The Counting Functions of Prime Numbers

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Abstract: The explicit analytical formula of the prime counting functions π(N) is derived for accurately calculating the number of primes less than or equal to N. The asymptotic behavior of π(N) at N→∞ is analyzed. π(N) is independent on the non-trivial zeroes of Riemann zeta function which were proved to do not exist.

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Riemann hypothesis[1] can be stated as two questions: (1) the distribution of the zeroes of Riemann zeta function ζ(0.5+it) could provide a solution to derive the prime counting function; and (2) the derivation of an explicit analytical formula of the prime counting function π(N) to accurately calculate the number of primes less than or equal to an integer N.

It was proved that Riemann zeta function ζ(0.5+it) has no zeroes[2, 3], therefore, the function ζ(0.5+it) is irrelevant to the prime counting. Here the prime counting function π(N) is derived based on the original sieve method.

If *x* is a non-negative real number, *flr*(*x*) and *ceil*(*x*) are integers that fulfill the following equations, respectively:

$$\begin{cases} 0 \le x - flr(x) < 1 \\ 0 \le ceil(x) - x < 1 \end{cases} \tag{1}$$

flr(*x*) and *ceil*(*x*) are called the floor function and the ceiling function of *x*, respectively.

Perform the original sieve operation on the complete set of all positive integers: eliminate 1, 2, 3, all the multiples of 2 and all the multiples of 3. The integers left form the sets $\mathbf{c}_-$, $\mathbf{c}_+$ and $\mathbf{c}$

$$\begin{cases} \boldsymbol{c}_- = \{(6k-1) \mid k = 1, 2, \cdots, \infty\} \\ \boldsymbol{c}_+ = \{(6k+1) \mid k = 1, 2, \cdots, \infty\} \\ \boldsymbol{c} = \boldsymbol{c}_- U \boldsymbol{c}_+ = \{5, 7, 11, 13, \cdots\} \end{cases} \tag{2}$$

The prime counting function π(N) calculates the number of primes less than or equal to N in the set **c**, while $\pi_-(N)$ and $\pi_+(N)$ count the number of primes less than or equal to N in the sets $\mathbf{c}_-$ and

c$_+$ respectively. This means that 2, the only even prime, and 3, the first odd prime, are excluded from the prime counting functions π(N), π_-(N) and π_+(N).

In fact, for a given finite integer N, the prime counting functions π(N), π_-(N) and π_+(N) count the number of primes contained in the finite sets **c**(N), **c**$_-$(N) and **c**$_+$(N), respectively

$$\begin{cases} \boldsymbol{c}_-(N) = \left\{(6k-1) \mid k = 1, 2, \cdots, flr\left(\frac{N+1}{6}\right)\right\} \\ \boldsymbol{c}_+(N) = \left\{(6k+1) \mid k = 1, 2, \cdots, flr\left(\frac{N-1}{6}\right)\right\} \\ \boldsymbol{c}(N) = \boldsymbol{c}_-(N) U \boldsymbol{c}_+(N) \end{cases} \tag{3}$$

The total number of elements in the sets **c**(N), **c**$_-$(N) and **c**$_+$(N) are

$$\begin{cases} |\boldsymbol{c}_-(N)| = flr\left(\frac{N+1}{6}\right) \\ |\boldsymbol{c}_+(N)| = flr\left(\frac{N-1}{6}\right) \\ |\boldsymbol{c}(N)| = flr\left(\frac{N-1}{6}\right) + flr\left(\frac{N+1}{6}\right) \end{cases} \tag{4}$$

To find out whether a given integer n∈**c**$_-$(N) or n∈**c**$_+$(N), the sieve functions $C_-(n)$ and $C_+(n)$ are defined in the equations (5) and (6).

$$C_-(n) = 1 + flr\left(\frac{n+1}{6}\right) - ceil\left(\frac{n+1}{6}\right) \tag{5}$$

$$C_+(n) = 1 + flr\left(\frac{n-1}{6}\right) - ceil\left(\frac{n-1}{6}\right) \tag{6}$$

$$\begin{cases} C_-(n) = \begin{cases} 1, & if\ n \in \boldsymbol{c}_-(N) \\ 0, & if\ n \notin \boldsymbol{c}_-(N) \end{cases} \\ C_+(n) = \begin{cases} 1, & if\ n \in \boldsymbol{c}_+(N) \\ 0, & if\ n \notin \boldsymbol{c}_+(N) \end{cases} \end{cases}$$

Let k be a positive integer variable, and the sieve functions $S_-(k)$ and $S_+(k)$ are defined by

$$S_-(k) = \prod_{i=1}^{flr\left(\frac{k+1}{5}\right)} \left\{1 - \sum_{j=1}^{ceil\left(\frac{6k-1}{6i-1}\right)} \delta(6k-1-6ij+j)\right\} \times$$

$$\times \left\{1 - \sum_{j=1}^{ceil\left(\frac{6k-1}{6i+1}\right)} \delta(6k-1-6ij-j)\right\} \tag{7}$$

$$S_+(k) = \prod_{i=1}^{flr\left(\frac{k+1}{5}\right)} \left\{1 - \sum_{j=1}^{ceil\left(\frac{6k+1}{6i-1}\right)} \delta(6k+1-6ij+j)\right\} \times$$

$$\times \left\{1 - \sum_{j=1}^{ceil\left(\frac{6k+1}{6i+1}\right)} \delta(6k+1-6ij-j)\right\} \tag{8}$$

The $\delta(x)$ function used in the equations (7) and (8) and hereafter is usually defined as

$$\delta(x) = \begin{cases} 1 \ \ if \ x = 0 \\ 0 \ \ if \ x \neq 0 \end{cases}$$

According to the definitions (7) and (8), it is obvious that

$$\begin{cases} S_-(k) = \begin{cases} 1 \ \ if \ (6k-1) is \ a \ prime \\ 0 \ \ if \ (6k-1) is \ a \ composite \end{cases} \\ S_+(k) = \begin{cases} 1 \ if \ (6k+1) is \ a \ prime \\ 0 \ \ if \ (6k+1) is \ a \ composite \end{cases} \end{cases}$$

The original sieve method is an iterative process of eliminating the multiples of the known primes in three finite sets **c**(N), $\mathbf{c}_-$(N) and $\mathbf{c}_+$(N). The mathematical description of the sieve operations can be established by counting the numbers of the eliminated composites. The total number of primes π(N), π_-(N) and π_+(N) can be obtained by subtracting the total number of the eliminated composites from the total number of elements in the three sets **c**(N), $\mathbf{c}_-$(N) and $\mathbf{c}_+$(N).

Using the above definitions of the sieve functions $S_-(k)$, $S_+(k)$, $C_-(n)$ and $C_+(n)$, the explicit analytical formulas for the prime counting functions π(N), π_-(N) and π_+(N) are expressed by

$$\pi(N) = flr\left(\frac{N-1}{6}\right) + flr\left(\frac{N+1}{6}\right)$$

$$-\sum_{k^-=1} S_-(k^-)\left\{flr\left[\frac{|N-6k^-+1|}{6(6k^--1)}\right] + flr\left[\frac{N+6k^--1}{6(6k^--1)}\right]\right\}$$

$$-\sum_{k^+=1} S_+(k^+)\left\{flr\left[\frac{|N-6k^+-1|}{6(6k^++1)}\right] + flr\left[\frac{N+6k^++1}{6(6k^++1)}\right]\right\}$$

$$+\sum_{n=2}^{\infty}(-1)^n \sum_{m=0}^{n} \sum_{\substack{1\le k_1^-<k_2^-<\cdots<k_{n-m}^- \\ 1\le k_1^+<k_2^+<\cdots<k_m^+}} \left\{\prod_{i=1}^{n-m} S_-(k_i^-) \prod_{j=1}^{m} S_+\left(k_j^+\right)\right\} \times$$

$$\times\left\{flr\left(\frac{N + \prod_{i=1}^{n-m}(6k_i^- - 1)\prod_{j=1}^{m}\left(6k_j^+ + 1\right)}{6\prod_{i=1}^{n-m}(6k_i^- - 1)\prod_{j=1}^{m}\left(6k_j^+ + 1\right)}\right) + flr\left(\frac{N + 5\prod_{i=1}^{n-m}(6k_i^- - 1)\prod_{j=1}^{m}\left(6k_j^+ + 1\right)}{6\prod_{i=1}^{n-m}(6k_i^- - 1)\prod_{j=1}^{m}\left(6k_j^+ + 1\right)}\right)\right\}$$

(9)

$$\pi_-(N) = flr\left(\frac{N+1}{6}\right) - \sum_{k^-=1} S_-(k^-) flr\left\{\frac{|N-6k^-+1|}{6(6k^--1)}\right\} - \sum_{k^+=1} S_+(k^+) flr\left\{\frac{N+6k^++1}{6(6k^++1)}\right\}$$

$$+\sum_{n=2}^{\infty}(-1)^n \sum_{m=0}^{n} \sum_{\substack{1\le k_1^-<k_2^-<\cdots<k_{n-m}^- \\ 1\le k_1^+<k_2^+<\cdots<k_m^+}} \left\{\prod_{i=1}^{n-m} S_-(k_i^-) \prod_{j=1}^{m} S_+\left(k_j^+\right)\right\} \times$$

$$\times\left\{C_-\left[\prod_{i=1}^{n-m}(6k_i^- - 1)\prod_{j=1}^{m}\left(6k_j^+ + 1\right)\right]flr\left(\frac{N+5\prod_{i=1}^{n-m}(6k_i^- - 1)\prod_{j=1}^{m}\left(6k_j^+ + 1\right)}{6\prod_{i=1}^{n-m}(6k_i^- - 1)\prod_{j=1}^{m}\left(6k_j^+ + 1\right)}\right)+\right.$$

$$\left.+C_+\left[\prod_{i=1}^{n-m}(6k_i^- - 1)\prod_{j=1}^{m}\left(6k_j^+ + 1\right)\right]flr\left(\frac{N+\prod_{i=1}^{n-m}(6k_i^- - 1)\prod_{j=1}^{m}\left(6k_j^+ + 1\right)}{6\prod_{i=1}^{n-m}(6k_i^- - 1)\prod_{j=1}^{m}\left(6k_j^+ + 1\right)}\right)\right\} \tag{10}$$

$$\pi_+(N) = flr\left(\frac{N-1}{6}\right) - \sum_{k^-=1} S_-(k^-)flr\left\{\frac{N+6k^- - 1}{6(6k^- - 1)}\right\} - \sum_{k^+=1} S_+(k^+)flr\left\{\frac{|N-6k^+ - 1|}{6(6k^+ + 1)}\right\}$$

$$+\sum_{n=2}^{\infty}(-1)^n\sum_{m=0}^{n}\sum_{\substack{1\le k_1^-<k_2^-<\cdots<k_{n-m}^- \\ 1\le k_1^+<k_2^+<\cdots<k_m^+}}\left\{\prod_{i=1}^{n-m}S_-(k_i^-)\prod_{j=1}^{m}S_+\left(k_j^+\right)\right\}\times$$

$$\times\left\{C_-\left[\prod_{i=1}^{n-m}(6k_i^- - 1)\prod_{j=1}^{m}\left(6k_j^+ + 1\right)\right]flr\left(\frac{N+\prod_{i=1}^{n-m}(6k_i^- - 1)\prod_{j=1}^{m}\left(6k_j^+ + 1\right)}{6\prod_{i=1}^{n-m}(6k_i^- - 1)\prod_{j=1}^{m}\left(6k_j^+ + 1\right)}\right)+\right.$$

$$\left.+C_+\left[\prod_{i=1}^{n-m}(6k_i^- - 1)\prod_{j=1}^{m}\left(6k_j^+ + 1\right)\right]flr\left(\frac{N+5\prod_{i=1}^{n-m}(6k_i^- - 1)\prod_{j=1}^{m}\left(6k_j^+ + 1\right)}{6\prod_{i=1}^{n-m}(6k_i^- - 1)\prod_{j=1}^{m}\left(6k_j^+ + 1\right)}\right)\right\} \tag{11}$$

For any given finite integer N, the number of primes π(N), π_-(N) and π_+(N) contained in the finite sets **c**(N), **c**$_-$(N) and **c**$_+$(N) can be accurately calculated by the equations (9), (10) and (11), respectively. The summations in the equations (9-11) are only over a finite number of terms, for all the k^- and all the k^+ only need to run in finite ranges as shown below:

$$\begin{cases} k^- = 1, 2, \cdots, flr\left(\dfrac{N+5}{30}\right) \\ k^+ = 1, 2, \cdots, flr\left(\dfrac{N-5}{30}\right) \end{cases}$$

Beyond the above ranges, the terms in the summations will be zero. Because of the definitions of the sieve functions in the equations (7) and (8) the non-zero terms in the summations are those terms only if all the (6k -1) and all the (6k^++1) in the product are primes.

If expressed using primes, the equations (7-9) can be simplified into (12-14) respectively

$$S_-(k) = \sum_{p_i=5}^{flr\left(\frac{6k-1}{5}\right)} \left\{ 1 - \sum_{l=1}^{ceil\left(\frac{6k-1}{p_i}\right)} \delta(6k-1-lp_i) \right\} \tag{12}$$

$$S_+(k) = \sum_{p_i=5}^{flr\left(\frac{6k+1}{5}\right)} \left\{ 1 - \sum_{l=1}^{ceil\left(\frac{6k+1}{p_i}\right)} \delta(6k+1-lp_i) \right\} \tag{13}$$

$$\pi(N) = \left\{ flr\left(\frac{N-1}{6}\right) + flr\left(\frac{N+1}{6}\right) \right\} - \left\{ \sum_{5 \le p_i} \left\{ flr\left(\frac{|N-p_i|}{6p_i}\right) + flr\left(\frac{N+p_i}{6p_i}\right) \right\} + \right.$$
$$\left. + \sum_{n=2}^{\infty} (-1)^{n-1} \sum_{5 \le p_{k_1} < p_{k_2} < \cdots < p_{k_n}} \left\{ flr\left(\frac{N+5p_{k_1}p_{k_2}\cdots p_{k_n}}{6p_{k_1}p_{k_2}\cdots p_{k_n}}\right) + flr\left(\frac{N+p_{k_1}p_{k_2}\cdots p_{k_n}}{6p_{k_1}p_{k_2}\cdots p_{k_n}}\right) \right\} \right\} \tag{14}$$

In the equation (14) the summations over primes only need to run in the region of

$$\left[5, flr\left(\frac{N}{5}\right)\right] \tag{15}$$

Beyond the region, the terms in the summations will be zero.

Let τ(N) denote the cardinality of the set **c**. τ(N) is the total number of integers contained in the finite set **c**, which can be accurately calculated by

$$\tau(N) = |\boldsymbol{c}| = flr\left(\frac{N-1}{6}\right) + flr\left(\frac{N+1}{6}\right) \tag{16}$$

If χ(N) is the total number of composites contained in the finite set **c** defined in the equation (2),

$$\pi(N) = \tau(N) - \chi(N) \tag{17}$$

For an individual prime p_i, the total number of composites consisting of p_i in the prime factors is

$$flr\left(\frac{|N-p_i|}{6p_i}\right) + flr\left(\frac{N+p_i}{6p_i}\right) \tag{18}$$

χ(N) can be calculated by adding together all the values of equation (18) for all the primes in the region of equation (15). But with respect to two specific primes $p_i < p_j$, the composites consisting of both p_i and p_j in the prime factors are counted for two times and the total number of those twice-counted composites is

$$flr\left(\frac{N+5p_ip_j}{6p_ip_j}\right) + flr\left(\frac{N+p_ip_j}{6p_ip_j}\right) \tag{19}$$

The correction can be made by subtracting all the values of equation (19) for all the possible two-prime combinations in the region of equation (15). Through continuing on by alternatively adding and subtracting the values arisen from the combinations of all odd and all even number of primes in the region of equation (15), an explicit analytical expression of χ(N) can be obtained

$$\chi(N) = \sum_{5\le p_i}\left\{flr\left(\frac{|N-p_i|}{6p_i}\right) + flr\left(\frac{N+p_i}{6p_i}\right)\right\} +$$

$$+\sum_{n=2}^{\infty}(-1)^{n-1}\sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}}\left\{flr\left(\frac{N+5p_{k_1}p_{k_2}\cdots p_{k_n}}{6p_{k_1}p_{k_2}\cdots p_{k_n}}\right) + flr\left(\frac{N+p_{k_1}p_{k_2}\cdots p_{k_n}}{6p_{k_1}p_{k_2}\cdots p_{k_n}}\right)\right\} \tag{20}$$

which is the accurate equation of the total number of composites contained in the finite set **c**. Substituting the equations (16) and (20) into the equation (17) simply proves the equation (14).

When N is infinite, the floor operation in the equation (14) can be replaced by the simple divide. Then the asymptotic behavior of π(N) at N→∞ is

$$\pi(N \to \infty) \approx \lim_{N\to\infty}\left\{\frac{N}{3} - \frac{N}{3}\sum_{5\le p_i}\frac{1}{p_i} - \frac{N}{3}\sum_{n=2}^{\infty}(-1)^{n-1}\sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}}\frac{1}{p_{k_1}p_{k_2}\cdots p_{k_n}}\right\}$$

$$\pi(N \to \infty) \approx \lim_{N\to\infty}\left\{\frac{N}{3}\prod_{p_i\ge 5}\left(1-\frac{1}{p_i}\right)\right\} = \lim_{N\to\infty}\left\{N\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\prod_{p_i\ge 5}\left(1-\frac{1}{p_i}\right)\right\}$$

$$= \lim_{N\to\infty}\left\{\frac{N}{\zeta(1)}\right\} \approx \lim_{N\to\infty}\frac{N}{\gamma + lnN} \tag{21}$$

where ζ(1) is the value of Riemann zeta function at s=1 and γ is the Euler constant.

$$\pi(N \to \infty) \approx \lim_{N\to\infty}\frac{N}{lnN} \tag{22}$$

Similarly, the equations (10) and (11) can be also simplified using primes in the equations as the followings:

$$\pi_-(N) = \tau_-(N) - \chi_-(N) \tag{23}$$

$$\pi_-(N) = flr\left(\frac{N+1}{6}\right) - \left\{\sum_{5\le p_i}\left\{C_-(p_i)flr\left(\frac{|N-p_i|}{6p_i}\right) + C_+(p_i)flr\left(\frac{N+p_i}{6p_i}\right)\right\} + \right.$$

$$+\sum_{n=2}^{\infty}(-1)^{n-1}\sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}}\left\{C_-(p_{k_1}p_{k_2}\cdots p_{k_n})flr\left(\frac{N+5p_{k_1}p_{k_2}\cdots p_{k_n}}{6p_{k_1}p_{k_2}\cdots p_{k_n}}\right) + \right.$$

$$\left.\left. +C_+(p_{k_1}p_{k_2}\cdots p_{k_n})flr\left(\frac{N+p_{k_1}p_{k_2}\cdots p_{k_n}}{6p_{k_1}p_{k_2}\cdots p_{k_n}}\right)\right\}\right\} \tag{24}$$

$$\tau_-(N) = flr\left(\frac{N+1}{6}\right) \tag{25}$$

$$\chi_-(N) = \sum_{5\le p_i}\left\{C_-(p_i)flr\left(\frac{|N-p_i|}{6p_i}\right) + C_+(p_i)flr\left(\frac{N+p_i}{6p_i}\right)\right\}$$

$$+\sum_{n=2}^{\infty}(-1)^{n-1}\sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}}\left\{C_-\left(p_{k_1}p_{k_2}\cdots p_{k_n}\right)flr\left(\frac{N+5p_{k_1}p_{k_2}\cdots p_{k_n}}{6p_{k_1}p_{k_2}\cdots p_{k_n}}\right)\right.$$

$$\left.+C_+\left(p_{k_1}p_{k_2}\cdots p_{k_n}\right)flr\left(\frac{N+p_{k_1}p_{k_2}\cdots p_{k_n}}{6p_{k_1}p_{k_2}\cdots p_{k_n}}\right)\right\} \quad (26)$$

$$\pi_+(N) = \tau_+(N) - \chi_+(N) \quad (27)$$

$$\pi_+(N) = flr\left(\frac{N-1}{6}\right) - \left\{\sum_{5\le p_i}\left\{C_+(p_i)flr\left(\frac{|N-p_i|}{6p_i}\right) + C_-(p_i)flr\left(\frac{N+p_i}{6p_i}\right)\right\} + \right.$$

$$+\sum_{n=2}^{\infty}(-1)^{n-1}\sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}}\left\{C_+\left(p_{k_1}p_{k_2}\cdots p_{k_n}\right)flr\left(\frac{N+5p_{k_1}p_{k_2}\cdots p_{k_n}}{6p_{k_1}p_{k_2}\cdots p_{k_n}}\right) + \right.$$

$$\left.\left.+C_-(p_{k_1}p_{k_2}\cdots p_{k_n})flr\left(\frac{N+p_{k_1}p_{k_2}\cdots p_{k_n}}{6p_{k_1}p_{k_2}\cdots p_{k_n}}\right)\right\}\right\} \quad (28)$$

$$\tau_+(N) = flr\left(\frac{N-1}{6}\right) \quad (29)$$

$$\chi_+(N) = \sum_{5\le p_i}\left\{C_+(p_i)flr\left(\frac{|N-p_i|}{6p_i}\right) + C_-(p_i)flr\left(\frac{N+p_i}{6p_i}\right)\right\}$$

$$+\sum_{n=2}^{\infty}(-1)^{n-1}\sum_{5\le p_{k_1}<p_{k_2}<\cdots<p_{k_n}}\left\{C_+\left(p_{k_1}p_{k_2}\cdots p_{k_n}\right)flr\left(\frac{N+5p_{k_1}p_{k_2}\cdots p_{k_n}}{6p_{k_1}p_{k_2}\cdots p_{k_n}}\right)\right.$$

$$+C_{-}(p_{k_1}p_{k_2}\cdots p_{k_n})flr\left(\frac{N+p_{k_1}p_{k_2}\cdots p_{k_n}}{6p_{k_1}p_{k_2}\cdots p_{k_n}}\right)\Bigg\} \tag{30}$$

The equations (14), (24) and (28) are equivalent to the equations (9), (10) and (11), respectively. For any given finite integer N, the number π(N), π_-(N) and π_+(N) of primes less than or equal to N can be accurately calculated through the equations (9), (10) and (11) or through the equations (14), (24) and (28). When N is infinite, the asymptotic behavior of π(N), π_-(N) and π_+(N) at N→∞ is described by the equations (22) and (31)

$$\lim_{N\to\infty}\pi_-(N)=\lim_{N\to\infty}\pi_+(N)=\lim_{N\to\infty}\pi(N)=\lim_{N\to\infty}\frac{N}{lnN} \tag{31}$$

References

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