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Study of Two Fractional Integrals

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***Abstract*: In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we study two fractional integral problems. Using some techniques, the solutions of these two fractional integrals can be obtained. In fact, our results are generalizations of the result in ordinary calculus.**

***Keywords*: Jumarie's modified R-L fractional calculus, New multiplication, Fractional analytic functions, Fractional integrals.**

I. INTRODUCTION

Fractional derivatives of non-integer orders are widely used in physics, mechanics, dynamics, and mathematical economics, electrical engineering, viscoelasticity, biology, and control theory [1-12]. Until now, the rules of fractional derivative are not unique. Many authors have given the definition of fractional derivative. The commonly used definition is the Riemann-Liouvellie (R-L) definition. Other useful definitions include Caputo definition of fractional derivative, Grunwald Letnikov (G-L) fractional derivative, conformable fractional derivative, and Jumarie's modified R-L fractional derivative [13-17]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.
In this paper, based on Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions, we study the following two α-fractional integrals:

$$\left({}_0I_x^{\alpha}\right)\left[2\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2}\otimes_\alpha\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 4}\right]^{\otimes_\alpha(-1)}\right], \tag{1}$$

$$\left({}_0I_x^{\alpha}\right)\left[2\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 4}\otimes_\alpha\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 6}\right]^{\otimes_\alpha(-1)}\right], \tag{2}$$

where $0<\alpha\leq 1$, $\Gamma(\ \ )$ is the gamma function. The solutions of these two fractional integrals can be obtained by using some techniques. On the other hand, our results are generalizations of classical calculus results.

II. PRELIMINARIES

At first, we introduce the fractional derivative used in this paper and its properties.

Definition 2.1 ([18])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\,. \tag{3}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{4}$$

where $\Gamma(\ \ )$ is the gamma function.

Proposition 2.2 ([19])**:** *If* α, β, x_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_0D_x^{\alpha}\right)\left[x^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, \tag{5}$$

and

$$\left({}_0D_x^{\alpha}\right)[C] = 0. \tag{6}$$

Next, the definition of fractional analytic function is introduced.

Definition 2.3 ([20])**:** Let x, x_0, and a_k be real numbers for all k, $x_0 \in (a, b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a, b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. In addition, if $f_\alpha: [a, b] \to R$ is continuous on closed interval $[a, b]$ and it is α-fractional analytic at every point in open interval (a, b), then f_α is called an α-fractional analytic function on $[a, b]$.

Next, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([21])**:** Let $0 < \alpha \leq 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}, \tag{7}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}. \tag{8}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \otimes_\alpha \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(x-x_0)^{k\alpha}. \end{aligned} \tag{9}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k} \otimes_\alpha \sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k} \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k}. \end{aligned} \tag{10}$$

Definition 2.5 ([22])**:** Suppose that $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions. Then $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha n} = f_\alpha(x^\alpha)\otimes_\alpha \cdots \otimes_\alpha f_\alpha(x^\alpha)$ is called the n-th power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha(-1)}$.

Definition 2.6 ([23])**:** If $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}, \tag{11}$$

$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \quad (12)$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha\circ g_\alpha)(x^\alpha)=f_\alpha(g_\alpha(x^\alpha))=\sum_{n=0}^{\infty}\frac{a_n}{n!}(g_\alpha(x^\alpha))^{\otimes_\alpha n}, \quad (13)$$

and

$$(g_\alpha\circ f_\alpha)(x^\alpha)=g_\alpha(f_\alpha(x^\alpha))=\sum_{n=0}^{\infty}\frac{b_n}{n!}(f_\alpha(x^\alpha))^{\otimes_\alpha n}. \quad (14)$$

Definition 2.7 ([24])**:** Let $0<\alpha\le 1$, and x be a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{x^{k\alpha}}{\Gamma(k\alpha+1)}=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha k}. \quad (15)$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$.

Theorem 2.8 ([25])**:** Let $0<\alpha\le 1$. Then the α-fractional arctangent function

$$arctan_\alpha(x^\alpha)=\left({}_0I_x^\alpha\right)\left[\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}\right]^{\otimes_\alpha(-1)}\right]. \quad (16)$$

Definition 2.9: The smallest positive real number T_α such that $E_\alpha(iT_\alpha)=1$, is called the period of $E_\alpha(ix^\alpha)$.

III. MAIN RESULTS

In this section, we solve two fractional integral problems.

Theorem 3.1: *If* $0<\alpha\le 1$, *then*

$$\left({}_0I_x^\alpha\right)\left[2\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}\otimes_\alpha\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 4}\right]^{\otimes_\alpha(-1)}\right]$$

$$=\frac{1}{\sqrt{2}}arctan_\alpha\left(\frac{1}{\sqrt{2}}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha-\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha(-1)}\right]\right)+\begin{cases}\frac{T_\alpha}{4} & \text{if } x\ge 0\\ -\frac{T_\alpha}{4} & \text{if } x<0\end{cases}$$

$$+\frac{1}{2\sqrt{2}}Ln_\alpha\left(\left|\left(\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}-\sqrt{2}\frac{1}{\Gamma(\alpha+1)}x^\alpha+1\right)\otimes_\alpha\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+\sqrt{2}\frac{1}{\Gamma(\alpha+1)}x^\alpha+1\right]^{\otimes_\alpha(-1)}\right|\right). \quad (17)$$

Proof $\left({}_0I_x^\alpha\right)\left[2\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}\otimes_\alpha\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 4}\right]^{\otimes_\alpha(-1)}\right]$

$$=\left({}_0I_x^\alpha\right)\left[\left[\left(\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+1\right)+\left(\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}-1\right)\right]\otimes_\alpha\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 4}\right]^{\otimes_\alpha(-1)}\right]$$

$$=\left({}_0I_x^\alpha\right)\left[\left(\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+1\right)\otimes_\alpha\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 4}\right]^{\otimes_\alpha(-1)}\right]$$

$$+\left({}_0I_x^\alpha\right)\left[\left(\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}-1\right)\otimes_\alpha\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 4}\right]^{\otimes_\alpha(-1)}\right]$$

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$$= \left({}_0I_x^\alpha\right)\left[\left(1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha(-2)}\right)\otimes_\alpha\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha(-2)}\right]^{\otimes_\alpha(-1)}\right]$$

$$+\left({}_0I_x^\alpha\right)\left[\left(1-\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha(-2)}\right)\otimes_\alpha\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha(-2)}\right]^{\otimes_\alpha(-1)}\right]$$

$$= \left({}_0I_x^\alpha\right)\left[\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha-\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha(-1)}\right]^{\otimes_\alpha 2}+2\right]^{\otimes_\alpha(-1)}\otimes_\alpha\left({}_0D_x^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha-\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha(-1)}\right]\right]$$

$$+\left({}_0I_x^\alpha\right)\left[\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha(-1)}\right]^{\otimes_\alpha 2}-2\right]^{\otimes_\alpha(-1)}\otimes_\alpha\left({}_0D_x^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha(-1)}\right]\right]$$

$$= \frac{1}{\sqrt{2}}arctan_\alpha\left(\frac{1}{\sqrt{2}}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha-\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha(-1)}\right]\right)+\begin{cases}\frac{T_\alpha}{4} & \text{if } x\geq 0\\ -\frac{T_\alpha}{4} & \text{if } x<0\end{cases}$$

$$+\frac{1}{2\sqrt{2}}Ln_\alpha\left(\left|\left(\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}-\sqrt{2}\frac{1}{\Gamma(\alpha+1)}x^\alpha+1\right)\otimes_\alpha\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+\sqrt{2}\frac{1}{\Gamma(\alpha+1)}x^\alpha+1\right]^{\otimes_\alpha(-1)}\right|\right). \quad \text{Q.e.d.}$$

Theorem 3.2: *Let* $0<\alpha\leq 1$, *then*

$$\left({}_0I_x^\alpha\right)\left[2\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 4}\otimes_\alpha\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 6}\right]^{\otimes_\alpha(-1)}\right]$$

$$= arctan_\alpha(x^\alpha)+\frac{1}{3}arctan_\alpha\left(\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 3}\right)$$

$$+\frac{1}{2\sqrt{3}}Ln_\alpha\left(\left|\left(\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}-\sqrt{3}\frac{1}{\Gamma(\alpha+1)}x^\alpha+1\right)\otimes_\alpha\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+\sqrt{3}\frac{1}{\Gamma(\alpha+1)}x^\alpha+1\right]^{\otimes_\alpha(-1)}\right|\right). \quad (18)$$

Proof $\left({}_0I_x^\alpha\right)\left[2\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 4}\otimes_\alpha\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 6}\right]^{\otimes_\alpha(-1)}\right]$

$$= \left({}_0I_x^\alpha\right)\left[\left[\left(\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 4}+1\right)+\left(\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 4}-1\right)\right]\otimes_\alpha\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 6}\right]^{\otimes_\alpha(-1)}\right]$$

$$= \left({}_0I_x^\alpha\right)\left[\left(\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 4}+1\right)\otimes_\alpha\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 6}\right]^{\otimes_\alpha(-1)}\right]$$

$$+\left({}_0I_x^\alpha\right)\left[\left(\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 4}-1\right)\otimes_\alpha\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 6}\right]^{\otimes_\alpha(-1)}\right]$$

$$= \left({}_0I_x^{\alpha}\right)\left[\left[\left(\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 4} - \left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2} + 1\right) + \left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2}\right] \otimes_\alpha \left[1 + \left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 6}\right]^{\otimes_\alpha (-1)}\right]$$

$$+\left({}_0I_x^{\alpha}\right)\left[\left(\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2} - 1\right) \otimes_\alpha \left(\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2} + 1\right) \otimes_\alpha \left[1 + \left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 6}\right]^{\otimes_\alpha (-1)}\right]$$

$$= \left({}_0I_x^{\alpha}\right)\left[\left[1 + \left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2}\right]^{\otimes_\alpha (-1)}\right] + \left({}_0I_x^{\alpha}\right)\left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2} \otimes_\alpha \left[1 + \left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 6}\right]^{\otimes_\alpha (-1)}\right]$$

$$+\left({}_0I_x^{\alpha}\right)\left[\left(\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2} - 1\right) \otimes_\alpha \left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 4} - \left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2} + 1\right]^{\otimes_\alpha (-1)}\right]$$

$$= \left({}_0I_x^{\alpha}\right)\left[\left[1 + \left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2}\right]^{\otimes_\alpha (-1)}\right] + \left({}_0I_x^{\alpha}\right)\left[\frac{1}{3}\left[1 + \left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 6}\right]^{\otimes_\alpha (-1)} \otimes_\alpha \left({}_0D_x^{\alpha}\right)\left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 3}\right]\right]$$

$$+\left({}_0I_x^{\alpha}\right)\left[\left(1 - \left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha (-2)}\right) \otimes_\alpha \left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2} - 1 + \left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha (-2)}\right]^{\otimes_\alpha (-1)}\right]$$

$$= arctan_\alpha(x^{\alpha}) + \frac{1}{3}arctan_\alpha\left(\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 3}\right)$$

$$+\left({}_0I_x^{\alpha}\right)\left[\left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha} + \left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha (-1)}\right]^{\otimes_\alpha 2} - 3\right]^{\otimes_\alpha (-1)} \otimes_\alpha \left({}_0D_x^{\alpha}\right)\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha} + \left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha (-1)}\right]\right]$$

$$= arctan_\alpha(x^{\alpha}) + \frac{1}{3}arctan_\alpha\left(\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 3}\right)$$

$$+\frac{1}{2\sqrt{3}}Ln_\alpha\left(\left|\left(\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2} - \sqrt{3}\frac{1}{\Gamma(\alpha+1)}x^{\alpha} + 1\right) \otimes_\alpha \left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2} + \sqrt{3}\frac{1}{\Gamma(\alpha+1)}x^{\alpha} + 1\right]^{\otimes_\alpha (-1)}\right|\right). \qquad \text{Q.e.d.}$$

IV. CONCLUSION

In this paper, we study two fractional integrals based on Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions. The solutions of these two fractional integrals can be obtained by using some techniques. In fact, our results are generalizations of traditional calculus results. In the future, we will continue to use Jumarie's modified R-L fractional calculus and the new multiplication of fractional analytic functions to solve the problems in fractional differential equations and engineering mathematics.

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