# Joint State, Disturbance and Fault Estimation for Weakly Output Redundant Discrete-Time Linear Systems

Guitao Yang, Angelo Barboni, Hamed Rezaee, Andrea Serrani, and Thomas Parisini

*Abstract*— **The problem considered in this paper is dual to the control problem for over-actuated systems found in the literature. We show that, due to a certain notion of weak output redundancy, there always exists an unobservability subspace containing the input subspace, which ensures that the original system can be partitioned into two subsystems, one of which is not affected by actuator faults. We use this fact to estimate the disturbance and the fault in a cascaded fashion: we first design a discrete-time filter on a properly designed residual signal, that can reconstruct the disturbance. The estimated disturbance can then be used to perform fault detection and estimation in a cascaded fashion. A numerical example verifies the efficacy of the proposed strategy.**

*Index Terms*— **Discrete-time observers, over-sensed systems, fault detection.**

## I. Introduction

Resilience in the presence of faults and unknown disturbances is a crucial requirement in many practical systems, and this has attracted a significant interest for several decades [1]–[3]. One viable and effective approach in the design of resilient and fault tolerant control systems consists in estimating the fault so that a controller can compensate for it [4]. In this regard, the problem of jointly estimating unknown inputs and the system state has received significant attention in both the continuous-time [5]–[7] and discrete-time settings [8]–[10].

The problem of fault estimation becomes more complex when one considers a system to be affected by exogenous signals and actuator faults simultaneously. This problem of robust fault estimation has been considered in [11] in continuous time, and has received additional attention in [12], where a robust framework is proposed for both continuous- and discrete-time systems, and in [13], where a discrete-time difference algebraic observer is proposed.

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Most approaches in the literature on robust fault estimation are based on attenuation of the effect of disturbances on the estimation error, attained, for instance, by minimizing in some sense (e.g., $\mathcal{H}_\infty$). However, this strategy may not always be efficient, as the attenuated effect of disturbance still may be significant. Differently from the aforementioned strategies, the underlying idea in this work is to separate the fault from the disturbances and split the estimation problem in two simpler tractable sub-problems, in which the disturbance and then the fault will be estimated. We should note that the same principle was investigated in [14] based on the idea of unknown input and sliding-mode observers, where feasibility and effectiveness of the design were restricted to some standard conditions on unknown input and sliding-mode observers. Here, we adopt a fully linear approach stemming from the geometric unobservability subspace formulation of the fault detection and isolation problem developed in [15], applied to the algebraic dual of the framework considered in [16] for over-actuated systems. The specific class of systems under consideration is that of output-redundant systems with monic input map and epic output map. The proposed method hinges on the appropriate design of an observer gain matrix and state and output transformations that enables to decompose the *estimation error dynamics* in a way that actuator faults and disturbance are (partially) decoupled. Of the two subsystems resulting from the decomposition, the first one is used to reconstruct the disturbance. We design an asymptotically stable filter whose estimation error depends solely on the difference of the unknown disturbance over a sampling interval. Given that the considered system is proper, this is the best that can be achieved. Once an estimate of the disturbance is obtained, we leverage the invertibility properties of the second subsystem, which are granted by construction, and invert the system dynamics thus obtaining a deadbeat reconstruction of the fault. Finally, the state estimate is corrected with the estimates obtained at the previous steps. In summary, the contributions of this paper are the following:

- We use the output feedback to unlock an unobservability subspace and decompose the system correspondingly. Thanks to the properties of output redundancy, such decomposition is nontrivial, and faults and disturbances are partially decoupled in the error dynamics.
- A cascaded architecture of *error observers* is devised in order to sequentially estimate disturbances and faults. Contrary to what is done in the robust estimation framework [12], [13], our solution does not seek to attenuate

disturbances, but to compensate for them directly by estimation.
- We propose an alternative filter design, as opposed to the more typical predictor form [12], for disturbance estimation. We achieve a filter similar in spirit to [7], but owing to a discrete time formulation, we do not require derivatives of the output.
- We exploit the geometric properties guaranteed by the decomposition to achieve deadbeat fault reconstruction.

The paper is organized as follows. In Section II, the needed geometric tools are briefly recalled. In Section III, we define the problem and state our standing assumptions. The system decomposition and its properties are presented in Section IV, whereas Section V separately presents the disturbance and fault reconstructors. A brief example is given in Section VI, and Section VII offers as.

## II. Notation and Preliminaries

Notation and preliminaries from the geometric theory of linear systems [17] are provided in this section.

### A. Notation

$I$ stands for the identity matrix with compatible dimension. $\mathbf{0}$ is an all-zeros matrix with compatible dimension. For a square matrix $M$, we denote $M \succ (\prec)\ 0$, if it is symmetric positive (negative) definite. $M^\dagger$ represents the pseudo inverse of $M$. For a signal point $x(k)$ at a discrete time $k$, $x^+$ denotes the next sample $x(k+1)$ and $x^-$ denotes the previous one $x(k-1)$. $\sigma(M)$ denotes the spectrum of $M$, and $\uplus$ denotes the union with any common elements repeated. A set of complex numbers is called symmetric if all non-real elements in the set occur in conjugate pairs. We use script letters $\mathscr{X}$, $\mathscr{U}$, $\mathscr{V}$, ... to denote linear vector spaces, and the zero space is written as 0.

### B. Preliminaries

Let $C : \mathscr{X} \to \mathscr{Y}$ be a map. If $\mathscr{S} \subseteq \mathscr{Y}$, $C^{-1}\mathscr{S}$ denotes the *inverse map* of $\mathscr{S}$ under $C$ [17, Section 0.4]. Let $C : \mathscr{X} \to \mathscr{Y}$, and let $\mathscr{V} \subset \mathscr{X}$ be a subspace with insertion map $V : \mathscr{V} \to \mathscr{X}$, i.e., $\mathscr{V} = \operatorname{Im} V$ and $V$ is monic. The *domain restriction* of $C$ to $\mathscr{V}$ is denoted by $C|\mathscr{V} = CV$. Moreover, suppose $\operatorname{Im} C \subseteq \mathscr{W} \subseteq \mathscr{Y}$. The *codomain restriction* of $C$ from $\mathscr{W}$, denoted as $\mathscr{W}|C$ satisfies $W(\mathscr{W}|C) = C$, where $W : \mathscr{W} \to \mathscr{Y}$ is the insertion map of $\mathscr{W}$ [17, Section 0.4].

A subspace $\mathscr{V} \subseteq \mathscr{X}$ is said to be *invariant* with respect to a map $A : \mathscr{X} \to \mathscr{X}$ if $A\mathscr{V} \subseteq \mathscr{V}$. For an invariant subspace $\mathscr{V}$, we denote by $A|\mathscr{V} : \mathscr{V} \to \mathscr{V}$ the *restriction of $A$ to $\mathscr{V}$*, i.e., the unique map satisfying $AV = V(A|\mathscr{V})$, where $V : \mathscr{V} \to \mathscr{X}$ is the insertion map of $\mathscr{V}$. Furthermore, we denote by $A|\mathscr{X}/\mathscr{V}$ or simply by $\bar{A} : \mathscr{X}/\mathscr{V} \to \mathscr{X}/\mathscr{V}$ the map induced on $\mathscr{X}/\mathscr{V}$ by $A$, that is, the unique map satisfying $\bar{A}P = PA$, where $P : \mathscr{X} \to \mathscr{X}/\mathscr{V}$ is the canonical projection on $\mathscr{X}/\mathscr{V}$, the factor space modulo $\mathscr{V}$ [17, Sections 0.5, 0.7]. For a map $A : \mathscr{X} \to \mathscr{X}$ and subspaces $\mathscr{B} \subseteq \mathscr{X}$, $\mathscr{K} \subseteq \mathscr{X}$, we define the smallest $A$-invariant subspace that contains $\mathscr{B}$ as $\langle A \,|\, \mathscr{B}\rangle$ and the largest $A$-invariant subspace that is contained in $\mathscr{K}$ as $\langle \mathscr{K} \,|\, A\rangle$ [15, Section 2.1].

Let $A : \mathscr{X} \to \mathscr{X}$ and $C : \mathscr{X} \to \mathscr{Y}$. We say a subspace $\mathscr{V} \subseteq \mathscr{X}$ is $(A, B)$*-invariant* if there exists a map $F : \mathscr{X} \to \mathscr{U}$ such that $(A + BF)\mathscr{V} \subseteq \mathscr{V}$. If this is the case, we say that the state-feedback map $F$ is a friend of $\mathscr{V}$, and denote the class of all friends of $\mathscr{V}$ by $\mathbf{F}(\mathscr{V})$. We say a subspace $\mathscr{W} \subseteq \mathscr{X}$ is $(C, A)$*-invariant* if there exists a map $G : \mathscr{Y} \to \mathscr{X}$ such that $(A + GC)\mathscr{W} \subseteq \mathscr{W}$. If this is the case, we also say that the output injection map $G$ is a friend of $\mathscr{W}$, and denote the class of all friends of $\mathscr{W}$ by $\mathbf{G}(\mathscr{W})$. Moreover, let $\mathscr{B}, \mathscr{K} \subseteq \mathscr{X}$. We write $\overline{\mathscr{V}}(\mathscr{K})$ to denote the class of $(A, B)$-invariant subspaces *contained in* $\mathscr{K}$, and by $\underline{\mathscr{W}}(\mathscr{B})$ the class of $(C, A)$-invariant subspaces *containing* $\mathscr{B}$. The supremal and infimal elements of $\overline{\mathscr{V}}(\mathscr{K})$ and $\underline{\mathscr{W}}(\mathscr{B})$ are denoted by $\mathscr{V}^*(A, B; \mathscr{K})$ and $\mathscr{W}^*(C, A; \mathscr{B})$, respectively, or simply by $\mathscr{V}^*$ and $\mathscr{W}^*$ [17, Section 4.2], [15, Section 2.2].

We say a subspace $\mathscr{R} \subseteq \mathscr{X}$ is a *controllability subspace* if $\mathscr{R} = \langle A + BF | \operatorname{Im} BN\rangle$ for some state feedback map $F : \mathscr{X} \to \mathscr{U}$ and input selection map $N : \mathscr{U} \to \mathscr{U}$. We say a subspace $\mathscr{S} \subseteq \mathscr{X}$ is an *unobservability subspace* if $\mathscr{S} = \langle \operatorname{Ker} HC \,|\, A + LC\rangle$ for some output injection map $L : \mathscr{Y} \to \mathscr{X}$ and measurement mixing map $H : \mathscr{Y} \to \mathscr{Y}$. For $\mathscr{B}, \mathscr{K} \subseteq \mathscr{X}$, we write $\overline{\mathscr{R}}(\mathscr{K})$ to denote the class of controllability subspaces *contained in* $\mathscr{K}$, and by $\underline{\mathscr{S}}(\mathscr{B})$ the class of unobservability subspaces *containing* $\mathscr{B}$. The supremal and infimal elements of $\overline{\mathscr{R}}(\mathscr{K})$ and $\underline{\mathscr{S}}(\mathscr{B})$ are denoted by $\mathscr{R}^*(A, B; \mathscr{K})$ and $\mathscr{S}^*(C, A; \mathscr{B})$, respectively, or simply by $\mathscr{R}^*$ and $\mathscr{S}^*$, respectively [17, Section 5.1], [15, Section 2.3], [18].

## III. Problem Formulation

Consider a discrete-time linear system of the form:

$$x^+ = Ax + B(u + f) + Dd, \quad y = Cx \tag{1}$$

where $x \in \mathscr{X} \cong \mathbb{R}^n$, $u \in \mathscr{U} \cong \mathbb{R}^m$ and, $y \in \mathscr{Y} \cong \mathbb{R}^p$ are the system states, input, and output, respectively. Moreover, $f \in \mathscr{U}$ denotes the fault that occurs at the actuators, $d \in \mathscr{D} \cong \mathbb{R}^q$ represents unknown exogenous disturbances, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{n\times q}$ are matrices of compatible dimensions representing the corresponding maps.

*Assumption 1:* System (1) is observable. ◁

The following assumptions strictly characterize the class of systems under consideration:

*Assumption 2:* System (1) is *weakly output redundant*, that is, $p = \operatorname{rank} C > \operatorname{rank} B = m$. ◁

*Assumption 3:* $(C, A, B)$ is left-invertible [19]. ◁

Finally, the next assumption is made for reasons that will become clear in the sequel:

*Assumption 4:* The dimension of the disturbance space $\mathscr{D}$ does not exceed $p - m$. ◁

Our objective is to design an observer that is capable of jointly estimating the state $x$, the disturbance $d$, and the actuator fault $f$. The correct reconstruction of the fault implies, in fact, that the fault is also detected and isolated. In the next section, we obtain an observer with these properties by decomposing the state space in the direct sum of two subspaces, one of which is not affected by $u$ and $f$. As the input only affects the dynamics restricted to one subspace, we can use the dynamics projected on the complementary subspace to construct an estimator for $d$, which can then be used in a cascaded fashion for the purpose of fault reconstruction.

## IV. Observer Design and System Decomposition

We consider the typical Luenberger observer for (1) as

$$\hat{x}^+ = (A + GC)\hat{x} - Gy + Bu, \tag{2}$$

where $\hat{x}$ denotes the estimated state vector and $G \in \mathbb{R}^{n\times p}$ is the observer gain. We define the estimation and output errors as $\tilde{x} := x - \hat{x}$ and $\tilde{y} := y - C\hat{x}$, respectively. Accordingly, by considering (1) and (2), the difference equation describing $\tilde{x}$ is as follows:

$$\tilde{x}^+ = (A + GC)\tilde{x} + Bf + Dd, \quad \tilde{y} = C\tilde{x}. \tag{3}$$

The point of departure is the selection of the output injection map $G$ that enforces a decomposition of the dynamics (3) which is conducive (under additional assumptions) to the estimation of the disturbance $d$ separately from the reconstruction of the fault $f$.

*Proposition 1:* Let $\mathscr{B} := \operatorname{Im} B$ and $\mathscr{K} := \operatorname{Ker} C$, and denote by $\mathscr{S}^*$ the infimal unobservability subspace containing $\mathscr{B}$. Let $\mathscr{Z} := C\mathscr{S}^*$ and define $A_G := A + GC$ for $G \in \mathbf{G}(\mathscr{S}^*)$. System $\Sigma := \{C, A_G, B\}$ is decomposed according to the commutative diagram in Fig. 1, where:

- System $\Sigma_1 := \{\mathscr{Z}|(C|\mathscr{S}^*), A_G|\mathscr{S}^*, \mathscr{S}^*|B\}$ has input space $\mathscr{U}$, state space $\mathscr{S}^*$ and output space $\mathscr{Z}$;
- System $\Sigma_2 := \{\bar{\bar{C}}, \bar{A}_G\}$ is autonomous, and has state space $\mathscr{X}/\mathscr{S}^*$ and output space $\mathscr{Y}/\mathscr{Z}$.

In the diagram, $S : \mathscr{S}^* \to \mathscr{X}$ denotes the insertion map of $\mathscr{S}^*$ in $\mathscr{X}$, $\tilde{Q} : \mathscr{Y} \to \mathscr{Z}$ denotes the natural projection on $\mathscr{Z}$, $P : \mathscr{X} \to \mathscr{X}/\mathscr{S}^*$ denotes the canonical projection modulo $\mathscr{S}^*$, and $P_Y : \mathscr{Y} \to \mathscr{Y}/\mathscr{Z}$ denotes the canonical projection modulo $\mathscr{Z}$. □

*Proof:* The proof follows directly from the fact that, by definition, $\mathscr{B} \subseteq \mathscr{S}^*$, hence the co-domain restriction $\mathscr{S}^*|B$ is well-defined. Furthermore, it can be easily verified that $\operatorname{Ker} \bar{C} = \operatorname{Ker} P_Y C = \mathscr{S}^* + \mathscr{K}$, and hence the map $\bar{\bar{C}} : \mathscr{X}/\mathscr{S}^* \to \mathscr{Y}/\mathscr{Z}$ is well-defined. ■

The following result yields fundamental properties of the two subsystems defined in Proposition 1:

*Proposition 2:* Let Assumptions 1–3 hold. Then, system $\Sigma_1$ is square and invertible.

*Remark 1:* Recall that the class $\underline{\mathscr{S}}(\mathscr{B})$ contains at least the trivial element $\mathscr{S} = \mathscr{X}$. The statement of Proposition 2 implies that $\dim \mathscr{Z} = \dim C\mathscr{S}^* = m < p$, hence $\mathscr{S}^*$ is different from the whole space $\mathscr{X}$. Since $\mathscr{S}^* \supseteq \mathscr{B}$, it follows that $m \le \dim \mathscr{S}^* < n$. Furthermore, as the map $\bar{\bar{C}}$ is epic and $\dim \mathscr{Y}/\mathscr{Z} = p - m$, it follows that $\dim \mathscr{X}/\mathscr{S}^* \ge p - m$. Consequently, $m \le \dim \mathscr{S}^* \le n - p + m$.

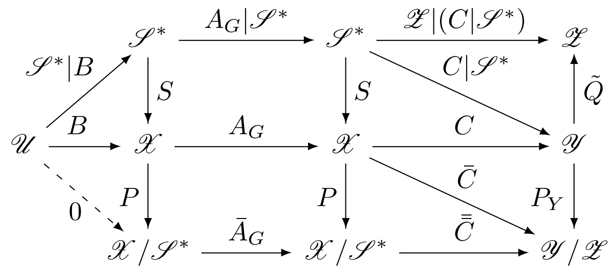


Fig. 1. Decomposition of $(C, A_G, B)$ induced by $G \in \mathbf{G}(\mathscr{S}^*)$.

*Proof:* Let $T = [T_1\ T_2] \in \mathbb{R}^{n\times n}$ be a state-space transformation representing a change of basis adapted to the decomposition $\mathscr{X} = \mathscr{T} \oplus \mathscr{S}^*$, where $\mathscr{T} \cong \mathscr{X}/\mathscr{S}^*$ is a generic complementary subspace to $\mathscr{S}^*$. Furthermore, let $\Gamma = [\Gamma_1\ \Gamma_2] \in \mathbb{R}^{p\times p}$ be an output-space transformation adapted to the decomposition $\mathscr{Y} = \mathscr{L} \oplus \mathscr{Z}$, where $\mathscr{L} \cong \mathscr{Y}/\mathscr{Z}$ is a generic complementary subspace to $\mathscr{Z}$. Note that, necessarily

$$\Gamma^{-1}C = \begin{bmatrix} H_1 C \\ H_2 C \end{bmatrix}$$

is such that $\operatorname{Ker} H_1 C = \mathscr{S}^* + \mathscr{K}$. In the given set of coordinates, system $\Sigma$ takes the form

$$\hat{A}_G := T^{-1}A_G T = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \ \hat{B} := T^{-1}B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$

$$\hat{C} := \Gamma^{-1}CT = \begin{bmatrix} H_1 CT \\ H_2 CT \end{bmatrix} = \begin{bmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{bmatrix}, \tag{4}$$

where $\{C_{22}, A_{22}, B_2\}$ is the representation of $\Sigma_1$ in the given basis, and $\{C_{11}, A_{11}\}$ is a representation of $\Sigma_2$. Let $\rho := \dim \mathscr{Z}$. The fact that $\rho = m$ is proved as follows: Recall that Assumption 3 is equivalent to $\mathscr{V}^* \cap \mathscr{B} = 0$, where $\mathscr{V}^*$ is the largest $(A, B)$-invariant subspace contained in $\mathscr{K}$. Since the class of $(A, B)$-invariant subspaces contained in $\mathscr{K}$ are invariant under output injection, system $\Sigma$ is left-invertible as well. As $\Sigma_1$ is a realization of the input-output map of $\Sigma$, it follows that $\Sigma_1$ is left-invertible as well, therefore $\rho \ge m$. Assume $\rho > m$. Then, the triplet $\{C_{22}, A_{22}, B_2\}$ is not right-invertible, hence for this system the infimal unobservability subspace $\mathscr{S}_2^* := \mathscr{S}^*(C_{22}, A_{22}; \operatorname{Im} B_2)$ does not satisfy $\mathscr{S}_2^* + \operatorname{Ker} C_{22} = \mathscr{S}^*$. Since $\operatorname{Im} B_2 = \operatorname{Im} B$, this contradicts minimality of $\mathscr{S}^*$ for $(C, A, B)$. Since $\Sigma_1$ has been shown to be square and left-invertible, it is necessarily right-invertible as well. ■

Finally, spectral assignability of the decomposition in Figure 1 is established as follows:

*Proposition 3:* Let Assumption 1 hold. Then, the spectra of $A_G|\mathscr{S}^*$ and $\bar{A}_G$ are assignable by $G \in \mathbf{G}(\mathscr{S}^*)$. □

*Proof:* The proof relies the following lemmas, which are the dual of [17, Prop. 4.1] and [17, Thm. 4.1, Cor. 5.2], respectively.

*Lemma 1:* [15, Proposition 13] Let $(C, A)$ be observable, $\mathscr{W}$ be a $(C, A)$-invariant subspace with $\dim(\mathscr{W}) = k$ and $P_W : \mathscr{X} \to \mathscr{X}/\mathscr{W}$ the canonical projection. If $G_0 \in \mathbf{G}(\mathscr{W})$

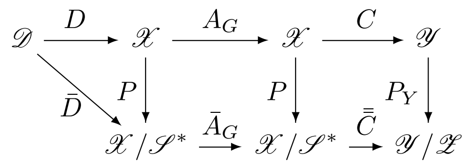


Fig. 2. Decomposition of $(C, A_G, D)$ induced by $G \in \mathbf{G}(\mathscr{S}^*)$.

and $\Lambda$ is an arbitrary symmetric set of $k$ complex numbers, there exists $G :\in \mathbf{G}(\mathscr{W})$ such that

$$P_W G = P_W G_0 \,, \qquad \sigma(A_G) = \sigma\left(A_G|\mathscr{X}/\mathscr{W}\right) \uplus \Lambda. \quad \square$$

*Lemma 2:* [15, Proposition 20] Let $\mathscr{W} \in \underline{\mathscr{W}}(\mathscr{B})$ and let $\mathscr{S}^*$ be the infimal element of $\underline{\mathscr{S}}(\mathscr{B})$. If $G \in \mathbf{G}(\mathscr{W})$ then

$$\sigma\left(A_G|\mathscr{X}/\mathscr{W}\right) = \sigma_G \uplus \sigma^*,$$

where $\sigma_G := \sigma\left(A_G|\mathscr{X}/\mathscr{S}^*\right)$ is freely assignable by choice of $G \in \mathbf{G}(\mathscr{W})$, and $\sigma^* := \sigma\left(A_G|\mathscr{S}^*/\mathscr{W}\right)$ is fixed for all $G \in \mathbf{G}(\mathscr{W})$. □

The proof of Proposition 3 then follows directly by taking $\mathscr{W} = \mathscr{S}^*$ in Lemma 2. ■

Next, we use the representation (4), where, for the sake of simplicity, we select $\mathscr{L} = \mathscr{Z}^\perp$. This yields the expression of error system (3) in the new coordinates $\tilde{\xi} = T^{-1}\tilde{x}$ and $\gamma = \Gamma^{-1}\tilde{y}$ as the interconnection of the fault-decoupled system (left-invertible with respect to the disturbance input $d$)

$$\tilde{\xi}_1^+ = A_{11}\tilde{\xi}_1 + D_1 d\,, \quad \gamma_1 = C_{11}\tilde{\xi}_1, \tag{5}$$

and the perturbed system (invertible with respect to $f$)

$$\tilde{\xi}_2^+ = A_{21}\tilde{\xi}_1 + A_{22}\tilde{\xi}_2 + B_2 f + D_2 d\,, \gamma_2 = C_{22}\tilde{\xi}_2. \tag{6}$$

Needless to say, $G \in \mathbf{G}(\mathscr{S}^*)$ has been selected by virtue of Proposition 3 so that $A_{11}$ and $A_{22}$ are Schur stable matrices.

Finally, we turn our attention to the *disturbance system* $\{C, A_G, D\}$, whose decomposition is shown in the diagram in Fig. 2. For the subsystem $\Sigma_d := \{\bar{\bar{C}}, \bar{A}_G, \bar{D}\}$, with input space $\mathscr{D}$, state space $\mathscr{X}/\mathscr{S}^*$ and output space $\mathscr{Y}/\mathscr{Z}$, we make the following assumption:

*Assumption 5:* System $\Sigma_d$ is left-invertible and has unitary vector relative degree. In particular, this implies that in the chosen basis adapted to $\mathscr{S}^*$ and $\mathscr{Z}$ $\operatorname{rank} C_{11}D_1 = p-m$, that is, $C_{11}D_1$ is full column rank. ◁

It is worth noting that Assumption 5 implies that $q \leq p - m$, hence the necessity of Assumption 4.

## V. Disturbance and Fault Reconstruction

In this section, we leverage the decomposition presented in the previous section and propose an estimation algorithm to reconstruct the disturbance and the fault in a cascaded fashion. In particular, error subsystem (5) is used for disturbance estimation, which is then used in (6) for fault reconstruction.

### A. Disturbance Estimation

Regarding disturbance estimation, we consider a disturbance observer for (5) as follows:

$$\hat{\tilde{\xi}}_1^+ = A_{11}\hat{\tilde{\xi}}_1 + D_1\hat{d}, \tag{7}$$

where $\hat{d}$ is the disturbance estimate. Notice that system (7) is intended to be an *error* observer and $A_{11}$ is already a closed-loop state transition matrix with arbitrarily assigned spectrum. A common approach in the literature of fault estimation, based on the idea of adaptation, is to reconstruct $d$ by integrating the output error with a suitably chosen gain [7], [12]. In the same spirit, we propose the following filter:

$$\hat{d} = \hat{d}^- + K_1(C_{11}D_1)^\dagger(\gamma_1 - C_{11}\hat{\tilde{\xi}}_1), \tag{8}$$

where $K_1 \in \mathbb{R}^{q\times(p-\varphi_c)}$ is an observer gain designed later ($\varphi_c := \dim(C\mathscr{S}^*)$). Let $\delta\tilde{\xi}_1 := \tilde{\xi}_1 - \hat{\tilde{\xi}}_1$. By inspection of (8) in forward form and by considering (5) and (7), one observes that

$$\begin{aligned}\hat{d}^+ &= \hat{d} + K_1(C_{11}D_1)^\dagger C_{11}\Big(A_{11}\tilde{\xi}_1 + D_1 d - A_{11}\hat{\tilde{\xi}}_1 - D_1\hat{d}\Big)\\ &= K_1M_1A_{11}\delta\tilde{\xi}_1 + (I-K_1)\hat{d} + K_1 d,\end{aligned} \tag{9}$$

where $M_1 := (C_{11}D_1)^\dagger C_{11}$. Thanks to Assumption 5, $M_1$ is full row rank. By introducing the variable $\delta d := d - \hat{d}$ and rewriting $d^+ = d + \Delta d$ in terms of its increments $\Delta d$, one gets

$$\delta d^+ = d + \Delta d - \hat{d}^+. \tag{10}$$

By virtue of (5), (7), (9), and (10), $\delta\tilde{\xi}_1^+$ and $\delta d^+$ can be stated into the following forms:

$$\begin{aligned}\delta\tilde{\xi}_1^+ &= A_{11}\delta\tilde{\xi}_1 + D_1\delta d\\ \delta d^+ &= -K_1M_1A_{11}\delta\tilde{\xi}_1 + (I-K_1)\delta d + \Delta d,\end{aligned} \tag{11}$$

whose state transition matrix is denoted by

$$F = \begin{bmatrix} A_{11} & D_1 \\ -K_1M_1A_{11} & I-K_1\end{bmatrix}. \tag{12}$$

In particular, notice that system (11) is driven by the disturbance "innovation" term $\Delta d$. Before showing a design procedure for $K_1$, we recall the following lemma.

*Lemma 3:* The eigenvalues of a given matrix $X \in \mathbb{R}^{n\times n}$ lie in a disc $D(c,\rho)$ of center $c+j0$ and radius $\rho$ if and only if there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} -P & P(X-cI) \\ \star & -\rho^2 P\end{bmatrix} \prec 0. \tag{13}$$

□

*Proof:* Inequality (13) can be obtained by applying the Schur complement to the formulation found in [20, Theorem 1]. ■

We can straightforwardly apply this result to obtain the following design procedure.

*Proposition 4:* Given $\rho \in (0,1)$, if there exist $P_1 \succ 0 \in \mathbb{R}^{n_1 \times n_1}, P_2 \succ 0 \in \mathbb{R}^{q\times q}$, and $Y \in \mathbb{R}^{q\times q}$ such that the following LMI is feasible:

$$\left[\begin{array}{cc|cc} -P_1 & \mathbf{0} & P_1 A_{11} & P_1 D_1 \\ \mathbf{0} & -P_2 & -YM_1A_{11} & P_2 - Y \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$\star$}} & -\rho^2 P_1 & \mathbf{0} \\ & & \mathbf{0} & -\rho^2 P_2 \end{array}\right] \prec 0, \qquad (14)$$

then the matrix $F$ defined in (12) is Schur-stable where the gain $K_1$ is given by $K_1 = P_2^{-1} Y$. □

*Proof:* Let $P := \mathrm{diag}(P_1, P_2)$, clearly we have $P \succ 0$. The off-diagonal blocks of (14) are obtained by multiplying $P$ and $F$, and by applying the linearizing change of variable $Y = P_2 K_1$. Hence, condition (14) is obtained by direct computation and by applying Lemma 3 to matrix $F$ for a given disc $D(0,\rho)$. ■

### B. Fault Estimation

In this section, we use the estimate $\hat{d}$ in subsystem (6) obtained in the previous part in order to reconstruct the actuator fault $f$. Recall that system (6) is invertible with respect to the input/output pair $(f, \gamma_2)$. This property lets us tackle the input reconstruction by means of inversion of the discrete-time dynamics [10], [21]. As this amounts to solving a least squares problem on a finite time window, the reconstruction is deadbeat [10].

Let $r$ be a non-negative integer representing the number of past samples of the signals $\hat{\tilde{\xi}}_1$, $\hat{d}$, $f$, and $\gamma_2$. Accordingly, at time $k$, we define the vectors $\underline{\hat{\tilde{\xi}}_1} \in \mathbb{R}^{(n-\varphi)(r+1)}$, $\underline{\hat{d}} \in \mathbb{R}^{q(r+1)}$, $\underline{f} \in \mathbb{R}^{m(r+1)}$, and $\underline{\gamma_2} \in \mathbb{R}^{\varphi_c(r+1)}$ each of which contains $r+1$ samples of the respective signal. For instance,

$$\underline{\hat{d}}(k) = \left[\hat{d}(k-r)^\top \ \hat{d}(k-r+1)^\top \ \dots \ \hat{d}(k)^\top\right]^\top.$$

We also define the matrices

$$\Phi_o = \begin{bmatrix} C_{22} \\ C_{22}A_{22} \\ \vdots \\ C_{22}A_{22}^r \end{bmatrix}$$

$$\Phi_f = \begin{bmatrix} \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ C_{22}B_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ C_{22}A_{22}^{r-1}B_2 & C_{22}A_{22}^{r-2}B_2 & \cdots & \mathbf{0} \end{bmatrix}$$

$$\Phi_\xi = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ C_{22}A_{21} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ C_{22}A_{22}^{r-1}A_{21} & C_{22}A_{22}^{r-2}A_{21} & \cdots & \mathbf{0} \end{bmatrix}$$

$$\Phi_d = \begin{bmatrix} \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ C_{22}D_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ C_{22}A_{22}^{r-1}D_2 & C_{22}A_{22}^{r-2}D_2 & \cdots & \mathbf{0} \end{bmatrix}.$$

Now, by replacing the unknown quantities $\tilde{\xi}_1$ and $d$ with their respective estimates, it follows that the dynamics of (6) can be expressed in matrix form as

$$\underline{\gamma_2}(k) = \Phi_2 \chi(k) + \Phi_1 \begin{bmatrix} \underline{\hat{\tilde{\xi}}_1}(k) \\ \underline{\hat{d}}(k) \end{bmatrix},$$

where $\chi(k) = \begin{bmatrix} \tilde{\xi}_2(k-r)^\top & \underline{f}(k)^\top \end{bmatrix}^\top$, $\Phi_2 = \begin{bmatrix} \Phi_o & \Phi_f \end{bmatrix}$, and $\Phi_1 = \begin{bmatrix} \Phi_\xi & \Phi_d \end{bmatrix}$. Then, the fault estimation problem can then be obtained by computing the following equation:

$$\chi(k) = \Phi_2^\dagger \left( \underline{\gamma_2}(k) - \Phi_1 \begin{bmatrix} \underline{\hat{\tilde{\xi}}_1}(k) \\ \underline{\hat{d}}(k) \end{bmatrix} \right), \qquad (15)$$

and an estimate $\hat{f}(k)$ of $f(k-r)$ can be obtained by selecting the appropriate components of $\chi$.

Finally, the corrected estimate $\hat{x}^c$ of $x$ can be computed by correcting $\hat{x}$ in (2) with the new estimated inputs as follows:

$$\hat{x}^{c+} = \hat{x}^+ + B\hat{f} + D\hat{d}.$$

In the next section, we provide a numerical example to show the effectiveness of the proposed methodology.

## VI. Numerical simulations

Consider a discrete-time linear system of the form (1) with the following parameters

$$A = \begin{bmatrix} -1.300 & 1 & 17 & 5 & 16 \\ 2 & -6 & 3 & -1 & -8 \\ 0 & 1 & -8 & -7 & 3 \\ 2 & -13 & -2 & -15 & 5 \\ 6 & -6 & 1 & 2 & -1 \end{bmatrix}, B = \begin{bmatrix} I \\ \mathbf{0} \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

The system is sampled using the forward Euler method, with a sampling time $T_s = 10^{-3}$ s. The unknown disturbance is given by the vector of sinusoidal signals:

$$d(k) = \begin{bmatrix} 0.3 + 0.6\sin(1.5k + 0.35\pi) \\ 0.4 + 0.5\sin(0.8k + 0.63\pi) \end{bmatrix}, \qquad (16)$$

and the actuator fault is given by $f(k) = \begin{bmatrix} 2 & 1 \end{bmatrix}^\top h(kT_s - 5)$ where $h$ denotes the (scalar) Heaviside step function. By considering $\rho = 0.2$, we solve the LMI (14) by using the CVX toolbox [22] and obtain the matrix gain $K_1$. Moreover, for the methodology described in Section V-B, we choose $r = 1$ (equivalent to the relative degree of subsystem (6)). The simulation runs for 20 s, and each component of the initial states is uniformly randomly initialized in the $[0,1]$ interval. Without loss of generality, given that the control inputs are known, the system is regulated via a stabilizing state feedback with preassigned poles. All observers and filters are initialized at 0.

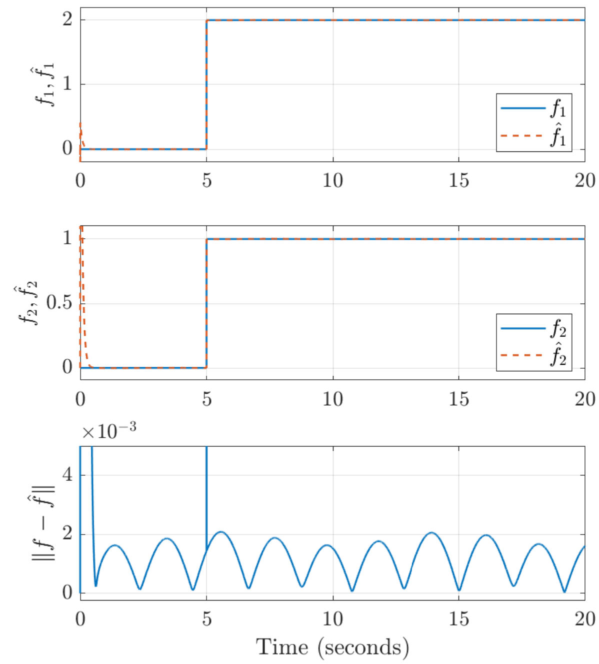


Fig. 3. Comparison of faults and the respective reconstructed signals (top 2 plots), and 2-norm of the fault estimation error (bottom plot).

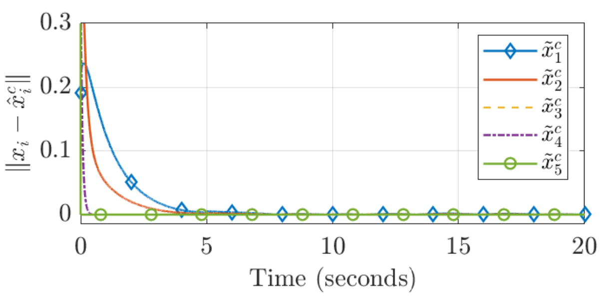


Fig. 4. Norm of the estimation error $\tilde{x}^c = x - \hat{x}^c$ components.

In Fig. 3, the fault and its estimates along with the 2-norm of the fault estimation error are shown. Notice that the trend of the 2-norm of the fault estimation error follows that of $\delta d$, as expect, since (15) is an exact solution to the reconstruction problem, and the only error sources come from the upstream filter (11). Furthermore, the peak amplitude in the bottom plot, omitted for scaling reasons, is $2.24 \approx \sqrt{5}$, which corresponds in fact to the amplitude of the fault step from $k = 5/T_s$ to $k = 5/T_s+1$. This is consistent with our choice of estimation delay $r = 1$, which cannot be smaller than the relative degree of the system. Finally, Fig. 4 shows the norm of the estimation error for the components of $x^c$.

## VII. Conclusions and Future Work

We have presented a method for joint state, unknown input, and actuator fault estimation for *over-sensed systems*, based on the geometric framework developed in [15]. In this regard, we designed a cascaded architecture consisting of a novel disturbance estimator filter—for which we provide a possible design method—and a deadbeat fault reconstructor leveraging the invertibility properties stemming from a suitable decomposition. The proposed architecture can be effortlessly extended to a system with random disturbances by integrating techniques presented in [9]. Current research is aimed at investigating the optimality and the robustness properties of the proposed filter.

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