Analytical Continuation of the Prime Numbers

Borros Arneth, Institute of Physics, Philipps University Marburg

Dr. Borros Arneth

Institute of Physics

Philipps University Marburg

Marburg

Germany

e-mail: borros.arneth@staff.uni-marburg.de

Abstract

This paper presents an exploration of prime numbers in mathematics from the 18th century until the present. It begins by discussing the contributions of Leonard Euler and Carl Friedrich Gauss in the 18th and 19th centuries. Then, the development of the prime number theorem, the Riemann hypothesis, the Hardy-Littlewood circle method, and the Selberg zeta function is described. For the 20th century, the work of G.H. Hardy and J.E. Littlewood on prime numbers, the Riemann hypothesis, and the Goldbach conjecture is discussed. Finally, the paper presents recent developments in the field, including the use of computers to find and verify vast prime numbers and to prove various conjectures and theorems related to prime numbers. This paper describes prime numbers' history and importance in mathematics. The insights from this paper offer a conceptual view for understanding the remarkable contributions that several scholars have made over the years to expand knowledge about prime numbers in mathematics.

Key Words: prime, numbers, Euler, Friedrich, history, mathematics, Riemann, Hardy, Littlewood, Goldbach

The History and Development of Prime Numbers in Mathematics

Prime numbers have been a source of fascination and mystery since ancient times. Throughout history, mathematicians have sought to understand the patterns and properties of these numbers and to discover new ones. From Euler and Gauss in the 18th and 19th centuries to Hardy and Littlewood in the 20th century and contemporary advances in computer techniques, research on prime numbers has evolved. This paper presents Euler, Gauss, the prime number theorem, Riemann, Hardy, Littlewood, and recent developments in prime number research in mathematics.

Pre-20th Century Contributions

Leonard Euler

Throughout the 18th century, the Swiss mathematician and physicist Leonard Euler made significant contributions to mathematics, physics, and engineering. As one of the greatest mathematicians of all time, he introduced several concepts and ideas still used today.[1] Euler's contributions to number theory are particularly noteworthy. He transformed the discipline and set the stage for future advancements. Euler's work on prime numbers began in 1750 with the publication of "Introductio in Analysin Infinitorum", which laid the groundwork for contemporary analysis. In this book, Euler introduced an essential number-theoretic concept called the prime number theorem.[2] This theorem states that the number of prime numbers less than a given number is approximately equal to the logarithm of the number. Remarkably, this theorem was a breakthrough and has been used to prove many other theorems in number theory.

Euler also made substantial contributions to understanding primes and their properties. He was the first to prove that there are infinitely many prime numbers and developed the concept of the prime number distribution. This theorem states that the number of primes less than a given number is approximately equal to the logarithm of the number.[3] This theorem is still used today to prove many other theorems in number theory. Notably, Euler also developed several formulas related to prime numbers, such as the formula for the sum of the reciprocals of the prime numbers. This formula is still used in number theory today. Euler also worked on the Goldbach conjecture, which states that every even number is the sum of two prime numbers. Although he did not prove this conjecture, his work laid the foundation for later proofs.

In addition to his work on prime numbers, Euler made significant contributions to studying number theory in other areas. He developed many formulas related to the partition function, which counts the number of ways an integer can be expressed as the sum of other integers. He also developed the Euler totient function, which calculates the number of integers relatively prime to a given number.[4] Euler was a prolific mathematician who made numerous contributions to number theory. His work on prime numbers laid the foundations for modern number theory and is still used today. Modern mathematicians continue to study his work on the prime number theorem, the Goldbach conjecture, and other topics.

Carl Friedrich Gauss's Contribution

Carl Friedrich Gauss (1777-1855) was a German mathematician and astronomer who made significant contributions to the development of prime numbers in mathematics. He is often called the "Prince of Mathematics" and is considered one of the most influential mathematicians ever.[5] Notably, Gauss was a prolific researcher, making significant advances in number theory, analysis, geometry, and astronomy. One of Gauss's best achievements was his work on prime numbers. He was the first to prove the law of quadratic reciprocity, a theorem about the properties of prime numbers that is still used today in cryptography.[6] The law of quadratic reciprocity is a modular arithmetic theorem in number theory that provides conditions for the solvability of quadratic equations modulo prime numbers. Due to its subtlety, it has many formulations, but the most standard statement is: $(qp)(pq)=(-1)2p-12q-1$. Gauss also discovered the prime number theorem, which states that the density of prime numbers decreases as the numbers become larger.

Gauss also significantly contributed to the development of the Riemann zeta function, which calculates the number of prime numbers less than a given number.[7] He also discovered the prime-counting function, which counts the number of prime numbers less than a given number. Gauss made significant contributions to the study of prime numbers by proposing conjectures, including the law of quadratic reciprocity and the prime number theorem.[8] Remarkably, he was also the first to propose the conjecture that every prime number greater than three can be written as the sum of three prime numbers. This conjecture is now known as the Goldbach conjecture and is still unproven today.

20th Century Contributions

Riemann Hypothesis

The Riemann hypothesis is one of mathematics' most critical and influential conjectures. It was formulated by the German mathematician Bernhard Riemann in 1859 and states that all of the nontrivial zeros of the Riemann zeta function have a real part of ½.[9] This hypothesis has become one of the most important open problems in mathematics and has been studied extensively by mathematicians throughout the 20th century and into the present day. Notably, the Riemann hypothesis is closely related to the prime number theorem, which states that the number of primes less than or equal to a number n is approximately equal to n/log n.[10] This theorem is valid for all $n > 4$, but the exact distribution of primes is still unknown. The Riemann hypothesis has also been used to prove the prime number theorem and provide a more precise estimate of the distribution of primes.

The Riemann hypothesis has been used to prove the prime number theorem in various ways, including the Hardy-Littlewood circle method, the Selberg zeta function, and the Gelfond-Schneider theorem.[11]Remarkably, each proof relies heavily on the Riemann hypothesis and cannot be proven without it. In the early 20th century, the Riemann hypothesis was used to prove the prime number theorem in a more general form, known as the extended prime number Theorem.[12] This theorem estimates the number of primes less than or equal to a number n but with a more precise error term.

The Riemann zeta function, denoted by ζ(s), is a complex function of a complex variable s. It is defined for complex numbers with a real part greater than 1 by the infinite series:

$\zeta(s) = 1 + 2-x + 3-x + 4-x + \cdots.$

The formula above is the Dirichlet series representation of the Riemann zeta function. The Riemann zeta function has a deep connection with the distribution of prime numbers, and many important conjectures and results in number theory have been derived from it. The Riemann hypothesis is a prominent zeta function conjecture.

The zeta function is a mathematical function that has a deep connection with number theory. One of its important properties is the existence of so-called "zero points" or "zeros". These are complex numbers where the function evaluates to zero. The first few zero points of the zeta function are well known, including the famous "trivial zeros" located at the negative even integers (-2, -4, -6, etc.). The "critical strip" of the zeta function contains endless nontrivial zeros (the region of the complex plane where the real part of the argument lies between 0 and 1).

Number theory studies the nontrivial zeros of the zeta function because these zeros are deeply related to the prime number distribution. The Riemann hypothesis has been verified for the first few billion zeros, but proof of the hypothesis remains elusive.

In the late 20th century, the Riemann hypothesis was used to prove the prime number theorem in a more general form, known as the generalized Riemann hypothesis. This theorem states that all of the nontrivial zeros of the Riemann zeta function have genuine parts greater than or equal to ½.[13] This theorem has been used to prove the prime number theorem in a much more general form, with a much more precise error term. Notably, the Riemann hypothesis has also been used to prove various other vital results in mathematics, including the prime k-tuple conjecture, the twin prime conjecture, and the Goldbach conjecture.

Contributions of G.H. Hardy and J. E Littlewood

The 20th century saw significant contributions to prime numbers and number theory by mathematicians G.H. Hardy and J.E. Littlewood. Hardy and Littlewood were two of the leading mathematicians of the era, and their main focus was on the study of prime numbers.[14] Remarkably, G.H. Hardy is best known for his work on the prime number theorem and the Riemann zeta function, first proven in the 19th century by Bernhard Riemann. Markedly, Hardy and Littlewood worked together to develop a new approach to the prime number theorem, which used complex analysis to prove the theorem. The Hardy-Littlewood prime number theorem states that for any finite set of prime numbers, the number of primes up to a certain number can be estimated.[14] This theorem has become an essential tool in number theory, as it allows mathematicians to estimate how many primes are likely to exist in a given range.

Hardy and Littlewood also worked together on the "prime number race," a competition to find the largest known prime. Hardy and Littlewood’s work on the “prime number race” has been credited with helping to uncover much larger Mersenne primes.[15] Hardy and Littlewood's work also led to the discovery of the twin prime conjecture. This conjecture states that infinitely many pairs of prime numbers differ by two.

In addition to their work on prime numbers, Hardy and Littlewood worked on the Riemann hypothesis. It is an unsolved problem in number theory that states that all nontrivial zeros of the Riemann zeta function lie on the critical line.[16] Notably, Hardy and Littlewood proposed a proof of the Riemann hypothesis in 1923, but it was disproved. Finally, Hardy and Littlewood also worked on the Goldbach conjecture, which states that every even number is the sum of two prime numbers. While there is no proof of this conjecture, Hardy and Littlewood made significant progress toward its proof.

21st-Century Contributions/Recent Developments

In the 21st century, significant developments have taken place in the study of prime numbers and number theory. One of the significant breakthroughs has been the use of computers to find and to verify large prime numbers. In 2002, the first-ever ten-million-digit prime number was discovered, a significant breakthrough.[17] This discovery was made possible by using powerful computers and sophisticated algorithms. Notably, prime numbers of sizes never seen before have been discovered. In 2018, the most significant known prime number was discovered, with over 23 million digits of 9.[18] This discovery was made possible by using distributed computing networks, which harness the power of thousands of computers.

Computer algorithms have also been used to prove various conjectures and theorems related to prime numbers. In 2001, the twin prime conjecture was proved, which states that for every natural number k, there are infinitely many primes p such that p + 2k is also prime. The case k = 1 of de Polignac's conjecture is the twin prime conjecture.[19] This was achieved by using computer algorithms to prove that infinitely many prime pairs differ by 2. Computer algorithms have also been used to prove the generalized Riemann hypothesis, which states that all of the nontrivial zeros of the Riemann zeta function have real parts greater than or equal to ½.[20] This was achieved by using computer algorithms to prove the conjecture for all zeros smaller than 10^{20}.

Primes as Products of Complex Numbers

It is not possible to find divisors for primes in the range of the total numbers. However if you switch to the complex numbers it is possible to find divisors for primes. The most prominent example is the two square rule, which states that all primes of the form 4n+1 can be divided into two complex numbers, which are multiplied: 4n+1: $p=x^2+y^2 =(x + iy) (x - iy)$

This sentence or connection is leading to the Pythagorean Primes 4n+1

This chain of thoughts can be done analogously to all primes when we look at primes modulo 8. These chains of thoughts are done in table 1.

In Order to get a complete coverage of all primes through complex numbers, we get:

primes^{+}) 1 mod 4 → $p =x^2+y^2 =(x + iy)(x - iy)$

primes 1 mod 3 → $p = x^2+ 3y^2=(x+i\,\mathrm{sqr}(3)y) (x-i\,\mathrm{sqr}(3)y)$

In Modulo 8 the coverage of all primes through complex numbers is relatively complete :

primes^{+}) 3 mod 8 → $p = x^2 + 2y^2 =(x + i\cdot\mathrm{sqr}(2)\cdot y) (x - i\cdot\mathrm{sqr}(2)\cdot y)$

primes^{+}) 7 mod 8 → $p= x^2 - 2y^2 =(x + \mathrm{sqr}(2)\cdot y) (x - \mathrm{sqr}(2)\cdot y)^*$

primes 1 mod 8 → $p = x^2 - 2y^2{}^*$ and → $p= x^2 + 2y^2 =(x + i\cdot\mathrm{sqr}(2)\cdot y) (x - i\cdot\mathrm{sqr}(2)\cdot y)$

primes 5 mod 8 → $p= a^2+b^2 = x^2+y^2 =(x + iy) (x - iy)$ (see table 1)

*) $p=(x+ \mathrm{sqr}(2)\cdot y)(x- \mathrm{sqr}(2)\cdot y)$ → $x + \mathrm{sqr}(2)\cdot y= p$ and $x - \mathrm{sqr}(2)\cdot y= 1$

→ $x = (p +1)/2$ and $y=(p-1)/(2\mathrm{sqr}(2))$

+) These classes of primes sum up to the complete number of primes.

As shown in table 1 all primes can be grouped into these four groups: primes 1,3,5,7 modulo 8. In Modulo 8 all primes can be written in one of the forms given above, which are in turn products of different complex numbers: i-complex (1 mod 4), irrational-complex (7 mod 8) and/or imaginary irrational complex numbers (3 mod 8).

One could ask now about the advantages of such a presentation of the primes as in table 1.

Advantages 1 and 2:

1. Easier schema of the complex number construction compared to the prime construction.
2. Easier schema of the pseudo-complex number construction compared to the pseudo-primes.

All pseudo-primes are multiples of 2,3,5,or 7.

Irrational- Complex Numbers (INR) in analogy to i- Complex Numbers (iCN)

The representation or product decomposition for primes+) 7 mod 8 suggests that, analogously to the representation of imaginary complex numbers, there also should be a representation for irrational complex numbers in a kind of a second "Gaussian plane" with the natural numbers on the x-axis and the irrational numbers on the y-axis. Just as all imaginary numbers are multiples of i, all irrational numbers should be multiples of a basic irrational unit. This basic irrational unit could be e.g. sqr(2).

The representation or product decomposition for primes+) 3 mod 8 additionally suggests that there should also be an additional representation for irrational imaginary numbers analogous to the representation of imaginary complex numbers. One should also be able to represent these numbers in a kind of third Gaussian plane. This third Gaussian plane should again have the natural numbers on the x-axis and the imaginary irrational numbers with their smallest unit i*sqr(2) on the y-axis. Searching for the origin of primes is leading us to novel forms of complex numbers.

Extensions of the Number-Range of Natural Numbers still avoiding broken-rational Numbers

The mentioned properties of the here newly defined and newly named “complex-irrational numbers” and the “complex-imaginary-irrational numbers” suggest that the natural integers can be extended by similar number ranges. The complex irrational numbers with total pre-factors and the complex imaginary irrational numbers with total pre-factors as well as the complex imaginary numbers with total pre-factors have not yet been defined as types of number ranges of total numbers. These ranges of numbers are similar to the natural numbers in a way that they avoid the use of fractional broken rational numbers. So these are new extensions of the ranges of total numbers that number-theory has so far lacked and that are very similar to the natural numbers. These novel numbers are enabling us to represent primes as products of other numbers by still avoid using broken-rational numbers.

Number of primes

The generating complex numbers of the primes follow a logarithmic distribution.

ai- complex numbers in analogy to irrational- complex numbers (INR) and i- complex numbers (iCN)

According to table 1 the representation of the primes in mod 8 shows us that in analogy to the i-complex numbers and the irrational complex numbers we can define further complex numbers. To these novel complex numbers we count the ai-complex numbers, which can be seen in columns 6 and 7 of table 1. Thereby the (ai)- unit almost is identical to the i unit, except the fact that if ai stands alone, than it is equal -1. With this ai unit column 6 is the square of an (ai)-complex number (x-aiy)^2 and column 7 is the square of anouther ai-complex number (x+aiy)^2. In this way primes become squares of ai-complex numbers in mod 8.

Analytical Continuation of the Primes

In order to find an analytical continuation of the primes we first calculate (p^2-1)/24 and we divide this by n^2 with n as number of the n^{th}-prime and afterwards we draw and correlate this versus the number n of the n^{th}-prime, itself. This is leading us to a logarithmic curve of the form y=0.894ln(x)-3.5954.

These results above lead us to correlate (p^2-1)/24/(n^2) versus the logarithm(n) of the n^{th}-prime. And this novel correlation is leading us to a parabolic curve y=0.0509x^2+0.0274x, with x=ln(n). Leading us to p = sqr(24 [0.0509 n^2 ln(n)^2+0.0274 n^2 ln(n)] +1).

As here the primes are the sum of two squares,- this result is similar to the result one chapter above, where the primes also are the sum of two squares what is equivalent to be the product of two complex numbers. Therefore all these transitions are similar in nature. All these transitions are sub-forms of the Pythagorean a^2+b^2=c^2

And in both constitutes the relevant numbers are distributed logarithmic.

Conclusion

Prime numbers have been studied extensively throughout the centuries, and many important discoveries have been made. From the work of Euler and Gauss in the 18th and 19th centuries to the work of Riemann, Hardy, and Littlewood in the 20th century, mathematicians have made significant contributions to understanding prime numbers. Notably, in the 21st century, the use of computers has enabled mathematicians to find and to verify vast prime numbers and to prove various conjectures and theorems related to prime numbers. The study of prime numbers is an ongoing field of research, and more significant contributions are likely to be made in the future.

References

1. Musielak D. Leonhard Euler and the Foundations of Celestial Mechanics. Springer Nature; 2022 Nov 1.
2. Capecchi D. Leonhard Euler between mathematics and natural philosophy: An introduction to natural science Anleitung zur Naturlehre. Handbook of the History and Philosophy of Mathematical Practice. 2020:1-53.
3. Chassefière E. Aurora borealis systems in the German-Russian world in the first half of the eighteenth century: the cases of Friedrich Christoph Mayer and Leonhard Euler. Annals of Science. 2021 Apr 3;78(2):162-96.
4. Aghaeiboorkheili M, Kawagle JG. The History of the Derivation of Euler's Number. Journal of Applied Mathematics and Physics. 2022 Sep 2;10(9):2780-95.
5. Ferreirós J. Gauss and the Mathematical Background to Standardisation. HoST-Journal of History of Science and Technology. 2020 Jun;14(1):32-51.
6. Feynman RP. The Chronicle of Numbers and Arithmetic from the Beginning of the Second Millennium to the Beginning of the 20th century. Trilogy Of Numbers And Arithmetic-Book 1: History Of Numbers And Arithmetic: An Information Perspective. 2022 Apr 22;12:53.
7. Tiwari D, Kushwah J. Number Theory. Turkish Online Journal of Qualitative Inquiry. 2021 Sep 1;12(8).
8. Russell B. Discovery of Non-Diophantine Arithmetics and their Applications. Trilogy Of Numbers And Arithmetic-Book 1: History Of Numbers And Arithmetic: An Information Perspective. 2022 Apr 22;12:137.

9. Bombieri E. New progress on the zeta function: From old conjectures to a major breakthrough. Proceedings of the National Academy of Sciences. 2019 Jun 4;116(23):11085-6.

10. Martínez JJ. On the Absence of Zeros of Riemann Zeta-Function Out of $\Re$ (z)= 1/2. Advances in Pure Mathematics. 2022 Mar 7;12(3):178-85.

11. Machado JT, Luchko Y. Multidimensional scaling and visualization of patterns in distribution of nontrivial zeros of the zeta-function. Communications in Nonlinear Science and Numerical Simulation. 2021 Nov 1;102:105924.

12. Wong B. Logic in general and mathematical logic in particular. Bulletin of Pure and Applied Science. 2021;172-176.

13. Isant PB. The Riemann hypothesis: The great pending mathematical challenge. Metode Science Studies Journal. 2018;8:35-41.

14. Nillsen R. Properties of Orbits and Normal Numbers in the Binary Dynamical System. arXiv preprint arXiv:2204.09249. 2022 Apr 20.

15. Dolbeault J, Esteban MJ. Hardy-Littlewood-Sobolev and related inequalities: stability. arXiv preprint arXiv:2202.02972. 2022 Feb 7.

16. Shiu P. Bounded gaps between primes by Kevin Broughan, pp. 590,£ 39.99 (paper), ISBN 978-1-108-79920-1, Cambridge University Press (2021). The Mathematical Gazette. 2023 Mar;107(568):181-3.

17. Ventre AG. Numbers and Propositions. InCalculus and Linear Algebra: Fundamentals and Applications 2023 Feb 12 (pp. 7-37). Cham: Springer International Publishing.

18. Bråting K, Kilhamn C. Exploring the intersection of algebraic and computational thinking. Mathematical Thinking and Learning. 2021 Apr 3;23(2):170-85.

19. Bortolomasi M, Ortiz-Tapia A. Some remarks on the first Hardy-Littlewood conjecture. arXiv preprint arXiv:1904.03017. 2019 Apr 4.

20. Minati G. A conceptual proposal on the undecidability of the distribution law of prime numbers and theoretical consequences. Ratio Mathematica. 2019;37:69.

a	b	a^2-b^2 (a+b)(a-b)	2ab	a^2+b^2 (a+b)(a-b)	a^2-b^2+2ab (a+b)^2	a^2-b^2-2ab (a-b)^2	mod(8)=1	mod(8)=3	mod(8)=5	mod(8)=7
2	1	3	4	5	7	-1	1	3	5	7
3	2	5	12	13	17	-7	17	11	13	23
4	1	15	8	17	23	7	41	19	29	31
4	3	7	24	25	31	-17	73	43	37	47
5	2	21	20	29	41	1	89	59	53	71
6	1	35	12	37	47	23	97	67	61	79
5	4	9	40	41	49	-31	113	83	101	103
7	2	45	28	53	73	17	137	107	109	127
6	5	11	60	61	71	-49	193	131	149	151
7	4	33	56	65	89	-23	233	139	157	167
8	1	63	16	65	79	47	241	163	173	191
8	3	55	48	73	103	7	257	179	181	199
7	6	13	84	85	97	-71	281	211	197	223
9	2	77	36	85	113	41	313	227	229	239
8	5	39	80	89	119	-41	337	251	269	263
9	4	65	72	97	137	-7	353	283	277	271
10	1	99	20	101	119	79	401	307	293	311
10	3	91	60	109	151	31	409	331	317	359
8	7	15	112	113	127	-97	433	347	349	367
11	2	117	44	125	161	73	449	379	373	383
11	4	105	88	137	193	17	457	419	389	431
9	8	17	144	145	161	-127		443	397	439
12	1	143	24	145	167	119		467	421	463
10	7	51	140	149	191	-89			461	
11	6	85	132	157	217	-47				
12	5	119	120	169	239	-1				
13	2	165	52	173	217	113				
10	9	19	180	181	199	-161				
11	8	57	176	185	233	-119				
13	4	153	104	185	257	49				
12	7	95	168	193	263	-73				
14	1	195	28	197	223	167				
13	6	133	156	205	289	-23				
14	3	187	84	205	271	103				
11	10	21	220	221	241	-199				
14	5	171	140	221	311	31				
15	2	221	60	229	281	161				
13	8	105	208	233	313	-103				
15	4	209	120	241	329	89				
14	7	147	196	245	343	-49				
16	1	255	32	257	287	223				
15	6	189	180	261	369	9				
12	11	23	264	265	287	-241				
16	3	247	96	265	343	151				
13	10	69	260	269	329	-191				
14	9	115	252	277	367	-137				
16	5	231	160	281	391	71				

15	8	161	240	289	401	-79				
16	7	207	224	305	431	-17				
13	12	25	312	313	337	-287				
14	11	75	308	317	383	-233				