Variance of a difference between variables

The variance of variables *x* and *y* of respective means $\bar{x}$ and $\bar{y}$, in a sample of size *N*, are:

$$\begin{cases} \sigma^2(x) = \frac{1}{N}\sum_i (x_i - \bar{x})^2 \\ \sigma^2(y) = \frac{1}{N}\sum_i (y_i - \bar{y})^2 \end{cases}$$

Consequently, the variance of the difference *x*-*y* is:

$$\sigma^2(x-y) = \frac{1}{N}\sum_i (x_i - y_i - \overline{x-y})^2$$

Since the average of a sum is necessarily the sum of averages, we can write:

$$\sigma^2(x-y) = \frac{1}{N}\sum_i [(x_i - \bar{x}) - (y_i - \bar{y})]^2$$

⇔

$$\sigma^2(x-y) = \frac{1}{N}\sum_i [(x_i - \bar{x})^2 + (y_i - \bar{y})^2 - 2(x_i - \bar{x})(y_i - \bar{y})]$$

⇔

$$\sigma^2(x-y) = \frac{1}{N}\sum_i (x_i - \bar{x})^2 + \frac{1}{N}\sum_i (y_i - \bar{y})^2 - 2\frac{1}{N}\sum_i (x_i - \bar{x})(y_i - \bar{y})$$

⇔

$$\sigma^2(x-y) = \sigma^2(x) + \sigma^2(y) - 2\left(\frac{1}{N}\sum_i x_i y_i - \frac{1}{N}\sum_i x_i \bar{y} - \frac{1}{N}\sum_i \bar{x} y_i + \frac{1}{N}\sum_i \bar{x}\bar{y}\right)$$

⇔

$$\sigma^2(x-y) = \sigma^2(x) + \sigma^2(y) - 2\left(\overline{xy} - \bar{y}\frac{1}{N}\sum_i x_i - \bar{x}\frac{1}{N}\sum_i y_i + \bar{x}\bar{y}\right)$$

⇔

$$\sigma^2(x-y) = \sigma^2(x) + \sigma^2(y) - 2(\overline{xy} - \bar{y}\bar{x} - \bar{x}\bar{y} + \bar{x}\bar{y})$$

⇔

$$\sigma^2(x-y) = \sigma^2(x) + \sigma^2(y)$$

QED