A Study On Split-Complex Vector Spaces

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Abstract:

The objective of this paper is to define and study split-complex vector spaces as modules over the ring of split-complex numbers. Where we study the elementary properties of this new algebraic class in terms of theorems, and we present many examples to clarify the validity of our work.

Keywords: split-complex vector space, split-complex number, split-complex basis

Introduction

The concept of split-complex numbers (hyperbolic numbers) is considered as a generalization of real numbers. These numbers have many applications in mathematics and physics [1-2,4], and they have a similar algebraic structure to the neutrosophic numbers [3]. From this point of view, we study the structure of vector spaces defined over split-complex numbers, where we find that they are modules in the algebraic meaning, with a strict basis can be found from the classical basis of the vector space V.

First, we recall some elementary definitions.

Definition

$R_D = \{a + b\varepsilon; a, b \in R, \varepsilon^2 = 1\}$ the ring of hyperbolic numbers. It can be understood as an extension of the real field R.

Main discussion

Definition.

Let $K = \{a + bJ;\ J^2 = 1; a, b \in R\}$ be the ring of split-complex numbers.

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Let V be a vector space over R, we define the split-complex space: $V_K = V + VJ = \{X + YJ; X, Y \in V\}$.

Definition.

Let $X = x_1 + x_2 J, Y = y_1 + y_2 J, A = a_1 + a_2 J \in K$ and $X, Y \in V_K$, we define:

$X + Y = (x_1 + y_1) + (x_2 + y_2)J$

$A.X = a_1 x_1 + a_2 x_2 + J(a_1 x_2 + a_2 x_1)$.

Remark.

The operations have the following properties:

Property 1.

$X + Y = Y + X, X + 0 = X, X + (-X) = 0$, and $X + (Y + Z) = (X + Y) + Z$.

Property 2.

$(A + B).X = A.X + B.X, A.(Y + X) = A.Y + A.X, (A.B).X = A.(B.X)$, and $1.X = X$.

This implies that $(V_K, +, .)$ Is a module over the ring K.

Definition.

Let V_K be a split-complex space over K, W_K be a nonempty subset of V_K, we say that W_K is a subspace if and only if $X + Y \in W_K, A.X \in W_K;\ \forall X, Y \in W_K, A \in K$.

Definition.

Let V_1, V_2 be two subspace of V, we define the split-complex AH-subspace of V_K as follows:

$T_K = V_1 + V_2 J = \{x + yJ; x \in V_1, y \in V_2\}$

Example.

Take the vector space $V = R^4$ over R we have:

$V_1 = \langle(1,0,0,0)\rangle = \{(x, 0,0,0); x \in R\}, V_2 = \langle(0,1,0,0)\rangle = \{(0, y, 0,0); y \in R\}$ are two subspace of V.

$T_K = V_1 + V_2 J = \{(x, 0,0,0) + (0, y, 0,0)J; x, y \in R\}$ is an AH-subspace.

Theorem.

Let $T_K = V_1 + V_2 J$ be an AH-subspace of V_K, then T_K is a subspace if and only if $V_1 = V_2$.

Proof.

Assume that T_K is a subspace, then:

For $A = a_1 + a_2 J \in K, X = x + yJ \in T_K$, we get:

$A.X = a_1 x + a_2 y + J(a_1 y + a_2 x) \in T_K$, thus:

$$\begin{cases} a_1 x + a_2 y \in V_1 \\ a_1 y + a_2 x \in V_2 \end{cases} \Rightarrow \begin{cases} a_2 y \in V_1 \\ a_2 x \in V_2 \end{cases} \Rightarrow \begin{cases} V_1 \subseteq V_2 \\ V_2 \subseteq V_1 \end{cases} \Rightarrow \{V_1 = V_2$$

The convers is clear.

Definition.

Let $L: V_K \to W_K$ be a mapping between to two split-complex spaces, we say that L is a split-complex linear transformation if:

$L(X + Y) = L(X) + L(Y), L(A.X) = A.L(X)$ for all $X, Y \in V_K, A \in K$.

Now, we will present an algebraic to build a split-complex linear transformation.

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Theorem.

Let $f: V \to W$ be a classical linear transformation between V and W. Let V_K, W_K be the corresponding split-complex spaces of V and W over K, then $L: V_K \to W_K$; $L(X + YJ) = f(X) + f(Y)J$ is a split-complex linear transformation.

Proof.

Take $X = x_1 + x_2J, Y = y_1 + y_2J \in V_K$, $A = a_1 + a_2J \ \in K$, we have:

$L(X + YJ) = f(x_1 + y_1) + f(x_2 + y_2)J = [f(x_1) + f(x_2)J] + [f(y_1) + f(y_2)J] = L(X) + L(Y)$

$L(A.X) = f(a_1x_1 + a_2x_2) + f(a_1x_2 + a_2x_1)J = a_1f(x_1) + a_2f(x_2) + [a_1f(x_2) + a_2f(x_1)]J$

$A.L(X) = (a_1 + a_2J) + (f(x_1) + f(x_2)J) = a_1f(x_1) + a_2f(x_2) + J[a_1f(x_2) + a_2f(x_1)] =$

$L(A.X)$.

Example.

Let $V = R^2, W = R^2, f: V \to W;\ f(x, y) = (2x, x - y)$ is a linear transformation.

Let $V_K = W_K = \{(x_1, y_1) + (x_2, y_2)J\ ;\ x_i, y_i \in R\}$ be the corresponding split-complex space.

We define $L: V_K \to W_K$ such that $L\big((x_1, y_1) + (x_2, y_2)J\big) = f(x_1, y_1) + f(x_2, y_2)J = (2x_1, x_1 - y_1)$ is a split- complex linear transformation.

Split-complex Inner products.

Definition.

Let $g: V \times V \to R$ be an inner product, we define the corresponding split-complex product as follows $f: V_K \times V_K \to K$ such that:

$f(x_1 + x_2J, y_1 + y_2J) = g(x_1, y_1) + g(x_2, y_2) + J[g(x_1, y_2) + g(x_2, y_1)]$

Now, we will discuss some properties of the split-complex inner product (f).

Property 1.

$f(X, Y) = f(Y, X)$ (the proof is clear).

Property 2.

$$f(X, X) = f(x_1 + x_2J, x_1 + x_2J) = g(x_1, x_1) + g(x_2, x_2) + J[g(x_1, x_2) + g(x_2, x_1)]$$
$$= \|x_1\|^2 + \|x_2\|^2 + 2Jg(x_1, x_2)$$

$f(X, X) = 0$ implies that:

$$\begin{cases} \|x_1\|^2 + \|x_2\|^2 = 0 \\ g(x_1, x_2) = 0 \end{cases} \Leftrightarrow \begin{cases} \|x_1\|^2 = -\|x_2\|^2 \\ x_1 \perp x_2 \end{cases} \Leftrightarrow x_1 = x_2 = 0$$

Thus $X = 0$.

Property 3.

$f(A.X, Y) = f(a_1x_1 + a_2x_2 + J(a_1x_2 + a_2x_1), y_1 + y_2J) = g(a_1x_1 + a_2x_2, y_1) + g(a_1x_2 + a_2x_1, y_2) + J[g(a_1x_1 + a_2x_2, y_2) + g(a_1x_2 + a_2x_1, y_1)] = a_1g(x_1, y_2) + a_2g(x_2, y_1) + a_1g(x_2, y_2) + a_2g(x_1, y_2) + J[a_1g(x_1, y_2) + a_2g(x_2, y_2) + a_1g(x_2, y_1) + a_2g(x_1, y_1)] = A.f(X, Y)$.

Property 4.

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$f(X+Y,Z)=f(X,Z)+f(Y,Z)$

Remark.

If $x_1 \perp x_2$, then $f(X,X)=\|x_1\|^2+\|x_2\|^2=\|x_1+x_2\|^2$.

Definition.

We define the norm of X as follows:

$$\|X\|=f(X,X)=\|x_1\|^2+\|x_2\|^2+2Jg(x_1,x_2)\in K$$

Remark.

1. If $x_1=0$, then $\|X\|=\|x_2\|^2$.
2. If $x_2=0$, then $\|X\|=\|x_1\|^2$.
3. If $x_1=x_2$, then $\|X\|=2\|x_1\|^2+2\varepsilon\|x_1\|^2$.
4. $\|A.X\|=A^2.\|X\|^2;\ \forall A\in K$.

Definition.

Let $X,Y\in V_K$, we say that $X\perp Y$ if and only if $f(X,Y)=0$.

Remark.

$X\perp Y$ if and only if $f(X,Y)=0$, thus:

$$\begin{cases} g(x_1,y_1)+g(x_2,y_2)=0 \\ g(x_1,y_2)+g(x_2,y_1)=0 \end{cases}$$

On the other hand, we have $g(x_1+x_2,y_1+y_2)=g(x_1,y_1)+g(x_1,y_2)+g(x_2,y_1)+g(x_2,y_2)=0$

Thus $X\perp Y$ if and only if $x_1+x_2\perp y_1+y_2$.

Conclusion

In this paper, we have defined and studied split-complex vector spaces as modules over the ring of split-complex numbers. Where we have presented the elementary properties of this new algebraic class in terms of theorems, and we present many examples to clarify the validity of our work.

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