A NEW UPPER BOUND ON THE STAR DISCREPANCY OF (0,1)-SEQUENCES

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Abstract

We study $(0,1)$-sequences in arbitrary base b and derive a new upper bound on the star discrepancy of these. Moreover, we show that the van der Corput sequence is the sequence with the highest star discrepancy among all $(0,1)$-sequences. The key property of the van der Corput sequence leading to this result is that its points are in a certain sense as close to the origin as possible. The main tool in our work is a recent finding on the discrepancy of $(0,m,2)$-nets.

1. Introduction and Statement of the Result

In many applications, notably numerical integration, point sets with good distribution properties in the unit cube are of interest. One way of measuring the quality of distribution of a point set in the s-dimensional unit cube $[0,1]^s$ is based on the discrepancy function. Let P_N be a point set in $[0,1]^s$ consisting of N points. Let $0 \le \alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(s)} \le 1$, then the discrepancy function Δ is given by

$$\Delta\left(P_N, \alpha^{(1)}, \ldots, \alpha^{(s)}\right) := A\left(P_N, \prod_{j=1}^{s}\left[0, \alpha^{(j)}\right)\right) - N\alpha^{(1)} \cdots \alpha^{(s)},$$

where $A\left(P_N, \prod_{j=1}^{s}\left[0, \alpha^{(j)}\right)\right)$ denotes the number of points of P_N in $\prod_{j=1}^{s}\left[0, \alpha^{(j)}\right)$. It is useful in the following to introduce a "closed version" of Δ, which is defined by

$$\overline{\Delta}\left(P_N, \alpha^{(1)}, \ldots, \alpha^{(s)}\right) := A\left(P_N, \prod_{j=1}^{s}\left[0, \alpha^{(j)}\right]\right) - N\alpha^{(1)} \cdots \alpha^{(s)}.$$

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By taking a norm of $\Delta(P_N, \alpha^{(1)}, \ldots, \alpha^{(s)})N^{-1}$, we obtain a measurement of the irregularity of distribution. In particular, the supremum norm has been studied extensively. In this case, we speak of the star discrepancy of the point set P_N, which is defined by

$$D^*(P_N) := \sup_{0 \le \alpha^{(1)}, \ldots, \alpha^{(s)} \le 1} \left| \Delta(P_N, \alpha^{(1)}, \ldots, \alpha^{(s)})N^{-1} \right|.$$

Note that, in the special case $s = 1$, we might also write

$$D^*(P_N) = \sup_{0 \le \alpha \le 1} \left| \overline{\Delta}(P_N, \alpha)N^{-1} \right|$$

(this follows from the fact that $\overline{\Delta}(P_N, 1) = 0$).

A broad class of point sets with small star discrepancy is provided by the concepts of (t, m, s)-nets and (t, s)-sequences. An extensive survey on this topic can be found in [7, 8]. We first give the definition of a (t, m, s)-net.

Definition 1 *Let $b \ge 2$, $s \ge 1$, and $0 \le t \le m$ be integers. A point set P consisting of b^m points in $[0, 1)^s$ forms a (t, m, s)-net in base b, if every subinterval $J = \prod_{j=1}^{s} [a_j b^{-d_j}, (a_j + 1)b^{-d_j})$ of $[0, 1)^s$, with integers $d_j \ge 0$ and integers $0 \le a_j < b^{d_j}$ for $1 \le j \le s$ and of volume b^{t-m}, contains exactly b^t points of P.*

Observe that a (t, m, s)-net is particularly well distributed if its quality parameter t is small. A very prominent example of a $(0, m, 2)$-net in base b is the so-called two-dimensional Hammersley net in base b consisting of the points

$$\widetilde{\boldsymbol{x}}_n = \left(\frac{n}{b^m}, \phi_b(n) \right), \quad 0 \le n \le b^m - 1.$$

Here, $\phi_b(n)$ is the radical-inverse function in base b, with

$$\phi_b(n) := \sum_{i=0}^{\infty} a_i(n) b^{-i-1}$$

for an integer $n \ge 0$, where n is given by its unique digit expansion in base b,

$$n = \sum_{i=0}^{\infty} a_i(n) b^i,$$

and where $a_i(n) \in \{0, \ldots, b-1\}$ for all $i \ge 0$, and $a_i(n) = 0$ for sufficiently large i. We denote, for given m, the Hammersley point set in base b by $H_{m,b}$. It has recently been outlined in [2, Lemma 1] that the Hammersley net $H_{m,b}$ plays a special role among the $(0, m, 2)$-nets in base b since it can be shown that

$$A(Y_{b^m}, [0, \alpha] \times [0, \beta]) \le A(H_{m,b}, [0, \alpha] \times [0, \beta]) \tag{1}$$

for any $(0, m, 2)$-net Y_{b^m} in base b and any $\alpha, \beta \in [0, 1]$. This result holds for all $m \ge 0$ and any choice of $b \ge 2$. Inequality (1) means that the Hammersley point set has its points as close to the origin as possible for a $(0, m, 2)$-net in base b. This is a property that causes relatively bad distribution properties of the Hammersley net; indeed, it can be shown that the net $H_{m,b}$ is the $(0, m, 2)$-net that has essentially the highest star discrepancy among all $(0, m, 2)$-nets in base b (see again [2]).

A class of infinite point sets that are in their structure based on (t, m, s)-nets are so-called (t, s)-sequences which are defined as follows (see [7, 8] for broader information).

Definition 2 *Let $b \geq 2$, $s \geq 1$, and $t \geq 0$ be integers. A sequence $(\boldsymbol{y}_n)_{n\geq 0}$ is a (t, s)-sequence in base b if for all $l \geq 0$ and $m > t$ the point set consisting of $\boldsymbol{y}_{lb^m}, \ldots, \boldsymbol{y}_{(l+1)b^m-1}$ is a (t, m, s)-net in base b.*

A popular example of a $(0, 1)$-sequence is the so-called van der Corput sequence in base b, denoted by C_b and consisting of points $x_0, x_1, \ldots$, where the n-th point x_n is the radical inverse function in base b of n $(n \geq 0)$. Observe that the points $\widetilde{\boldsymbol{x}}_n$ of $H_{m,b}$ and the points x_n of C_b are related to each other via

$$\widetilde{\boldsymbol{x}}_n = \left(\frac{n}{b^m}, x_n\right), \quad 0 \leq n \leq b^m - 1. \tag{2}$$

This relation between $H_{m,b}$ and C_b is a special case of a more general situation described by Niederreiter. In fact, Niederreiter shows in [7, Lemma 5.15] that, given a (t, s)-sequence $(\boldsymbol{y}_n)_{n\geq 0}$ in base b, the point set consisting of

$$\widetilde{\boldsymbol{y}}_n := \left(\frac{n}{b^m}, \boldsymbol{y}_n\right), \quad 0 \leq n \leq b^m - 1,$$

forms a $(t, m, s + 1)$-net in base b, provided that $m \geq t$.

In this note, it is our aim to derive a new upper bound on the star discrepancy of $(0, 1)$-sequences. We also show that C_b is the sequence with the highest star discrepancy among all $(0, 1)$-sequences in base b. The star discrepancy of $(0, 1)$-sequences in general and of the van der Corput sequence in particular has been studied extensively in the literature. For example, Niederreiter derived good general upper bounds on the star discrepancy of arbitrary (t, s)-sequences in base b (see [7, 8]). In the case of a $(0, 1)$-sequence Y in base b, Niederreiter showed

$$ND^*(Y_N) \leq (\log N)\frac{b-1}{2\log b} + O(1), \tag{3}$$

where Y_N denotes the collection of the first N $(N \in \mathbb{N})$ points of Y, and the constant in the O-notation does not depend on N. Restricting himself to a special case, Pillichshammer proved in [9] that for any digital $(0, 1)$-sequence Y over $\mathbb{Z}_2$ we have

$$ND^*(Y_N) \leq ND^*(C_{2,N}) \leq \frac{\log N}{3\log 2} + 1, \tag{4}$$

where Y_N denotes the collection of the first N $(N \in \mathbb{N})$ points of Y, and $C_{2,N}$ is the collection of the first N points of the van der Corput sequence in base 2. The latter result means that the van der Corput sequence is the worst distributed digital $(0,1)$-sequence over $\mathbb{Z}_2$ with respect to the star discrepancy (for the definition of digital sequences see [9] or, more generally, [8]). Further interesting results concerning the discrepancy of C_b are due to Béjian and Faure [1], Drmota, Larcher, and Pillichshammer [3], and Faure, who gave formulas for the star discrepancy of the van der Corput and related sequences and studied their asymptotical behavior (see, e. g., among many papers, [4, 5, 6]).

In this paper, we are going to generalize the inequalities in (4) to a broader class of $(0,1)$-sequences, to be more precise to all $(0,1)$-sequences, thereby improving on (3). This will be achieved by finding an analogue to inequality (1), which will show that the van der Corput sequence is the $(0,1)$-sequence with its points as close to the origin as possible. As it is the case with the Hammersley point set in two dimensions, the latter property is the reason why C_b will turn out to be the sequence with the highest star discrepancy among all $(0,1)$-sequences. In the next section we prove the subsequent theorem.

Theorem *Let Y be an arbitrary $(0,1)$-sequence in base b and denote by Y_N the first N elements of Y. Moreover, let $C_{b,N}$ be the first N terms of the van der Corput sequence in base b. Then*

$$ND^*(Y_N) \leq ND^*(C_{b,N}) \leq (\log N) f(b) + c(b),$$

where $c(b)$ is a constant depending only on b and where

$$f(b) = \begin{cases} \frac{b^2}{4(b+1)\log b}, & \text{if } b \text{ is even,} \\ \frac{b-1}{4\log b}, & \text{if } b \text{ is odd.} \end{cases}$$

Remark Note that the bound in the theorem is, in the special case $b = 2$, the same as the bound in (4) with respect to the leading term. Note also that the bound improves on (3).

We obtain the following corollary, which immediately follows by using our theorem and Théorème 6 in [4].

Corollary *For given $b \geq 2$, we have*

$$\limsup_{N\to\infty} \sup_{Y} \frac{ND^*(Y_N)}{\log N} = f(b),$$

where the supremum is extended over all $(0,1)$-sequences Y in base b, where Y_N denotes the collection of the first N points of Y, and where $f(b)$ is defined as above.

2. The Proof

We start with some auxiliary results. The subsequent lemma is motivated by Lemma 2 in [9].

Lemma 1 *Let $Y = (y_n)_{n=0}^{\infty}$ be a $(0,1)$-sequence in base b. Let $m \geq 0$, $b^{m-1} < N \leq b^m$, and denote the collection of the first N points $y_0, \ldots, y_{N-1}$ of Y by Y_N. Moreover, define $\widetilde{Y}_{b^m}$ as the $(0,m,2)$-net with points $\widetilde{\boldsymbol{y}}_0, \ldots, \widetilde{\boldsymbol{y}}_{b^m-1}$, where*

$$\widetilde{\boldsymbol{y}}_n := \left(\frac{n}{b^m}, y_n\right), \quad 0 \leq n \leq b^m - 1.$$

Then for $\alpha \in [0,1]$ we have

(a)

$$A(Y_N, [0,\alpha)) = A(\widetilde{Y}_{b^m}, [0, Nb^{-m}) \times [0,\alpha)),$$

(b)

$$A(Y_N, [0,\alpha]) = A(\widetilde{Y}_{b^m}, [0, (N-1)b^{-m}] \times [0,\alpha]),$$

(c)

$$\Delta(Y_N, \alpha) = \Delta(\widetilde{Y}_{b^m}, Nb^{-m}, \alpha),$$

(d)

$$\overline{\Delta}(Y_N, \alpha) = \overline{\Delta}(\widetilde{Y}_{b^m}, (N-1)b^{-m}, \alpha) - \alpha.$$

Proof. The formulas in (a) and (b) are obvious. Concerning (c), note that, due to (a),

$$\begin{aligned} \Delta(Y_N, \alpha) &= A(Y_N, [0,\alpha)) - N\alpha \\ &= A(\widetilde{Y}_{b^m}, [0, Nb^{-m}) \times [0,\alpha)) - b^m N b^{-m} \alpha \\ &= \Delta(\widetilde{Y}_{b^m}, Nb^{-m}, \alpha). \end{aligned}$$

The proof of (d) is similar to that of (c), making use of (b). □

Let us now define a function $S : [0,1] \to [0,1]$ by $S(x) := 1 - x$ and denote, for a point set P_N consisting of N points $p_0, \ldots, p_{N-1} \in [0,1]$, by $S(P_N)$ the point set consisting of the collection of the $S(p_n)$, $0 \leq n \leq N-1$.

We also need some further notation. Let $Y = (y_n)_{n=0}^{\infty}$ be an arbitrary $(0,1)$-sequence in base b and let $m \geq 0$. Let for a point $y_n \in Y$, with $0 \leq n \leq b^m - 1$, $\overline{y}_n$ be the point that is obtained by moving y_n into the upper endpoint $(r+1)b^{-m}$ of the interval $[rb^{-m}, (r+1)b^{-m})$, $0 \leq r < b^m$, it lies in. Furthermore, denote the collection of the points $\overline{y}_0, \ldots, \overline{y}_{b^m-1}$ by $\overline{Y}_{b^m}$. We now have

Lemma 2 *For an arbitrary $(0,1)$-sequence $Y = (y_n)_{n=0}^{\infty}$ in base b and $m \geq 0$, let S and $\overline{Y}_{b^m}$ be defined as above. Then the points of $S(\overline{Y}_{b^m})$ satisfy the properties of the first b^m points of a $(0,1)$-sequence, that is, for any $l \geq 0$ and any $k \in \{1, \ldots, m\}$ with $(l+1)b^k \leq b^m$ the points $S(\overline{y}_{lb^k}), \ldots, S(\overline{y}_{(l+1)b^k-1})$ form a $(0,k,1)$-net.*

Proof. Let Y be a $(0,1)$-sequence in base b and let $m \geq 0$. We need to show that for any $l \geq 0$ and any $k \in \{1,\ldots,m\}$ with $(l+1)b^k \leq b^m$ the points $S(\overline{y}_{lb^k}),\ldots,S(\overline{y}_{(l+1)b^k-1})$ form a $(0,k,1)$-net. This can be seen as follows. Since Y is a $(0,1)$-sequence, it follows that the points $y_{lb^k},\ldots,y_{(l+1)b^k-1}$ are a $(0,k,1)$-net. Thus, for any nonnegative integer $a < b^k$, there is exactly one point among $y_{lb^k},\ldots,y_{(l+1)b^k-1}$ that lies in the interval $[ab^{-k},(a+1)b^{-k})$. Consequently, there is exactly one point among $\overline{y}_{lb^k},\ldots,\overline{y}_{(l+1)b^k-1}$ that lies in $(ab^{-k},(a+1)b^{-k}]$ for each $a \in \{0,1,\ldots,b^k-1\}$. This, however, implies that there is exactly one point among $S(\overline{y}_{lb^k}),\ldots,S(\overline{y}_{(l+1)b^k-1})$ in each interval $[ab^{-k},(a+1)b^{-k})$, $a \in \{0,1,\ldots,b^k-1\}$, which means that the points $S(\overline{y}_{lb^k}),\ldots,S(\overline{y}_{(l+1)b^k-1})$ form a $(0,k,1)$-net. □

The following auxiliary result will be essential in the proof of our main result.

Lemma 3 *Denote by $C_{b,N}$ the first N elements of the van der Corput sequence in base b. Moreover, let Y be an arbitrary $(0,1)$-sequence in base b and denote the collection of its first N elements by Y_N. Then we have*

$$A(Y_N,[0,\alpha]) \leq A(C_{b,N},[0,\alpha])$$

and

$$A(Y_N,[0,\alpha)) \geq A(S(C_{b,N}),[0,\alpha))$$

for any $\alpha \in [0,1]$, where S is defined as above.

Proof. We start with showing the first inequality. Let Y be an arbitrary $(0,1)$-sequence in base b and let $m \geq 0$ be such that $b^{m-1} < N \leq b^m$. Denote by $\widetilde{Y}_{b^m}$ the $(0,m,2)$-net in base b with points

$$\widetilde{\boldsymbol{y}}_n := \left(\frac{n}{b^m}, y_n\right), \quad 0 \leq n \leq b^m - 1,$$

where y_n is the n-th point of Y. By (2) and by Lemma 1,

$$A(Y_N,[0,\alpha]) = A(\widetilde{Y}_{b^m},[0,(N-1)b^{-m}] \times [0,\alpha]),$$

$$A(C_{b,N},[0,\alpha]) = A(H_{m,b},[0,(N-1)b^{-m}] \times [0,\alpha]).$$

Due to (1),

$$A(\widetilde{Y}_{b^m},[0,\gamma] \times [0,\delta]) \leq A(H_{m,b},[0,\gamma] \times [0,\delta])$$

for any choice of $\gamma,\delta \in [0,1]$. The first inequality follows.

We show the second inequality by making use of the first inequality. For an arbitrary $(0,1)$-sequence Y in base b and $b^{m-1} < N \leq b^m$, define $\overline{Y}_{b^m}$ as above. By Lemma 2, $S(\overline{Y}_{b^m})$ is such that these points satisfy the properties of the first b^m points of a $(0,1)$-sequence. Note that by the construction outlined in the proof of Lemma 2 the collection of the points $S(\overline{y}_0),\ldots,S(\overline{y}_{N-1})$, let us denote it by $S(\overline{Y}_N)$, satisfies the analog properties of the first N points of a $(0,1)$-sequence. For given $\alpha \in [0,1]$ it thus follows by the first inequality that

$$A(\overline{Y}_N, [\alpha, 1]) = A(S(\overline{Y}_N), [0, 1-\alpha]) \leq A(C_{b,N}, [0, 1-\alpha]) = A(S(C_{b,N}), [\alpha, 1]).$$

By the fact that

$$A(\overline{Y}_N, [0,1]) = N = A(S(C_{b,N}), [0,1]),$$

we obtain

$$A(\overline{Y}_N, [0, \alpha)) \geq A(S(C_{b,N}), [0, \alpha)).$$

Since

$$A(Y_N, [0, \alpha)) \geq A(\overline{Y}_N, [0, \alpha)),$$

the assertion is shown. □

We now deduce

Proposition 1 *Let Y be an arbitrary $(0,1)$-sequence in base b. Further, let Y_N, $C_{b,N}$, and S be defined as above. Then we have*

$$ND^*(Y_N) \leq \max\{ND^*(C_{b,N}), ND^*(S(C_{b,N}))\}.$$

Proof. To begin with, note that the first inequality in Lemma 3 also yields

$$A(Y_N, [0, \alpha)) \leq A(C_{b,N}, [0, \alpha))$$

for any $\alpha \in [0,1]$. From this inequality together with the second inequality in Lemma 3 we obtain

$$A(S(C_{b,N}), [0, \alpha)) \leq A(Y_N, [0, \alpha)) \leq A(C_{b,N}, [0, \alpha)),$$

which gives

$$\Delta(S(C_{b,N}), \alpha) \leq \Delta(Y_N, \alpha) \leq \Delta(C_{b,N}, \alpha)$$

for any $\alpha \in [0,1]$.

Consequently,

$$\sup_{0 \leq \alpha \leq 1} |\Delta(Y_N, \alpha)| \leq \sup_{0 \leq \alpha \leq 1} \max\left\{|\Delta(C_{b,N}, \alpha)|, |\Delta(S(C_{b,N}), \alpha)|\right\}.$$

Interchanging supremum and maximum in the right hand side of the latter inequality (which causes no problems since Δ is piecewise linear in α) yields the result. □

We can now give the proof of our theorem.

Proof. For the first inequality, it is by Proposition 1 sufficient to show that

$$ND^*(S(C_{b,N})) \leq ND^*(C_{b,N}).$$

Let $\alpha \in [0,1]$ be given. Observe that

$$\begin{aligned}\Delta(S(C_{b,N}),\alpha) &= A(S(C_{b,N}),[0,\alpha)) - N\alpha \\ &= A(C_{b,N},(1-\alpha,1]) - N\alpha \\ &= A(C_{b,N},[0,1]) - A(C_{b,N},[0,1-\alpha]) - N\alpha \\ &= -A(C_{b,N},[0,1-\alpha]) + N - N\alpha \\ &= -\overline{\Delta}(C_{b,N},1-\alpha).\end{aligned}$$

Similarly it can be shown that $\overline{\Delta}(S(C_{b,N}),\alpha) = -\Delta(C_{b,N},1-\alpha)$. From this it even follows that

$$ND^*(S(C_{b,N})) = ND^*(C_{b,N})$$

and the first inequality is shown.

The second inequality is shown as follows. Since it is true that the star discrepancy of $C_{b,N}$ equals the extreme discrepancy of $C_{b,N}$ (see [4, Corollary to Théorème 1 (p. 147) and Section 5.5.1 (p. 178)]), we can use results by Faure who showed that

$$D^*(C_{b,N}) \leq \frac{a_b}{\log b}\log N + \max\left\{2, 1 + \frac{1}{b} + a_b\right\}$$

(cf. [4, Théorème 2]), where

$$a_b = \begin{cases} \frac{b^2}{4(b+1)}, & \text{if } b \text{ is even,} \\ \frac{b-1}{4}, & \text{if } b \text{ is odd} \end{cases}$$

(cf. [4, Section 5.5]). The result follows. □

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