# The Counting Functions of Prime Pairs

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**Abstract**: The prime pair counting functions $\pi_{2,\Delta}(N)$ with the gap $\Delta$=2, 4 and $P_x(1, 1)$ are derived. The asymptotic behavior of $\pi_{2,\Delta}(N)$ with the gap $\Delta$=2, 4 and $P_x(1, 1)$ are also analyzed.

It was indicated[1] that the original sieve operation can be expressed in mathematical equations and the mathematical equations describing the original sieve method lead to the derivation of prime counting functions $\pi(N)$, $\pi_-(N)$ and $\pi_+(N)$. Here the original sieve method is used to derive the counting functions of prime pairs.

For a given finite integer N, the prime counting functions $\pi(N)$, $\pi_-(N)$ and $\pi_+(N)$ count the number of primes contained in the finite sets $\mathbf{c}(N)$, $\mathbf{c}_-(N)$ and $\mathbf{c}_+(N)$, respectively

$$\begin{cases} \boldsymbol{c}_-(N) = \left\{(6k-1) \mid k = 1, 2, \cdots, flr\left(\frac{N+1}{6}\right)\right\} \\ \boldsymbol{c}_+(N) = \left\{(6k+1) \mid k = 1, 2, \cdots, flr\left(\frac{N-1}{6}\right)\right\} \\ \boldsymbol{c}(N) = \boldsymbol{c}_-(N) U \boldsymbol{c}_+(N) \end{cases} \tag{1}$$

$\pi(N)$ is the number of primes contained in the set $\mathbf{c}(N)$:

$$\pi(N) = \tau(N) - \chi(N) \tag{2}$$

$$\pi(N) = \left\{flr\left(\frac{N-1}{6}\right) + flr\left(\frac{N+1}{6}\right)\right\} - \left\{\sum_{5\le p_i}\left\{flr\left(\frac{|N-p_i|}{6p_i}\right) + flr\left(\frac{N+p_i}{6p_i}\right)\right\} + \right.$$

$$\left. + \sum_{n=2}^{\infty}(-1)^{n-1} \sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}} \left\{flr\left(\frac{N+5p_{k_1}p_{k_2}\cdots p_{k_n}}{6p_{k_1}p_{k_2}\cdots p_{k_n}}\right) + flr\left(\frac{N+p_{k_1}p_{k_2}\cdots p_{k_n}}{6p_{k_1}p_{k_2}\cdots p_{k_n}}\right)\right\}\right\} \tag{3}$$

$$\tau(N) = |\boldsymbol{c}(N)| = flr\left(\frac{N-1}{6}\right) + flr\left(\frac{N+1}{6}\right) \tag{4}$$

$$\chi(N) = \sum_{5\le p_i}\left\{flr\left(\frac{|N-p_i|}{6p_i}\right) + flr\left(\frac{N+p_i}{6p_i}\right)\right\} +$$

$$+\sum_{n=2}^{\infty}(-1)^{n-1}\sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}}\left\{flr\left(\frac{N+5p_{k_1}p_{k_2}\cdots p_{k_n}}{6p_{k_1}p_{k_2}\cdots p_{k_n}}\right)+flr\left(\frac{N+p_{k_1}p_{k_2}\cdots p_{k_n}}{6p_{k_1}p_{k_2}\cdots p_{k_n}}\right)\right\} \quad (5)$$

$\chi$(N) is the total number of composites contained in the set **c**(N). *flr*(*x*) is the floor function.

$\pi_-$(N) is the number of primes contained in the sets $\mathbf{c}_-$(N):

$$\pi_-(N)=\tau_-(N)-\chi_-(N) \quad (6)$$

$$\pi_-(N)=flr\left(\frac{N+1}{6}\right)-\left\{\sum_{5\le p_i}\left\{C_-(p_i)flr\left(\frac{|N-p_i|}{6p_i}\right)+C_+(p_i)flr\left(\frac{N+p_i}{6p_i}\right)\right\}+\right.$$

$$+\sum_{n=2}^{\infty}(-1)^{n-1}\sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}}\left\{C_-\left(p_{k_1}p_{k_2}\cdots p_{k_n}\right)flr\left(\frac{N+5p_{k_1}p_{k_2}\cdots p_{k_n}}{6p_{k_1}p_{k_2}\cdots p_{k_n}}\right)+\right.$$

$$\left.\left.+C_+(p_{k_1}p_{k_2}\cdots p_{k_n})flr\left(\frac{N+p_{k_1}p_{k_2}\cdots p_{k_n}}{6p_{k_1}p_{k_2}\cdots p_{k_n}}\right)\right\}\right\} \quad (7)$$

$$\tau_-(N)=|\boldsymbol{c}_-(N)|=flr\left(\frac{N+1}{6}\right) \quad (8)$$

$$\chi_-(N)=\sum_{5\le p_i}\left\{C_-(p_i)flr\left(\frac{|N-p_i|}{6p_i}\right)+C_+(p_i)flr\left(\frac{N+p_i}{6p_i}\right)\right\}+$$

$$+\sum_{n=2}^{\infty}(-1)^{n-1}\sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}}\left\{C_-\left(p_{k_1}p_{k_2}\cdots p_{k_n}\right)flr\left(\frac{N+5p_{k_1}p_{k_2}\cdots p_{k_n}}{6p_{k_1}p_{k_2}\cdots p_{k_n}}\right)+\right.$$

$$\left.+C_+\left(p_{k_1}p_{k_2}\cdots p_{k_n}\right)flr\left(\frac{N+p_{k_1}p_{k_2}\cdots p_{k_n}}{6p_{k_1}p_{k_2}\cdots p_{k_n}}\right)\right\} \quad (9)$$

$\pi_+$(N) is the number of primes contained in the set $\mathbf{c}_+$(N):

$$\pi_+(N)=\tau_+(N)-\chi_+(N) \quad (10)$$

$$\pi_+(N)=flr\left(\frac{N-1}{6}\right)-\left\{\sum_{5\le p_i}\left\{C_+(p_i)flr\left(\frac{|N-p_i|}{6p_i}\right)+C_-(p_i)flr\left(\frac{N+p_i}{6p_i}\right)\right\}+\right.$$

$$+\sum_{n=2}^{\infty}(-1)^{n-1}\sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}}\left\{C_+\left(p_{k_1}p_{k_2}\cdots p_{k_n}\right)flr\left(\frac{N+5p_{k_1}p_{k_2}\cdots p_{k_n}}{6p_{k_1}p_{k_2}\cdots p_{k_n}}\right)+\right.$$

$$\left.\left.+C_-(p_{k_1}p_{k_2}\cdots p_{k_n})flr\left(\frac{N+p_{k_1}p_{k_2}\cdots p_{k_n}}{6p_{k_1}p_{k_2}\cdots p_{k_n}}\right)\right\}\right\} \quad (11)$$

$$\tau_+(N) = |\boldsymbol{c}_+(N)| = flr\left(\frac{N-1}{6}\right) \tag{12}$$

$$\chi_+(N) = \sum_{5\le p_i}\left\{C_+(p_i)flr\left(\frac{|N-p_i|}{6p_i}\right) + C_-(p_i)flr\left(\frac{N+p_i}{6p_i}\right)\right\} +$$
$$+\sum_{n=2}^{\infty}(-1)^{n-1}\sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}}\left\{C_+(p_{k_1}p_{k_2}\cdots p_{k_n})flr\left(\frac{N+5p_{k_1}p_{k_2}\cdots p_{k_n}}{6p_{k_1}p_{k_2}\cdots p_{k_n}}\right) +\right.$$
$$\left.+C_-(p_{k_1}p_{k_2}\cdots p_{k_n})flr\left(\frac{N+p_{k_1}p_{k_2}\cdots p_{k_n}}{6p_{k_1}p_{k_2}\cdots p_{k_n}}\right)\right\} \tag{13}$$

$\chi_-(N)$ is the total number of composites contained in the set $\mathbf{c}_-(N)$. $\chi_+(N)$ is the total number of composites contained in the set $\mathbf{c}_+(N)$. The functions $C_-(x)$ and $C_+(x)$ are defined by

$$C_-(x) = 1 + flr\left(\frac{x+1}{6}\right) - ceil\left(\frac{x+1}{6}\right) \tag{14}$$

$$C_+(x) = 1 + flr\left(\frac{x-1}{6}\right) - ceil\left(\frac{x-1}{6}\right) \tag{15}$$

$$\begin{cases} C_-(x) = \begin{cases} 1, & if\ x \in \boldsymbol{c}_-(N) \\ 0, & if\ x \notin \boldsymbol{c}_-(N) \end{cases} \\ C_+(x) = \begin{cases} 1, & if\ x \in \boldsymbol{c}_+(N) \\ 0, & if\ x \notin \boldsymbol{c}_+(N) \end{cases} \end{cases}$$

The two primes $p_i$ and $p_j$ are called to form a prime pair if one or the other of the following two equations is satisfied.

$$\begin{cases} p_j - p_i = \Delta \quad for\ p_i < p_j \in \boldsymbol{c} & (16.1) \\ p_i + p_j = x \quad for\ p_i \le p_j \in \{3, \boldsymbol{c}\} & (16.2) \end{cases}$$

The counting functions $\pi_{2,\Delta}(N)$ counts the total number of prime pairs in the set $\mathbf{c}(N)$ with a constant gap $\Delta$ between the two primes in every pair and $P_x(1, 1)$ defines the total number of prime pairs in the set of $\{3, \mathbf{c}(N)\}$ with a constant sum $x$ of the two primes in every pair.

The sieve functions $S_-^c(k|p_i)$ and $S_+^c(k|p_i)$ defined in the equations (17) and (18) will make sure that the prime $p_i$ appears or does not appear in a term of an equation if $p_i$ can or cannot divide $(6k-1)$ and $(6k+1)$ evenly, respectively.

$$S_-^c(k|p_i) = \sum_{l=1}^{ceil\left(\frac{6k-1}{p_i}\right)} \delta(6k-1-lp_i) \tag{17}$$

$$S_+^c(k|p_i) = \sum_{l=1}^{ceil\left(\frac{6k+1}{p_i}\right)} \delta(6k+1-lp_i) \tag{18}$$

The sieve functions $S_-(k)$ and $S_+(k)$ defined in the equations (19) and (20) will tell whether (6*k*-1) and (6*k*+1) is a prime, respectively.

$$S_-(k) = \sum_{p_i=5}^{flr\left(\frac{6k-1}{5}\right)} \left\{1 - \sum_{l=1}^{ceil\left(\frac{6k-1}{p_i}\right)} \delta(6k-1-lp_i)\right\} \tag{19}$$

$$S_+(k) = \sum_{p_i=5}^{flr\left(\frac{6k+1}{5}\right)} \left\{1 - \sum_{l=1}^{ceil\left(\frac{6k+1}{p_i}\right)} \delta(6k+1-lp_i)\right\} \tag{20}$$

The δ function defined in equation (21) is to make sure that the composite (6*k*-1) or (6*k*+1) is the least integer which consists of:

$$\left\{\prod_{m=1}^{n-z} p_{i_m}\right\}$$

in the prime factors and of which the partner in the composite pair consists of:

$$\left\{\prod_{l=1}^{z} p_{j_l}\right\}$$

in the prime factors at the same time, respectively.

$$C_\pm\left(k, \left\{\prod_{m=1}^{n-z} p_{i_m}\right\}\left\{\prod_{l=1}^{z} p_{j_l}\right\}\right) = \delta\left\{1 + \frac{6k \pm 1 - 6\left\{\prod_{m=1}^{n-z} p_{i_m}\right\}\left\{\prod_{l=1}^{z} p_{j_l}\right\}}{\left|6k \pm 1 - 6\left\{\prod_{m=1}^{n-z} p_{i_m}\right\}\left\{\prod_{l=1}^{z} p_{j_l}\right\}\right|}\right\} \tag{21}$$

If one or more primes appear among the prime factors of a prime-composite pair or a composite-composite pair simultaneously, their powers in the whole prime product must be 1 as assured by the corrections defined in the equations (22) and (23).

$$(6k \pm 1)^{1-c_k^{\pm}} \prod_{m=1}^{n} p_{i_m} = \begin{cases} \prod_{m=1}^{n} p_{i_m} & if\, c_k^{\pm} = \sum_{m=1}^{n} \delta(6k \pm 1 - p_{i_m}) = 1 \\ (6k \pm 1) \prod_{m=1}^{n} p_{i_m} & if\ c_k^{\pm} = \sum_{m=1}^{n} \delta(6k \pm 1 - p_{i_m}) = 0 \end{cases} \tag{22}$$

$$p_{i_m}^{1-c_m^0} \prod_{l=1}^{z} p_{j_l} = \begin{cases} \prod_{l=1}^{z} p_{j_l} & if\, c_m^0 = \sum_{l=1}^{z} \delta(p_{i_m} - p_{j_l}) = 1 \\ p_{i_m} \prod_{l=1}^{z} p_{j_l} & if\ c_m^0 = \sum_{l=1}^{z} \delta(p_{i_m} - p_{j_l}) = 0 \end{cases} \tag{23}$$

The original sieve method can be employed to derive the explicit analytical formulas of the prime pair counting functions $\pi_{2,\Delta}(N)$ and $P_x(1, 1)$ as defined in the equation (16). Using the definition of $S_-(k)$, the explicit analytical formula of $\pi_{2,2}(N)$ is expressed by

$$\pi_{2,2}(N) = \tau_{2,2}(N) - \chi(N) + \chi_{2,2}(N) + flr\left(\frac{N}{6flr\left(\frac{N-1}{6}\right)+5}\right)\left[1 - S_-\left(flr\left(\frac{N-1}{6}\right)+1\right)\right] \tag{24}$$

in which $\tau_{2,2}(N)$ is the total number of (6$k$-1) and (6$k$+1) pairs, $\chi(N)$ is the total number of composites that is calculated by the equation (5), the last term will be 1 or 0 if the unpaired integer $(6k\text{-}1)\leq N$ is a composite or a prime and $\chi_{2,2}(N)$ is the total number of composite pairs.

$$\tau_{2,2}(N) = flr\left(\frac{N-1}{6}\right)$$

$$\pi_{2,2}(N) = \left\{\tau_{2,2}(N) + flr\left(\frac{N+1}{6}\right) - \chi(N)\right\} -$$

$$- \left\{flr\left(\frac{N+1}{6}\right) - \chi_{2,2}(N) - flr\left(\frac{N}{6flr\left(\frac{N-1}{6}\right)+5}\right)\left[1 - S_-\left(flr\left(\frac{N-1}{6}\right)+1\right)\right]\right\}$$

$$\pi_{2,2}(N) = \pi(N) - \left\{flr\left(\frac{N+1}{6}\right) - \chi_{2,2}(N) - flr\left(\frac{N}{6flr\left(\frac{N-1}{6}\right)+5}\right)\left[1 - S_-\left(flr\left(\frac{N-1}{6}\right)+1\right)\right]\right\} \tag{25}$$

In the equation (25), $\pi(N)$ is the prime counting function in the equation (3). Using the above definition of the sieve functions $S_-(k)$, $S_+(k)$, $S_-^c(k|p_i)$ and $S_+^c(k|p_i)$ and the $\delta$ function defined in equation (21), the explicit analytical formula of $\chi_{2,2}(N)$ is expressed by

$$\chi_{2,2}(N) = \sum_{k=1} \left\{ S_-(k)S_+(k)\left\{flr\left[\frac{|N-6k-1|}{6(6k-1)(6k+1)}\right] + flr\left[\frac{N+6k-1}{6(6k-1)(6k+1)}\right]\right\}\right.$$

$$+\sum_{n=1}(-1)^{n+1}\sum_{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_n}\le flr\left(\frac{6k-1}{5}\right)}\left\{\prod_{m=1}^{n} S_-^c\left(k|p_{i_m}\right)\right\}S_+(k)\times$$

$$\times\left\{flr\left[\frac{|N-6k-1|}{6(6k+1)\prod_{m=1}^{n}p_{i_m}}\right]+flr\left[\frac{N+6k-1}{6(6k+1)\prod_{m=1}^{n}p_{i_m}}\right]\right\}$$

$$+\sum_{n=1}(-1)^{n+1}\sum_{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_n}\le flr\left(\frac{6k+1}{5}\right)}S_-(k)\left\{\prod_{m=1}^{n} S_+^c\left(k|p_{i_m}\right)\right\}\times$$

$$\times\left\{flr\left[\frac{|N-6k-1|}{6(6k-1)\prod_{m=1}^{n}p_{i_m}}\right]+flr\left[\frac{N+6k-1}{6(6k-1)\prod_{m=1}^{n}p_{i_m}}\right]\right\}$$

$$+\sum_{n=2}(-1)^{n}\sum_{z=1}^{n-1}\sum_{\substack{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_{n-z}}\le flr\left(\frac{6k-1}{5}\right)\\ 5\le p_{j_1}<p_{j_2}<\cdots<p_{j_z}\le flr\left(\frac{6k+1}{5}\right)}}\left\{\prod_{m=1}^{n-z} S_-^c\left(k|p_{i_m}\right)\right\}\left\{\prod_{l=1}^{z} S_+^c\left(k|p_{j_l}\right)\right\}\times$$

$$\left.\times C_-\left(k,\prod_{m=1}^{n-z}p_{i_m},\prod_{j=1}^{z}p_{j_l}\right)\left\{1+flr\left[\frac{|N-6k-1|}{6\left\{\prod_{m=1}^{n-z}p_{i_m}\right\}\left\{\prod_{l=1}^{z}p_{j_l}\right\}}\right]+flr\left[\frac{N+6k-1}{6\left\{\prod_{m=1}^{n-z}p_{i_m}\right\}\left\{\prod_{l=1}^{z}p_{j_l}\right\}}\right]\right\}\right\} \quad (26)$$

For any given finite integer N, the number of prime pairs $\pi_{2,2}(N)$ contained in the finite sets **c**(N) can be accurately calculated by the equation (25) in combination with the equations (3) and (26).

When $N\to\infty$, the floor operation in the equation (26) can be replaced by the simple divide and the finite terms can be neglected. The asymptotic behavior of $\pi_{2,2}(N)$ at $N\to\infty$ is

$$\pi_{2,2}(N \to \infty) = \lim_{N\to\infty}\left\{\pi(N) - \frac{N}{6}\left(1 + \sum_{n=2}^{\infty}(-1)^{n+1}\sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}}\frac{2}{p_{k_1}p_{k_2}\cdots p_{k_n}}\right)\right\} \tag{27}$$

For the Riemann zeta function at s=1, there holds the following equivalence

$$\frac{1}{\zeta(1)} = \frac{1}{3}\left(1 - \sum_{n=1}^{\infty}(-1)^{n+1}\sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}}\frac{1}{p_{k_1}p_{k_2}\cdots p_{k_n}}\right)$$

If letting

$$\lambda = \sum_{n=1}^{\infty}(-1)^{n+1}\sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}}\frac{1}{p_{k_1}p_{k_2}\cdots p_{k_n}}$$

then there are the followings:

$$\frac{1}{\zeta(1)} = \frac{1}{3}(1-\lambda)$$

$$\frac{9}{[\zeta(1)]^2} = 1 - 2\lambda + \lambda^2$$

$$\frac{9}{[\zeta(1)]^2} = 1 - \sum_{n=1}^{\infty}(-1)^{n+1}\sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}}\frac{2}{p_{k_1}p_{k_2}\cdots p_{k_n}} + \lambda^2$$

$$\frac{9}{[\zeta(1)]^2} = \left(-1 - \sum_{n=2}^{\infty}(-1)^{n+1}\sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}}\frac{2}{p_{k_1}p_{k_2}\cdots p_{k_n}}\right) + \left(2 - \sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}}\frac{2}{p_k} + \lambda^2\right)$$

If denoting

$$\lambda^2 = \sum_{n=2}^{\infty}(-1)^{n}\sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}}\frac{2}{p_{k_1}p_{k_2}\cdots p_{k_n}} + O\left\{\frac{1}{[\zeta(1)]^3}\right\}$$

$$O\left\{\frac{1}{[\zeta(1)]^3}\right\} = \sum_{n=1}^{\infty}\sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}}\left\{(-1)^n\sum_{m=1}^{\infty}\sum_{5\le p_{l_1}<p_{l_2}<\cdots<p_{l_m}}^{all\ p_{l_i}\neq any\ p_{k_j}}\frac{2}{(p_{k_1}p_{k_2}\cdots p_{k_n})(p_{l_1}p_{l_2}\cdots p_{l_m})^2}\right\} +$$

$$+\sum_{n=1}^{\infty}\sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}}\left\{\frac{1}{(p_{k_1}p_{k_2}\cdots p_{k_n})^2}\right\}$$

then there hold the following equations:

$$\frac{9}{[\zeta(1)]^2} = -\left(1 + \sum_{n=2}^{\infty}(-1)^{n+1}\sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}}\frac{2}{p_{k_1}p_{k_2}\cdots p_{k_n}}\right) +$$

$$+\left(2 - \sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}}\frac{2}{p_k} + \sum_{n=2}^{\infty}(-1)^{n}\sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}}\frac{2}{p_{k_1}p_{k_2}\cdots p_{k_n}}\right) + O\left\{\frac{1}{[\zeta(1)]^3}\right\}$$

$$\frac{9}{[\zeta(1)]^2} = -\left(1 + \sum_{n=2}^{\infty}(-1)^{n+1} \sum_{5 \le p_{k_1} < p_{k_2} < \cdots < p_{k_n}} \frac{2}{p_{k_1} p_{k_2} \cdots p_{k_n}}\right) + \frac{6}{\zeta(1)} + O\left\{\frac{1}{[\zeta(1)]^3}\right\} \quad (28)$$

Substituting the equation (28) into the equation (27), then:

$$\pi_{2,2}(N \to \infty) = \lim_{N\to\infty}\left\{\pi(N) - \frac{N}{6}\left(\frac{6}{\zeta(1)} - \frac{9}{[\zeta(1)]^2} + O\left\{\frac{1}{[\zeta(1)]^3}\right\}\right)\right\}$$

$$\pi_{2,2}(N \to \infty) = \lim_{N\to\infty}\left\{\frac{N}{\zeta(1)} - \frac{N}{6}\left(\frac{6}{\zeta(1)} - \frac{9}{[\zeta(1)]^2} + O\left\{\frac{1}{[\zeta(1)]^3}\right\}\right)\right\}$$

$$\pi_{2,2}(N \to \infty) = \lim_{N\to\infty}\left(\frac{3N}{2[\zeta(1)]^2} - O\left\{\frac{1}{[\zeta(1)]^3}\right\}\right)$$

$$\pi_{2,2}(N \to \infty) \approx \lim_{N\to\infty}\frac{N}{(lnN)^2} \quad (29)$$

Following the derivations of the equations (25), (26), (27) and (28) then the explicit analytical formula of $\pi_{2,4}(N)$ can be obtained.

$$\pi_{2,4}(N) = \tau_{2,4}(N) - \chi(N) + \chi_{2,4}(N) + flr\left(\frac{N}{6flr\left(\frac{N-5}{6}\right)+1}\right)\left[1 - S_+\left(flr\left(\frac{N-5}{6}\right)\right)\right] \quad (30)$$

$$\tau_{2,4}(N) = flr\left(\frac{N-5}{6}\right) = flr\left(\frac{N+1}{6}\right) - 1$$

$$\pi_{2,4}(N) = \left\{\tau_{2,4}(N) + 1 + flr\left(\frac{N-1}{6}\right) - \chi(N)\right\} -$$

$$-\left\{flr\left(\frac{N-1}{6}\right) - \chi_{2,4}(N) - flr\left(\frac{N}{6flr\left(\frac{N-5}{6}\right)+1}\right)S_+\left(flr\left(\frac{N-5}{6}\right)\right) + 1\right\}$$

$$\pi_{2,4}(N) = \pi(N) - \left\{flr\left(\frac{N-1}{6}\right) - \chi_{2,4}(N) - flr\left(\frac{N}{6flr\left(\frac{N-5}{6}\right)+1}\right)S_+\left(flr\left(\frac{N-5}{6}\right)\right) + 1\right\} \quad (31)$$

$$\chi_{2,4}(N) = \sum_{k=1}\left\{S_-(k+1)S_+(k)\left\{flr\left[\frac{|N-6k-5|}{6(6k+1)(6k+5)}\right] + flr\left[\frac{N+6k+1}{6(6k+1)(6k+5)}\right]\right\}\right.$$

$$+\sum_{n=1}(-1)^{n+1}\sum_{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_n}\le flr\left(\frac{6k+5}{5}\right)}\left\{\prod_{m=1}^{n}S_{-}^{c}\left((k+1)|p_{i_m}\right)\right\}S_{+}(k)\times$$

$$\times\left\{flr\left[\frac{|N-6k-5|}{6(6k+1)\prod_{m=1}^{n}p_{i_m}}\right]+flr\left[\frac{N+6k+1}{6(6k+1)\prod_{m=1}^{n}p_{i_m}}\right]\right\}$$

$$+\sum_{n=1}(-1)^{n+1}\sum_{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_n}\le flr\left(\frac{6k+1}{5}\right)}S_{-}(k+1)\left\{\prod_{m=1}^{n}S_{+}^{c}\left(k|p_{i_m}\right)\right\}\times$$

$$\times\left\{flr\left[\frac{|N-6k-5|}{6(6k+5)\prod_{m=1}^{n}p_{i_m}}\right]+flr\left[\frac{N+6k+1}{6(6k+5)\prod_{m=1}^{n}p_{i_m}}\right]\right\}$$

$$+\sum_{n=2}(-1)^{n}\sum_{z=1}^{n-1}\sum_{\substack{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_{n-z}}\le flr\left(\frac{6k+5}{5}\right)\\ 5\le p_{j_1}<p_{j_2}<\cdots<p_{j_z}\le flr\left(\frac{6k+1}{5}\right)}}\left\{\prod_{m=1}^{n-z}S_{-}^{c}\left((k+1)|p_{i_m}\right)\right\}\left\{\prod_{l=1}^{z}S_{+}^{c}\left(k|p_{j_l}\right)\right\}\times$$

$$\times C_{+}\left(k,\prod_{m=1}^{n-z}p_{i_m},\prod_{j=1}^{z}p_{j_l}\right)\left\{1+flr\left[\frac{|N-6k-5|}{6\left\{\prod_{m=1}^{n-z}p_{i_m}\right\}\left\{\prod_{l=1}^{z}p_{j_l}\right\}}\right]+flr\left[\frac{N+6k+1}{6\left\{\prod_{m=1}^{n-z}p_{i_m}\right\}\left\{\prod_{l=1}^{z}p_{j_l}\right\}}\right]\right\}\Bigg\}\tag{32}$$

For any given finite integer N, the number of prime pairs $\pi_{2,4}(N)$ contained in the finite sets **c**(N) can be accurately calculated by the equation (31) in combination with the equations (3) and (32). As the same as $\pi_{2,2}(N)$, the asymptotic behavior of $\pi_{2,4}(N)$ at $N\rightarrow\infty$ is

$$\pi_{2,4}(N\rightarrow\infty)=\lim_{N\rightarrow\infty}\left\{\pi(N)-\frac{N}{6}\left(1+\sum_{n=2}^{\infty}(-1)^{n+1}\sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}}\frac{2}{p_{k_1}p_{k_2}\cdots p_{k_n}}\right)\right\}\tag{33}$$

$$\pi_{2,4}(N\rightarrow\infty)\approx\lim_{N\rightarrow\infty}\frac{N}{(lnN)^{2}}\tag{34}$$

When the integer N is finite, the number of prime pairs $\pi_{2,2}(N)$ and $\pi_{2,4}(N)$ are accurately calculated by the equations (25), (3) and (26) and by the equations (31), (3) and (32), respectively. When N is infinite, the asymptotic equations of $\pi_{2,2}(N)$ and $\pi_{2,4}(N)$ are the same as described by the equations (29) and (34). From the asymptotic equations of $\pi_{2,2}(N)$ and $\pi_{2,4}(N)$, it can be concluded that there are infinite number of prime pairs with the gap of $\Delta$=2 and $\Delta$=4.

The prime pair counting function $P_x(1, 1)$ can be discussed in three separate cases: $x$=(6N-2), $x$=6N and $x$=(6N+2). The original sieve method used in deriving π(N), $\pi_-$(N), $\pi_+$(N), $\pi_{2,2}$(N) and $\pi_{2,4}$(N) can be directly employed to derive the equations of $P_x(1, 1)$. The explicit formulas for $P_{6N-2}(1, 1)$, $P_{6N+2}(1, 1)$ and $P_{6N}(1, 1)$ are summarized below.

$$P_{6N-2}(1,1) = ceil\left(\frac{N-1}{2}\right) - \chi_-(6N-2) + \chi_{6N-2}(1,1) + S_+(N-1) \tag{35}$$

$$\chi_{6N-2}(1,1) = \sum_{k=1} \Bigg\{ S_-(k)S_-(N-k)flr\left\{\frac{\left|6ceil\left(\frac{N-1}{2}\right)-6k\right|}{6(6k-1)[6(N-k)-1]}\right\}$$

$$+S_+(k)S_-(N+k)flr\left\{\frac{6ceil\left(\frac{N-1}{2}\right)+6k}{6(6k+1)[6(N+k)-1]}\right\}$$

$$+\sum_{n=1}(-1)^{n+1}\left\{\sum_{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_n}}^{flr\left(\frac{6(N-k)-1}{5}\right)} S_-(k)\left\{\prod_{m=1}^{n} S_-^c\big((N-k)|p_{i_m}\big)\right\} flr\left[\frac{\left|6ceil\left(\frac{N-1}{2}\right)-6k\right|}{6(6k-1)^{1-c_k^-}\prod_{m=1}^{n} p_{i_m}}\right]\right.$$

$$+\sum_{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_n}\le flr\left(\frac{6k-1}{5}\right)} \left\{\prod_{m=1}^{n} S_-^c\big(k|p_{i_m}\big)\right\} S_-(N-k)\times flr\left[\frac{\left|6ceil\left(\frac{N-1}{2}\right)-6k\right|}{6(6(N-k)-1)^{1-c_{N-k}^-}\prod_{m=1}^{n} p_{i_m}}\right]$$

$$+\sum_{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_n}\le flr\left(\frac{6(N+k)-1}{5}\right)} S_+(k)\left\{\prod_{m=1}^{n} S_-^c\big((N+k)|p_{i_m}\big)\right\} flr\left[\frac{6ceil\left(\frac{N-1}{2}\right)+6k}{6(6k+1)^{1-c_k^+}\prod_{m=1}^{n} p_{i_m}}\right]$$

$$\left.+\sum_{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_n}\le flr\left(\frac{6k+1}{5}\right)} \left\{\prod_{m=1}^{n} S_+^c\big(k|p_{i_m}\big)\right\} S_-(N+k)\,flr\left[\frac{6ceil\left(\frac{N-1}{2}\right)+6k}{6(6(N+k)-1)^{1-c_{N+k}^-}\prod_{m=1}^{n} p_{i_m}}\right]\right\}\Bigg\}$$

$$+\sum_{n=2}\sum_{z=1}^{n-1}(-1)^n\left\{\sum_{\substack{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_{n-z}}\le flr\left(\frac{6k-1}{5}\right)\\ 5\le p_{j_1}<p_{j_2}<\cdots<p_{j_z}\le flr\left(\frac{6(N-k)-1}{5}\right)}} C_-\left(k,\prod_{m=1}^{n-z}p_{i_m},\prod_{j=1}^{z}p_{j_l}\right)\left\{\prod_{m=1}^{n-z}S_-^c(k|p_{i_m})\right\}\right.$$

$$\times\left\{\prod_{j=1}^{z}S_-^c((N-k)|p_{j_l})\right\}\left\{1+flr\left[\frac{\left|6ceil\left(\frac{N-1}{2}\right)-6k\right|}{6\left\{\prod_{m=1}^{n-z}p_{i_m}^{1-c_m^0}\right\}\left\{\prod_{l=1}^{z}p_{j_l}\right\}}\right]\right\}$$

$$+\sum_{\substack{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_{n-z}}\le flr\left(\frac{6k+1}{5}\right)\\ 5\le p_{j_1}<p_{j_2}<\cdots<p_{j_z}\le flr\left(\frac{6(N+k)-1}{5}\right)}} C_-\left(k,\prod_{m=1}^{n-z}p_{i_m},\prod_{j=1}^{z}p_{j_l}\right)\left\{\prod_{m=1}^{n-z}S_+^c(k|p_{i_m})\right\}$$

$$\left.\times\left\{\prod_{j=1}^{z}S_-^c((N+k)|p_{j_l})\right\}flr\left[\frac{6ceil\left(\frac{N-1}{2}\right)+6k}{6\left\{\prod_{m=1}^{n-z}p_{i_m}^{1-c_m^0}\right\}\left\{\prod_{l=1}^{z}p_{j_l}\right\}}\right]\right\} \tag{36}$$

$$P_{6N+2}(1,1)=ceil\left(\frac{N-1}{2}\right)-\chi_+(6N+2)+\chi_{6N+2}(1,1)+S_-(N) \tag{37}$$

$$\chi_{6N+2}(1,1)=\sum_{k=1}\left\{S_+(k)S_+(N-k)flr\left\{\frac{\left|6ceil\left(\frac{N-1}{2}\right)-6k\right|}{6(6k+1)[6(N-k)+1]}\right\}\right.$$

$$+S_-(k)S_+(N+k)flr\left\{\frac{6ceil\left(\frac{N-1}{2}\right)+6k}{6(6k-1)[6(N+k)+1]}\right\}$$

$$\sum_{n=1}(-1)^{n+1}\left\{\sum_{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_n}}^{flr\left(\frac{6(N-k)+1}{5}\right)} S_+(k)\left\{\prod_{m=1}^{n} S_+^c\left((N-k)|p_{i_m}\right)\right\} flr\left[\frac{\left|6ceil\left(\frac{N-1}{2}\right)-6k\right|}{6(6k+1)^{1-c_k^+}\prod_{m=1}^{n} p_{i_m}}\right]\right.$$

$$+\sum_{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_n}\le flr\left(\frac{6k+1}{5}\right)}\left\{\prod_{m=1}^{n} S_+^c\left(k|p_{i_m}\right)\right\} S_+(N-k) flr\left[\frac{\left|6ceil\left(\frac{N-1}{2}\right)-6k\right|}{6(6(N-k)+1)^{1-c_{N-k}^+}\prod_{m=1}^{n} p_{i_m}}\right]$$

$$+\sum_{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_n}\le flr\left(\frac{6(N+k)+1}{5}\right)} S_-(k)\left\{\prod_{m=1}^{n} S_+^c\left((N+k)|p_{i_m}\right)\right\} flr\left[\frac{6ceil\left(\frac{N-1}{2}\right)+6k}{6(6k-1)^{1-c_k^-}\prod_{m=1}^{n} p_{i_m}}\right]$$

$$\left.+\sum_{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_n}\le flr\left(\frac{6k-1}{5}\right)}\left\{\prod_{m=1}^{n} S_-^c\left(k|p_{i_m}\right)\right\} S_+(N+k)\, flr\left[\frac{6ceil\left(\frac{N-1}{2}\right)+6k}{6(6(N+k)+1)^{1-c_{N+k}^+}\prod_{m=1}^{n} p_{i_m}}\right]\right\}$$

$$+\sum_{n=2}\sum_{z=1}^{n-1}(-1)^n\left\{\sum_{\substack{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_{n-z}}\le flr\left(\frac{6k+1}{5}\right)\\ 5\le p_{j_1}<p_{j_2}<\cdots<p_{j_z}\le flr\left(\frac{6(N-k)+1}{5}\right)}} C_+\left(k,\prod_{m=1}^{n-z} p_{i_m},\prod_{j=1}^{z} p_{j_l}\right)\left\{\prod_{m=1}^{n-z} S_+^c\left(k|p_{i_m}\right)\right\}\right.$$

$$\times\left\{\prod_{j=1}^{z} S_+^c\left((N-k)|p_{j_l}\right)\right\}\left\{1+flr\left[\frac{\left|6ceil\left(\frac{N-1}{2}\right)-6k\right|}{6\left\{\prod_{m=1}^{n-z} p_{i_m}^{1-c_m^0}\right\}\left\{\prod_{l=1}^{z} p_{j_l}\right\}}\right]\right\}$$

$$+\sum_{\substack{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_{n-z}}\le flr\left(\frac{6k-1}{5}\right)\\ 5\le p_{j_1}<p_{j_2}<\cdots<p_{j_z}\le flr\left(\frac{6(N+k)+1}{5}\right)}} C_+\left(k,\prod_{m=1}^{n-z} p_{i_m},\prod_{j=1}^{z} p_{j_l}\right)\left\{\prod_{m=1}^{n-z} S_-^c\left(k|p_{i_m}\right)\right\}$$

$$\left.\times\left\{\prod_{j=1}^{z} S_+^c\left((N+k)|p_{j_l}\right)\right\} flr\left[\frac{6ceil\left(\frac{N-1}{2}\right)+6k}{6\left\{\prod_{m=1}^{n-z} p_{i_m}^{1-c_m^0}\right\}\left\{\prod_{l=1}^{z} p_{j_l}\right\}}\right]\right\}\right\} \quad (38)$$

$$P_{6N}(1,1) = (N-1) - \chi(6N) + \chi_{6N}(1,1) + \delta(N-1) \tag{39}$$

$$\chi_{6N}(1,1) = \sum_{k=1} \left\{ S_-(k)S_+(N-k)flr\left\{\frac{\left|6ceil\left(\frac{N-1}{2}\right) - 6k + 1\right|}{6(6k-1)[6(N-k)+1]}\right\} \right.$$

$$+S_+(k)S_+(N+k)flr\left\{\frac{6ceil\left(\frac{N-1}{2}\right) + 6k + 1}{6(6k+1)[6(N+k)+1]}\right\} + S_+(k)S_-(N-k)flr\left\{\frac{\left|6ceil\left(\frac{N-1}{2}\right) - 6k - 1\right|}{6(6k+1)[6(N-k)-1]}\right\}$$

$$+S_-(k)S_-(N+k)flr\left\{\frac{6ceil\left(\frac{N-1}{2}\right) + 6k - 1}{6(6k-1)[6(N+k)-1]}\right\}$$

$$\sum_{n=1}(-1)^{n+1}\left\{\sum_{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_n}}^{flr\left(\frac{6(N-k)+1}{5}\right)} S_-(k)\left\{\prod_{m=1}^{n} S_+^c\left((N-k)|p_{i_m}\right)\right\} flr\left[\frac{\left|6ceil\left(\frac{N-1}{2}\right) - 6k + 1\right|}{6(6k-1)^{1-c_k^-}\prod_{m=1}^{n} p_{i_m}}\right]\right.$$

$$+\sum_{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_n}\le flr\left(\frac{6k-1}{5}\right)} \left\{\prod_{m=1}^{n} S_-^c\left(k|p_{i_m}\right)\right\} S_+(N-k) flr\left[\frac{\left|6ceil\left(\frac{N-1}{2}\right) - 6k + 1\right|}{6(6(N-k)+1)^{1-c_{N-k}^+}\prod_{m=1}^{n} p_{i_m}}\right]$$

$$+\sum_{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_n}\le flr\left(\frac{6(N+k)+1}{5}\right)} S_+(k)\left\{\prod_{m=1}^{n} S_+^c\left((N+k)|p_{i_m}\right)\right\} flr\left[\frac{6ceil\left(\frac{N-1}{2}\right) + 6k + 1}{6(6k+1)^{1-c_k^+}\prod_{m=1}^{n} p_{i_m}}\right]$$

$$+\sum_{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_n}\le flr\left(\frac{6k+1}{5}\right)} \left\{\prod_{m=1}^{n} S_+^c\left(k|p_{i_m}\right)\right\} S_+(N+k) flr\left[\frac{6ceil\left(\frac{N-1}{2}\right) + 6k + 1}{6(6(N+k)+1)^{1-c_{N+k}^+}\prod_{m=1}^{n} p_{i_m}}\right]$$

$$+\sum_{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_n}\le flr\left(\frac{6(N-k)-1}{5}\right)} S_+(k)\left\{\prod_{m=1}^{n} S_-^c\left((N-k)|p_{i_m}\right)\right\} flr\left[\frac{\left|6ceil\left(\frac{N-1}{2}\right) - 6k - 1\right|}{6(6k+1)^{1-c_k^+}\prod_{m=1}^{n} p_{i_m}}\right]$$

$$+\sum_{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_n}\le flr\left(\frac{6k+1}{5}\right)}\left\{\prod_{m=1}^{n}S_{+}^{c}(k|p_{i_m})\right\}S_{-}(N-k)flr\left[\frac{\left|6ceil\left(\frac{N-1}{2}\right)-6k-1\right|}{6(6(N-k)-1)^{1-c_{N-k}^{-}}\prod_{m=1}^{n}p_{i_m}}\right]$$

$$+\sum_{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_n}\le flr\left(\frac{6(N+k)-1}{5}\right)}S_{-}(k)\left\{\prod_{m=1}^{n}S_{-}^{c}((N+k)|p_{i_m})\right\}flr\left[\frac{6ceil\left(\frac{N-1}{2}\right)+6k-1}{6(6k-1)^{1-c_{k}^{-}}\prod_{m=1}^{n}p_{i_m}}\right]$$

$$+\sum_{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_n}\le flr\left(\frac{6k-1}{5}\right)}\left\{\prod_{m=1}^{n}S_{-}^{c}(k|p_{i_m})\right\}S_{-}(N+k)\,flr\left[\frac{6ceil\left(\frac{N-1}{2}\right)+6k-1}{6(6(N+k)-1)^{1-c_{N+k}^{-}}\prod_{m=1}^{n}p_{i_m}}\right]\Bigg\}$$

$$\sum_{n=2}\sum_{z=1}^{n-1}(-1)^{n}\left\{\sum_{\substack{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_{n-z}}\le flr\left(\frac{6k-1}{5}\right)\\ 5\le p_{j_1}<p_{j_2}<\cdots<p_{j_z}\le flr\left(\frac{6(N-k)+1}{5}\right)}}C_{-}\left(k,\prod_{m=1}^{n-z}p_{i_m},\prod_{j=1}^{z}p_{j_l}\right)\left\{\prod_{m=1}^{n-z}S_{-}^{c}(k|p_{i_m})\right\}\right.$$

$$\times\left\{\prod_{j=1}^{z}S_{+}^{c}((N-k)|p_{j_l})\right\}\left\{1+flr\left[\frac{\left|6ceil\left(\frac{N-1}{2}\right)-6k+1\right|}{6\left\{\prod_{m=1}^{n-z}p_{i_m}^{1-c_m^0}\right\}\left\{\prod_{l=1}^{z}p_{j_l}\right\}}\right]\right\}$$

$$+\sum_{\substack{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_{n-z}}\le flr\left(\frac{6k+1}{5}\right)\\ 5\le p_{j_1}<p_{j_2}<\cdots<p_{j_z}\le flr\left(\frac{6(N+k)+1}{5}\right)}}C_{-}\left(k,\prod_{m=1}^{n-z}p_{i_m},\prod_{j=1}^{z}p_{j_l}\right)\left\{\prod_{m=1}^{n-z}S_{+}^{c}(k|p_{i_m})\right\}$$

$$\times\left\{\prod_{j=1}^{z}S_{+}^{c}((N+k)|p_{j_l})\right\}flr\left[\frac{6ceil\left(\frac{N-1}{2}\right)+6k+1}{6\left\{\prod_{m=1}^{n-z}p_{i_m}^{1-c_m^0}\right\}\left\{\prod_{l=1}^{z}p_{j_l}\right\}}\right]$$

$$+\sum_{\substack{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_{n-z}}\le flr\left(\frac{6k+1}{5}\right)\\ 5\le p_{j_1}<p_{j_2}<\cdots<p_{j_z}\le flr\left(\frac{6(N-k)-1}{5}\right)}}C_{+}\left(k,\prod_{m=1}^{n-z}p_{i_m},\prod_{j=1}^{z}p_{j_l}\right)\left\{\prod_{m=1}^{n-z}S_{+}^{c}(k|p_{i_m})\right\}$$

$$\times\left\{\prod_{j=1}^{z} S_{-}^{c}\big((N-k)|p_{j_l}\big)\right\}\left\{1+flr\left[\frac{\left|6ceil\left(\frac{N-1}{2}\right)-6k-1\right|}{6\left\{\prod_{m=1}^{n-z} p_{i_m}^{1-c_m^0}\right\}\left\{\prod_{l=1}^{z} p_{j_l}\right\}}\right]\right\}$$

$$+\sum_{\substack{5\le p_{i_1}<p_{i_2}<\cdots<p_{i_{n-z}}\le flr\left(\frac{6k-1}{5}\right)\\ 5\le p_{j_1}<p_{j_2}<\cdots<p_{j_z}\le flr\left(\frac{6(N+k)-1}{5}\right)}} C_{+}\left(k,\prod_{m=1}^{n-z} p_{i_m},\prod_{j=1}^{z} p_{j_l}\right)\left\{\prod_{m=1}^{n-z} S_{-}^{c}\big(k|p_{i_m}\big)\right\}$$

$$\times\left\{\prod_{j=1}^{z} S_{-}^{c}\big((N+k)|p_{j_l}\big)\right\}flr\left[\frac{6ceil\left(\frac{N-1}{2}\right)+6k-1}{6\left\{\prod_{m=1}^{n-z} p_{i_m}^{1-c_m^0}\right\}\left\{\prod_{l=1}^{z} p_{j_l}\right\}}\right]\Bigg\}\Bigg) \qquad (40)$$

When the integer N is finite, the number of prime pairs $P_{6N-2}(1, 1)$, $P_{6N+2}(1, 1)$ and $P_{6N}(1, 1)$ are accurately calculated by the equations (35), (9) and (36), by the equations (37), (13) and (38) and by the equations (39), (5) and (40), respectively.

When N is infinite, no simple asymptotic equation such as the equations (27) and (33) can be obtained for $P_{6N-2}(1, 1)$, $P_{6N+2}(1, 1)$ and $P_{6N}(1, 1)$. But the values (finite or infinite) of $P_{6N-2}(1, 1)$, $P_{6N+2}(1, 1)$ and $P_{6N}(1, 1)$ at $N\to\infty$ do not indicate that Goldbach conjecture holds true or not. As a matter of fact, if using $P_x(1, 1)$ to prove Goldbach conjecture holds true or not, it is to find out either $P_x(1, 1)\ge 1$ for all even integer $x$ or those even integer $x$ satisfying $P_x(1, 1)=0$.

It was proved that[2]

$$\lim_{N\to\infty}\sum_{M=1}^{flr\left(\frac{N-2}{6}\right)}\left\{P_{6M-2}(1,1)+P_{6M}(1,1)+P_{6M+2}(1,1)\right\}=\lim_{N\to\infty}\frac{\pi(N)[\pi(N)+1]}{2} \qquad (41)$$

which is equivalent to the equation (11) in reference [2].

If the number of odd prime integers is finite, that is, the odd prime set $P$ consisting of N primes

$$P=\{p_1,p_2,\cdots,p_N\} \qquad (42)$$

The even integers of which every is the sum of two odd primes from $P$ form a finite set of even integers. It is obvious that

$$\sum_{M=1}^{\infty}\left\{P_{2M+4}(1,1)(N)\right\}=\frac{N(N+1)}{2}$$

which becomes the equation (41) when N is infinite.

From the equation (41), the asymptotic formula of $P_x(1, 1)$ is

$$\begin{cases}\lim\limits_{N\to\infty}\sum\limits_{M=1}^{flr\left(\frac{N-2}{6}\right)}\left\{P_{6M-2}(1,1)+P_{6M}(1,1)+P_{6M+2}(1,1)\right\}\approx\lim\limits_{N\to\infty}\frac{N^2}{(lnN)^2}\\ \lim\limits_{N\to\infty}\left[\frac{1}{N}\times\sum\limits_{M=1}^{flr\left(\frac{N-2}{6}\right)}\left\{P_{6M-2}(1,1)+P_{6M}(1,1)+P_{6M+2}(1,1)\right\}\right]\approx\lim\limits_{N\to\infty}\frac{N}{(lnN)^2}\end{cases}\qquad(43)$$

**References**

[1] K. Ding, “*The Counting Functions of Prime Numbers*”, (non-peer reviewed), DOI: 10.5281/zenodo.7955168, (2023)
[2] K. Ding, “*The Solution of Goldbach Conjecture*”, (non-peer reviewed), DOI: 10.5281/zenodo.7856921, (2023)