$(ab)c = [m - n + p - q + u, (x\theta^{t'-n_d} y\varphi^{t'-p_d})\theta^{u'-a_d} z\varphi^{u'-r_d}, s - r + u]$, [3.02]

$a = q - p + t, t = \max(n,p), u = \max(q - p + t, r) = \max(a, r)$, and so let:

$M_1 = m - n + p - q + u$ [3.03]

$X = (x\theta^{t'-n_d} y\varphi^{t'-p_d})\theta^{u'-a_d} z\varphi^{u'-r_d}$ [3.04]

$N_1 = s - r + u$ [3.05]

Similarly:

$a(bc) = (m, x, n)[(p, y, q)(r, z, s)] = (m, x, n)(p - q + v, y\theta^{v'-q_d} z\varphi^{v'-r_d}, s - r + v), v = \max(q, r), v' = \max(q_d, r_d)$.

$a(bc) = (m - n + w, x\theta^{w'-n_d}(y\theta^{v'-q_d} z\varphi^{v'-r_d})\varphi^{w'-b_d}, s - r + q - p + w)$ [3.06]

Let $b = p - q + v, w = \max(n, b), w' = \max(n_d, b_d)$, and so let:

$M_2 = m - n + w$ [3.07]

$Y = x\theta^{w'-n_d}(y\theta^{v'-q_d} z\varphi^{v'-r_d})\varphi^{w'-b_d}$[3.08]

$N_2 = s - r + q - p + w$[3.09]

But the outer coordinates are bicyclic and so from [3.03], [3.05], [3.07] and [3.09] we have:

$m - n + p - q + u = M_1 = M_2 = m - n + w$ and so

$w = p - q + u$[3.10]

and so, it follows that:

$u = \max(q - p + t, r) = \max(q - p + max(n, p), r)$ so that:

$w = \max(n, b_d) = \max\{n, p - q + v\} = \max\{n, p - q + max(q, r)\}$ [3.11]

We now establish the equality for the middle coordinates. However, in view of lemma 2.3, we do so via the following observations:

Now observe the following:

If m, n, p are natural numbers such that $m < p$, let $r = n - m + p$, then obviously,

$r = r_d.d + \bar{r} = (n - m + p)_d.d + \overline{n - m + p} = (n_d - m_d + p_d).d + \bar{n} - \bar{m} + \bar{p}$.

Thus, we compare and take that out that;

$$r_d - n_d = \begin{cases} p_d - m_d & if\ \bar{n} - \bar{m} + \bar{p} < d \\ p_d - m_d + 1 & if\ \bar{n} - \bar{m} + \bar{p} > d \\ p_d - m_d - 1 & if\ \bar{n} - \bar{m} + \bar{p} < 0 \end{cases}$$

We now attempt to establish an equality on the middle co-ordinate by considering the following cases;

A. CASE 1

$$w = \max\{n, p - q + max(q, r)\} = n$$

Then [3.08] becomes $n = p - q + u$ or $u = q - p + n$ and then $w' = n_d$.

Also, $w = n = \max\{n, p - q + max(q, r)\} \Longrightarrow n > p - q + max(q, r) > p$ implying that

$t = \max(n, p) = n$.

However, with $u = \max(q - p + t, r) = \max(q - p + n, r) = q - p + n$ implies that

$q - p + n > r$.

Now with $a = q - p + t = q - p + \max(n, p) = q - p + n = u$, then clearly $a = u' = u_d$. More so using the value $u = q - p + n$ we have that $u' = (q - p + n)_d = q_d - p_d + n_d + \gamma = u_d$, and we impose that:

$$\gamma = \begin{cases} 0 \text{ if } 0 < \bar{q} - \bar{p} + \bar{n} < d \\ 1 \text{ if } \quad \bar{q} - \bar{p} + \bar{n} \geq d \\ -1 \text{ if } \quad \bar{q} - \bar{p} + \bar{n} < 0 \end{cases} \text{[3.12]}$$

Now if $t' = \max(n_d, p_d) = n_d$ then [3.04] becomes;

$X = (x\theta^0 y\varphi^{n_d - p_d})\theta^0 z\varphi^{u' - r_d} = (x.y\varphi^{n_d - p_d})z\varphi^{u_d - r_d}$.

By letting $\alpha = n_d - p_d$, $\beta = r_d - q_d$, $\gamma = 0, 1, -1$. In view of [3.12] and then we observe that:

$u_d - r_d = q_d - p_d + n_d + \gamma - r_d = n_d - p_d - r_d + q_d + \gamma = (n_d - p_d) - (r_d - q_d) + \gamma$, and so:

$$u_d - r_d = \alpha - \beta + \gamma$$

Then $X = (x.y\varphi^{\alpha})z\varphi^{\alpha - \beta + \gamma}$ [3.13]

Also recall from [3.08] that $Y = x\theta^{w' - n_d}\left(y\theta^{v' - q_d} z\varphi^{v' - r_d}\right)\varphi^{w' - b_d}$

But $w' = n_d$, and so

$Y = x\theta^{n_d - n_d}\left(y\theta^{v' - q_d} z\varphi^{v' - r_d}\right)\varphi^{n_d - b_d} = x.\left(y\theta^{v' - q_d} z\varphi^{v' - r_d}\right)\varphi^{n_d - b_d}$ [3.14]

Now, to obtain a more precise expression for Y, two cases are considered.

Subcase 1a:

$$\max(q, r) = q$$

Now for $v = \max(q, r) = q$, then $v' = q' = q_d$ and so [3.14] becomes:

$Y = x.(y.z\varphi^{q_d - r_d})\varphi^{n_d - b_d}$ [3.15]

But $b = p - q + v = p - q + \max(q, r) = p$ and so $b_d = p_d$, so that [3.15] becomes:

$Y = x.(y.z\varphi^{q_d - r_d})\varphi^{n_d - p_d}$[3.16]

$$Y = x.(y\varphi^{n_d - p_d}.z\varphi^{q_d - r_d + n_d - p_d}) = x.\left(y\varphi^{\alpha}.z\varphi^{(n_d - p_d) - (r_d - q_d)}\right)$$

Thus, $Y = x.\left(y\varphi^{\alpha}.z\varphi^{\alpha - \beta}\right)$ [3.17]

Thus, if $0 < \bar{q} - \bar{p} + \bar{n} < d$, then $X = (x.y\varphi^{\alpha})z\varphi^{\alpha - \beta} = x.\left(y\varphi^{\alpha}.z\varphi^{\alpha - \beta}\right) = Y$.

However, if $\bar{q} - \bar{p} + \bar{n} > d$, then $a = q - p + t = q - p + \max(n, p) = q - p + n$, so that

$a_d = (q - p + n)_d = q_d - p_d + n_d + \gamma$,

Thus $a_d - r_d = q_d - p_d + n_d + \gamma - r_d = (n_d - p_d) - (r_d - q_d) + \gamma = \alpha - \beta + 1$, and therefore:

$$X = (x.y\varphi^{n_d - p_d})z\varphi^{a_d - r_d} = (x.y\varphi^{\alpha})z\varphi^{\alpha - \beta + 1} = x.(y\varphi^{n_d - p_d}.z\varphi^{q_d - r_d + n_d - p_d})$$
$$= x.\left(y\varphi^{\alpha}.z\varphi^{\alpha - \beta + 1}\right) = Y$$

But if $\bar{q} - \bar{p} + \bar{n} < 0 \Longrightarrow p - q < d - r$ and by [3.03bii], it follows that $X = Y$.

- *Subcase 1b*

$$\text{Max}(q, r) = r$$

Now if $Max(q, r) = r$, then $v' = \max(q_d, r_d) = r_d$,

$b = p - q + v = p - q + \max(q, r) = p - q + r$, and so: $b_d = p_d - q_d + r_d + \gamma$,thus, [3.13] becomes:

$$Y = x.\left(y\theta^{v'-q_d} z \varphi^{v'-r_d}\right)\varphi^{n_d-b_d} = x.\left(y\theta^{r_d-q_d} z\right)\varphi^{n_d-b_d} = x.\left(y\theta^{r_d-q_d} z\right)\varphi^{n_d-(p_d-q_d+r_d+\gamma)}$$

$$Y = x.\left(y\theta^{r_d-q_d} z\right)\varphi^{n_d-p_d+q_d-r_d+\gamma} = x.\left(y\theta^{r_d-q_d} z\right)\varphi^{(n_d-p_d)-(r_d-q_d)+\gamma}$$

$$Y = x.\left(y\theta^{\beta}.z\right)\varphi^{\alpha-\beta+\gamma}. \qquad [3.18]$$

But $y\theta^{\beta}\varphi^{\alpha-\beta+\gamma} = y\varphi^{\alpha+\gamma}$ by lemma [2.3] and so [3.18] becomes:

$$Y = x.\left(y\varphi^{\alpha+\gamma}.z\varphi^{\alpha-\beta+\gamma}\right) \quad [3.19]$$

- *Remarks*

We remark the following which led to the conclusion of [3.19].

We observed that since

$n - b = n - (p - q + r) = n - p + q - r = (q - p + n) - r = a - r$ so that

$$n_d - b_d = n_d - (p_d - q_d + r_d + \gamma) = (n_d - p_d) - (r_d - q_d) - \gamma = \alpha - \beta - \gamma$$

Similarly:

$$a_d - r_d = q_d - p_d + n_d + \gamma - r_d = (n_d - p_d) - (r_d - q_d) + \gamma = \alpha - \beta + \gamma$$

Where we have by [3.12], that is:

$$\gamma = \begin{cases} 0 \text{ if } 0 < \bar{p} - \bar{q} + \bar{r} < d \text{ or } 0 < \bar{q} - \bar{p} + \bar{n} < d \\ 1 \text{ if } \quad \bar{p} - \bar{q} + \bar{r} > d \text{ or} \bar{q} - \bar{p} + \bar{n} > d \\ -1 \text{ if } \quad \bar{p} - \bar{q} + \bar{r} < 0 \text{ or} \bar{q} - \bar{p} + \bar{n} < 0 \end{cases}$$

Now we observe the correspondence between $a_d - r_d$ and $n_d - b_d$. This can be viewed from the fact that if $\bar{q} > \bar{p}$ then clearly $\bar{q} - \bar{p} + \bar{n} > 0$ or $0 < \bar{q} - \bar{p} + \bar{n} < d$. However, if $\bar{p} > \bar{q}$, then $0 < \bar{p} - \bar{q} + \bar{r} < d$ or $\bar{p} - \bar{q} + \bar{r} > d$. Thus, we can see that the value $\bar{q} - \bar{p} + \bar{n} < 0$ corresponds to

$0 < \bar{p} - \bar{q} + \bar{r} < d$ or $\bar{p} - \bar{q} + \bar{r} > d$. In a similar way if $\bar{q} > \bar{p}$, then clearly $\bar{q} - \bar{p} + \bar{n} < d$ corresponds to the values $\bar{p} - \bar{q} + \bar{r} < 0$ or $0 < \bar{p} - \bar{q} + \bar{r} < d$. Thus, we summarize the following conditions

(C1)

i	$a_d - r_d = \alpha - \beta, 0 < \bar{q} - \bar{p} + \bar{n} < d$	$n_d - b_d = \alpha - \beta, 0 < \bar{p} - \bar{q} + \bar{r} < d$
ii	$a_d - r_d = \alpha - \beta, 0 < \bar{q} - \bar{p} + \bar{n} < d$	$n_d - b_d = \alpha - \beta - 1, \bar{p} - \bar{q} + \bar{r} < d$
iii	$a_d - r_d = \alpha - \beta, 0 < \bar{q} - \bar{p} + \bar{n} < d$	$n_d - b_d = \alpha - \beta + 1, \bar{p} - \bar{q} + \bar{r} < d$
iv	$a_d - r_d = \alpha - \beta + 1,$ $\bar{q} - \bar{p} + \bar{n} > d$	$n_d - b_d = \alpha - \beta, 0 < \bar{p} - \bar{q} + \bar{r} < d$
v	$a_d - r_d = \alpha - \beta + 1,$ $\bar{q} - \bar{p} + \bar{n} > d$	$n_d - b_d = \alpha - \beta + 1, \bar{p} - \bar{q} + \bar{r} < 0$
vi	$a_d - r_d = \alpha - \beta - 1,$ $\bar{q} - \bar{p} + \bar{n} < 0$	$n_d - b_d = \alpha - \beta, 0 < \bar{p} - \bar{q} + \bar{r} < d$
vii	$a_d - r_d = \alpha - \beta - 1,$ $\bar{q} - \bar{p} + \bar{n} < 0$	$n_d - b_d = \alpha - \beta - 1, \bar{p} - \bar{q} + \bar{r} < d$

Thus, in respect of conditions (C1) of table 1 above, with $a_d - r_d = \alpha - \beta$ and

$n_d - b_d = \alpha - \beta$, if $\bar{p} > \bar{q}$ then equations [3.13] and [3.18], that is:

$X = (x.y\varphi^{\alpha})z\varphi^{\alpha-\beta}$ and $Y = x.(y\theta^{\beta}.z)\varphi^{\alpha-\beta} = x.(y\varphi^{\alpha}.z\varphi^{\alpha-\beta}) = X$.

Also, if $\bar{q} > \bar{p}$, then [3.18] becomes:

$Y = x.(y\theta^{\beta}.z)\varphi^{\alpha-\beta} = x.(y\varphi^{\alpha-1}.z\varphi^{\alpha-\beta}) = x.(y\varphi^{\beta-1}.z)\varphi^{\alpha-\beta} = X$.

To verify (CI) ii

$a_d - r_d = \alpha-\beta, 0 < \bar{q}-\bar{p}+\bar{n} < d$ and $n_d - b_d = \alpha-\beta-1, \bar{p}-\bar{q}+\bar{r} < d$, so that

$Y = x.(y\theta^{\beta}.z)\varphi^{\alpha-\beta-1}$. But recall that $y\theta \in M_{p-q,0}$, $y\theta^2 \in M_{p-q,0}$ and $y\theta^{\beta} \in M_{p-q,0}$, so that

$y\theta^{\beta}.z \in M_{p-q+d,s-r+d}, d = \max(0,r) = r \Longrightarrow y\theta^{\beta}.z \in M_{p-q+r,s}, \bar{p}-\bar{q}+\bar{r} > d$, but by lemma [2.3], we have:

$y\theta^{\beta}.z = y\theta^{\beta}\varphi.z = y\theta^{\beta+1}\varphi$ and so $Y = x.(y\theta^{\beta}.z)\varphi^{\alpha-\beta-1} = Y = x.(y\theta^{\beta+1}.z)\varphi^{\alpha-\beta-1}$.

But

$X = (x.y\varphi^{\alpha})z\varphi^{\alpha-\beta} = x.(y\varphi^{\beta}.z)\varphi^{\alpha-\beta} = x.(y\varphi^{\beta}\varphi.z\varphi)\varphi^{\alpha-\beta-1} = x.(y\theta^{\beta+1}.z)\varphi^{\alpha-\beta-1} = Y$. (in view of condition [3.03bii].

Now for (C1) iii

$X = (x.y\varphi^{\alpha})z\varphi^{\alpha-\beta} = x.(y\varphi^{\alpha}.z\varphi^{\alpha-\beta})$ but for $\bar{p} > \bar{q}$, then

$$Y = x.(y\theta^{\beta}.z)\varphi^{\alpha-\beta+1} = x.(y\varphi^{\beta}.z\varphi^{\alpha-\beta+1}) = X$$

For (C1) iv

$X = (x.y\varphi^{\alpha})z\varphi^{\alpha-\beta+1}$ thus with $\bar{q} > \bar{p}, 0 < \bar{p}-\bar{q}+\bar{r} < \bar{d}$, then

$$Y = x.(y\theta^{\beta}.z)\varphi^{\alpha-\beta} = x.(y\varphi^{\alpha-1}.z)\varphi^{\alpha-\beta} = X.$$

For (C1) v

$X = (x.y\varphi^{\alpha})z\varphi^{\alpha-\beta+1}$ and with $\bar{q} > \bar{p}$, then:

$$Y = x.(y\theta^{\beta}.z)\varphi^{\alpha-\beta} = x.(y\varphi^{\alpha}.z\varphi^{\alpha-\beta+1}) = X$$

For (C1) vi

$X = (x.y\varphi^{\alpha})z\varphi^{\alpha-\beta-1}$, but $\bar{p} > \bar{q}$, and so $r < p-q+r < d$. Thus

$$Y = x.(y\theta^{\beta}.z)\varphi^{\alpha-\beta} = x.(y\varphi^{\alpha}.z\varphi^{\alpha-\beta}) = X$$

For (C1) vii

$X = (x.y\varphi^{\alpha})z\varphi^{\alpha-\beta-1}$, but $\bar{p} > \bar{q}$, and so $\bar{p}-\bar{q}+\bar{r} > d$. Thus

$Y = x.(y\theta^{\beta}.z)\varphi^{\alpha} = x.(y\varphi^{\alpha}.z\varphi^{\alpha-\beta-1}) = X$.

- *Case II*
  If $w = p-q+maxq$,

ɼall that $w = \max(n,b) = \max\{n, p-q+max(q,r)\} = p-q+max(q,r)$ then it implies that;

$p-q+max(q,r) > n$. But $a = q-p+t = q-p+max(n,p)$.

Also recall by [3.10] that $w = p-q+u$ so that $w = p-q+u = p-q+max(q,r)$ and so

$u = max(q,r) = v$, thus $u' = u_d = max(q_d,r_d) = v'$. Thus equation [3.08] becomes:

$$Y = x\theta^{w'-n_d}(y\theta^{v'-q_d}z\varphi^{v'-r_d})\varphi^{w'-b_d} = Y = x\theta^{b_d-n_d}(y\theta^{u_d-q_d}z\varphi^{u_d-r_d}) \qquad [3.20]$$

In order to establish the equality between [3.04] and [3.20], we consider the following subcases:

- **Subcase II (a)**
  If $v = u = max(q,r) = q$, but $u = q = q - p + max(n,p) \Longrightarrow p = \max(n,p) = t$ so that

  $t' = p_d$.

$$a_d = u_d = q_d$$

Thus $X = (x\theta^{p_d - n_d}.y)z\varphi^{q_d - r_d}$

Now let $p_d - n_d = \alpha$, $q_d - r_d = \beta$ and so

$X = (x\theta^{\alpha}.y)z\varphi^{\beta} = x\theta^{\alpha}.(y.z\varphi^{\beta})$.

Also observe that $b_d = p_d$ and so [3.20] becomes:

$$Y = x\theta^{p_d - n_d}(yz\varphi^{q_d - r_d}) = x\theta^{\alpha}.(yz\varphi^{\beta}) = X$$

- **Subcase II** (b)

$$v = max(q,r) = r$$

Now observe that if $v = max(q,r) = r$, then $w = w = p - q + max(q,r) = p - q + r$ so that

$p - q + r > n$. But by [3.10], $w = p - q + u$ and then $u = r$ so that $u_d = r_d$. Also see that

$b = p - q + v = p - q + \max(q,r) = p - q + r$, so that $b_d = (p - q + r)_d = p_d - q_d + r_d + \gamma$ with the imposed conditions:

$$\gamma = \begin{cases} 0 \text{ if } 0 < \bar{p} - \bar{q} + \bar{r} < d \\ 1 \text{ if} \bar{p} - \bar{q} + \bar{r} > d \\ -1 \quad \text{if} \bar{p} - \bar{q} + \bar{r} < 0 \end{cases}$$

and so let $\alpha = (p_d - n_d), \beta = (r_d - q_d)$, thus $b_d - n_d = (p_d - n_d) + (r_d - q_d) + \gamma$ $b_d - n_d = \alpha + \beta + \gamma$.

Thus [3.20] becomes:

$Y = x\theta^{b_d - n_d}(y\theta^{r_d - q_d}.z) = x\theta^{\alpha+\beta+\gamma}(y\theta^{\beta}.z)$ [3.21]

To obtain a more precise expression of the values of X we remodel [3.04] as in the following subcases.

✓ **Subcase II b(i)** $\boldsymbol{t = \max(n,p) = p}$

If so then $t' = p_d a = q - p + \max(n,p) = q - p + p = q$ and so $a_d = q_d$ and then [3.04] becomes:

$X = (x\theta^{p_d - n_d}y)\theta^{r_d - q_d}z = (x\theta^{p_d - n_d + r_d - q_d}y\theta^{r_d - q_d}).z = (x\theta^{\alpha+\beta}y\theta^{\beta}).z$ [3.22]

Clearly observe that if $\gamma = 0{,}1$ then $X = Y$. However, if $\gamma = -1$ which is of course true when

$\bar{p} - \bar{q} + \bar{r} < 0$ so that $0 < q - p < d - n$. Now observe:

Recall that for $x \in B_{m,n}$, then $x\theta \subseteq B_{m,n} = B_{\overline{m-n},0}$, for $\overline{m-n} = m - n$ if $m > n$ and

$m - n + d$ if $m < n$, thus $(x\theta)\theta = x\theta^2 \subseteq B_{\overline{m-n},0}$ and so $x\theta^{\alpha} \subseteq B_{\overline{m-n},0}$. Also observe that

$x\theta \in B_{\overline{m-n},0} \Longrightarrow x\theta^{\alpha} \in B_{\overline{m-n},0}$, so that if let $x_1 = x\theta^{\alpha} \in B_{\overline{m-n},0}$, and so that $x_1\theta^{\beta} = x\theta^{\alpha+\beta}$

Thus $X = (x\theta^{\alpha+\beta}y\theta^{\beta}).z = (x_1\theta^{\beta}y\theta^{\beta}).z = (x_1\theta^{\beta-1}y\theta^{\beta}).z = Y$

✓ **Subcase II** b(ii) $t = \max(n,p) = n$

Should this be the case, then $a = q - p + \max(n,p) = q - p + n$, so that $t' = \max(n_d, p_d) = n_d$ and $a_d = (q - p + n)_d = q_d - p_d + n_d + \gamma$ so that [3.04] becomes:

$$X = (x\theta^{t' - n_d}y\varphi^{t' - p_d})\theta^{u' - a_d}z\varphi^{u' - r_d} = (x.y\varphi^{n_d - p_d})\theta^{r_d - (q_d - p_d + n_d + \gamma)}.z$$

$$X = (x.y\varphi^{n_d - p_d})\theta^{(r_d - q_d) + (p_d - n_d) - \gamma)}.z = (x.y\varphi^{n_d - p_d})\theta^{(r_d - q_d) - (n_d - p_d) + \gamma}.z$$

$X = (x.y\varphi^{\alpha})\theta^{\beta-\alpha+\gamma}.z$, for $\beta-\alpha+\gamma \geq 0$, $\gamma = 0,1,-1$. Thus, generally if $\beta-\alpha+\gamma \geq 0$, then the values of $b_d - n_d$ and the corresponding values of $r_d - a_d$ expressed in terms of α, β and γ are summerised in the table below.

(C2)

i	$r_d - a_d = \beta - \alpha$, $0 < \bar{q} - \bar{p} + \bar{n} < d$	$b_d - n_d = \beta - \alpha$, $0 < \bar{p} - \bar{q} + \bar{r} < d$
ii	$r_d - a_d = \beta - \alpha$, $0 < \bar{q} - \bar{p} + \bar{n} < d$	$b_d - n_d = \beta - \alpha + 1$, $\bar{p} - \bar{q} + \bar{r} > d$
iii	$r_d - a_d = \beta - \alpha$, $0 < \bar{q} - \bar{p} + \bar{n} < d$	$b_d - n_d = \beta - \alpha - 1$, $\bar{p} - \bar{q} + \bar{r} < 0$
iv	$r_d - a_d = \beta - \alpha - 1$, $\bar{q} - \bar{p} + \bar{n} > d$	$b_d - n_d = \beta - \alpha, 0 < \bar{p} - \bar{q} + \bar{r} < d$
v	$r_d - a_d = \beta - \alpha - 1$, $\bar{q} - \bar{p} + \bar{n} > d$	$b_d - n_d = \beta - \alpha - 1$, $\bar{p} - \bar{q} + \bar{r} < 0$
vi	$r_d - a_d = \beta - \alpha + 1$, $\bar{q} - \bar{p} + \bar{n} > 0$	$b_d - n_d = \beta - \alpha$, $0 < \bar{p} - \bar{q} + \bar{r} < d$
vii	$r_d - a_d = \beta - \alpha + 1$, $\bar{q} - \bar{p} + \bar{n} < 0$	$b_d - n_d = \beta - \alpha + 1$, $\bar{p} - \bar{q} + \bar{r} < d$

Therefore, table 2 above shows the values of X with the corresponding values of Y as in [3.20], for $\gamma = 0, 1, -1$. Evidently, the direct application of lemma [2.3] then shows that

$$X = (x.y\varphi^{\alpha})\theta^{\beta-\alpha+\gamma}.z = Y = \left(x\theta^{\beta-\alpha+\gamma}.y\theta^{\beta+\gamma}\right).z.$$

Thus, above verified that $(ab)c = a(bc)$, and we have proved:

- *Theorem 3.1*
  $S = S(\boldsymbol{B}, d, \theta, \varphi)$ is a semigroup. □

Let $e = (m, x, n)$, now suppose that $e = e^2$so that $(m, x, n)(m, x, n) = (m, x, n)$. So that

$(m, x, n)(m, x, n) = \left(m - n + t, x\theta^{t'-n_d}x\theta^{t'-n_d}, n - m + t\right) = (m, x, n)$ [3.23]

$t' = \max(m_d, n_d)$.

Now observe that $m - n + t = m \implies n = t$ [3.24]

Similarly,

$n - m + t = n \implies t = m$ [3.25]

By [3.24] and [3.25], we have that $t = m = n$ and so $t' = n_d$

$(m, x, n)(m, x, n) = (m, x^2, n) = (m, x, n)$.

But $x^2 = x \implies x \in M_m$, that is x is an idempotent.

Conversely, if $m = n$ and $x = e_m$, then observe that $(m, e_m, m)(m, e_m, m) = (m, {e_m}^2, m) = (m, e_m, m)$ since $e_m \in M_m$. And certainly (m, e_m, m) is an idempotent and we have shown

- *Lemma 3.2*

The idempotent of S is of the form (m, e_m, m).

Suppose that $f_m = (m, e_m, m)$ and $f_n = (n, e_n, n), m, n \in N$. Observe that

$f_m f_n = (m, e_m, m)(n, e_n, n) = \left(t, e_m \theta^{t'-m_d} e_m \varphi^{t'-n_d}, t\right)$[3.26]

Where $t = \max(m, n)\,, t' = \max(m_d, n_d)$. See that if $t = m, t' = m_d$ then [3.26] becomes;

$$f_m f_n = (m, e_0 e_n \varphi^{m_d - n_d}, m) = (m, e_0 e_n, m)$$

Similarly,

if $t = n$, then $t' = n_d$, then [3.26] becomes:

$f_m f_n = (n, e_m \theta^{n_d - m_d} e_0, n) = (m, e_m e_0, m)$, and if $t = m = n$, then

$f_m f_m = (m, e_m, m) = f_m$.

Thus:

$$f_m f_n = \begin{cases} (m, e_m e_0, m) = (m, e_m, m)\,, t = m > n \\ (n, e_0 e_n, n) = (m, e_m, m), t = m = n \end{cases} \qquad [3.27]$$

Thus, we can now define a partial order as follows:

$f_m \leq f_n \implies f_m f_n = f_n f_m = f_m$ if $m \geq n$

Thus, we observe the following:

- $f_m \leq f_m$
- $f_m \leq f_n$
- $f_m \leq f_n, f_n \leq f_l \implies f_m f_n = f_m$ and $f_n \leq f_l \implies f_n f_l = f_n$. Thus $f_m(f_n f_l) = (f_m f_n)\, f_l\, f_m f_l = f_m$.

Thus, the relation $\leq$ is an equivalence relation.

If f_0 is an idempotent in S we have:

$f_0 = (0, e, 0) \geq f_1 = (1, e, 1) \geq f_2 = (2, e, 2) \ldots \geq f_d = (d-1, e, d-1)$ and in general,

$$f_d = (d, e, d) \geq f_{d+1} = (d+1, e, d+1) \geq \ldots \geq f_{2d-1} = (2d-1, e, 2d-1) \geq \cdots$$

Evidently, $f_0 \in S$ is an identity.

Let $(m, x, n), (n, y, m) \in S$. Observe that

$(m, x, n)(n, y, m) = (m - n + n, x\theta^{n_d - n_d} y \varphi^{n_d - n_d}, m - n + n) = (m, xy, m)$so that

$$(m, x, n)(n, y, m)(m, x, n) = [(m, x, n)(n, y, m)](m, x, n) = (m, xy, m)(m, x, n)$$

$= (m, xy\theta^{m_d - m_d}\, x\varphi^{m_d - m_d}, n - m + m) = (m, (xy)x, n)$. But recall that for $\bar{m} = \bar{n}$ then $x \in M_m$, and so $(xy)x = e_m x = x$, since $xy = e_m$. Thus, we have proved:

- *Lemma3.3*

Let $a = (m, x, n) \in S$, and let $x \in M_m$, where x is a unit, then the inverse of a, is of the form

$a^{-1} = (n, y, m)$, where $y = x^{-1}$ and $m = n(\mathrm{mod}\, d)$.

Let $a = (m, x, n), f_n = (n, e_{\bar{n}}, n), f_m = (m, e_{\bar{m}}, m) \in S$. Then for all $u = (h, y, k),\ v = (f, z, g) \in S$, observe that;

$au = (m, x, n)(h, y, k) = \left(m - n + t, x\theta^{t'-n_d} y \varphi^{t'-h_d}, k - h + t\right)$ [3.28] $t = \max(n, h)\,, t' = \max(n_d, h_d)$

Similarly,

$av = (m,x,n)(f,z,g) = \left(m-n+w, x\theta^{w'-n_d}z\varphi^{w'-f_d},\ g-f+w\right)$ [3.29]

$w = \max(n,f), w' = \max(n_d, f_d)$

Suppose that $au = av$, then:

$\left(m-n+t, x\theta^{t'-n_d}y\varphi^{t'-h_d}, k-h+t\right) = \left(m-n+w, x\theta^{w'-n_d}z\varphi^{w'-f_d},\ g-f+w\right)$[3.30]

Thus;

- $m-n+t = m-n+w$ [3.31]
- $k-h+t = g-f+w$ [3.32]
- $x\theta^{t'-n_d}y\varphi^{t'-h_d} = x\theta^{w'-n_d}z\varphi^{w'-f_d}$ [3.33]

From [3.31] $t = \max(n,h) = w = \max(n,f)$ and so $t' = w' \Longrightarrow \max(n_d,h_d) = \max(n_d,f_d)$ and so, we have

$h_d = f_d$. Thus [3.33] becomes:

$x\theta^{t'-n_d}y\varphi^{t'-h_d} = x\theta^{t'-n_d}z\varphi^{t'-f_d}$ and for a particular case where $x = e_n$, then

$e_n\theta^{t'-n_d}y\varphi^{t'-h_d} = e_n\theta^{t'-n_d}z\varphi^{t'-f_d}$ [3.34]

If $t' = n_d$, then [3.34] becomes:

$e_n.y\varphi^{n_d-h_d} = e_n.z\varphi^{n_d-f_d}$ [3.35]

But if $t' > n_d$, then $t' = h_d = f_d$ and we have:

$e_n\theta^{h_d-n_d}y = e_n\theta^{h_d-n_d}z = e_0y = e_0z$ for $e_n\theta^{t'-n_d} = e_0$.

Thus, $y = z$, and then $(n,e_n,n)(h,y,k) = (n,e_n,n)(f,z,g)$.

Thus, we have proved.

➢ *Theorem3.4*

Let $a = (m,x,n), f_n = (n,e_{\bar{n}},n), f_m = (m,e_{\bar{m}},m) \in S$. Then for all $u = (h,y,k), v = (f,z,g) \in S$, then:

- $a\mathscr{L}^* f_n$ and
- $a\mathscr{R}^* f_m$

V. CONCLUSION

In this study, we have seen that with the binary array of bisystems closed and certain rules imposed as in conditions [2.01ai] through [2.01aiii]and the dual [2.01bi] through [2.01biii] along lemmas [2.1], [2.2] and [2.3], the closure of multiplication of elements in the binary array of bisystems was ensured and then the construction as in [3.01]onthe set$S(\boldsymbol{B},d,\theta,\varphi) = S$was seen to be associative, hence a semigroup, as seen in **theorem 3.1**. However, such a class of $*$ - bisimple Ample ω-semigroup are characterized as an extension of the binary array of bisystems of cancellative monoids. Thus, we obtained few of its properties, namely: the nature of its idempotents (**lemma 3.2**), it inverses (**lemma 3.3**) and the $\mathcal{L}^*(\mathcal{R}^*)$-relations with respect to it idempotents (**lemma 3.4**).

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