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Non-unique Fixed Points of Self Mappings in Bi-metric Spaces

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ABSTRACT

In this paper, we prove a few non-unique fixed-point results of mapping on a set with bi-metrics using θ – contraction. We also give an example that justifies our results. In the literature, our result generalized many results.

Keywords: *Fixed point, θ – contraction, Bi-metric, Non-unique fixed point, Bi-metric space.*

Introduction

This paper involves an extension of some non-unique fixed point theorems of mappings in bi-metric space using θ – contraction. First, we discuss a brief history of fixed points and applications of the fixed points. Fixed point theory plays an essential role in various applied and theoretical fields, such as differential and integral equations, theory of dynamic systems, game theory, approximation theory, and nonlinear analysis. In the research aspect, it is considered one of the most active and favorable subjects, thus numerous mathematicians such as Cauchy, Liouville, Peano, Lipschitz and Picard did work in its related field. The significant result in fixed-point theory is the famous result by Brouwer in 1910. He also proved the result for n–dimensional Euclidean space. Birkhoff and Kellogg investigated the first infinite dimensional fixed point theorem. Later, Banach [2] gave the idea of a fixed point for contraction mappings which is known as the "Banach Contraction Principle" in 1922. This principle is useful for the uniqueness and existence of fixed points. There have been many generalizations of this principle in distinct directions [1, 3, 4, 6, 7]. Marr gave the first important result for non-expensive mappings. In 1941, Kakutani studied the fixed point problem of multifunction and generalizations of this type of mapping discussed in [5, 8, 9, 10].

Main Body

Firstly, consider a theorem about non-unique fixed points of self mappings in bi-metric spaces.

Theorem-1: Let $S \neq \theta$ with bi-metrics ρ and δ on S and $T : S \to S$ be T – orbitally continuous mapping which satisfies

(i) $\theta\big(\rho(Tg,Th)\big) \leq \big[\theta\big(\delta(g\,,h)\big)\big]^{\alpha}, \quad for\ all\ \ g,h \in S\ and\ \alpha \in (0,1)$

(ii) (S,ρ) is T – orbitally complete metric space

(iii) $\frac{\theta[min\{\rho(Tg,Th),\rho(g,Tg),\rho(h,Th)\}]}{\theta[min\{\rho(g,Th),\rho(h,Tg)\}]} \leq (\theta[\delta(g,h)])^{c} \quad for \quad c \in (0,1)$

(iv) $\theta[\delta(g,h)] \leq (\theta[\rho(g,h)])^{c1}, \quad for\ all\ g \neq h \in S\ and\ c_1 \in (0,1)$

Then T has fixed point.

Proof: Suppose $g \in S$ be any arbitrary point. We wish to prove that the sequence of iterates

$g_0 = g,$

$g_1 = Tg_0,$

$g_2 = Tg_1,$

… … …

$g_w = Tg_{w-1}$

$at\ x$ is Cauchy sequence. If $g_{w-1} = g_w$ for some $w \in I^+$ where I^+ is a positive integer then $\{g_w\}$ is a Cauchy sequence, we may suppose that $g_{w-1} \neq g_w$ for each $w = 1,2,3,\ldots$ let $g = g_{w-1}$ and $h = g_w$ we have

$$\frac{\theta[min\{\rho(Tg_{w-1},Tg_w),\rho(g_{w-1},Tg_{w-1}),\rho(g_w,Tg_w)\}]}{\theta[min\{\rho(g_{w-1},Tg_w),\rho(g_w,Tg_{w-1})\}]} \leq (\theta[\delta(g_{w-1},g_w)])^c$$

$$\frac{\theta[min\{\rho(g_w,g_{w+1}),\rho(g_{w-1},g_w),\rho(g_w,g_{w+1})\}]}{\theta[min\{\rho(g_{w-1},g_{w+1}),\rho(g_w,g_w)\}]} \leq (\theta[\delta(g_{w-1},g_w)])^c$$

$$\frac{\theta[min\{\rho(g_{w-1},g_w),\rho(g_w,g_{w+1})\}]}{\theta[min\{\rho(g_{w-1},g_{w+1}),0\}]} \leq (\theta[\delta(g_{w-1},g_w)])^c$$

which implies

$$\theta[min\{\rho(g_{w-1},g_w),\rho(g_w,g_{w+1})\}] \leq (\theta[\delta(g_{w-1},g_w)])^c$$

By (iv) $\quad \theta[\delta(g_{w-1},g_w)] \leq \theta[\rho(g_{w-1},g_w)]_{c_1} \quad for \quad c_1 \in (0,1)$

$$\theta[min\{\rho(g_{w-1},g_w),\rho(g_w,g_{w+1})\}] \leq (\theta[\rho(g_{w-1},g_w)])^{cc_1}$$

Let $cc_1 = k \in (0,1)$ and suppose $min\{\rho(g_w,g_{w+1}),\rho(g_{w-1},g_w)\} = \rho(g_{w-1},g_w)$

$$\theta[min\{\rho(g_{w-1},g_w)\}] \leq (\theta[\rho(g_{w-1},g_w)])^k$$

$$\theta[\rho(g_{w-1},g_w)] \leq (\theta[\rho(g_{w-1},g_w)])^k < \theta[\rho(g_{w-1},g_w)]$$

$$\theta[\rho(g_{w-1},g_w)] < \theta[\rho(g_{w-1},g_w)]$$

which is contradiction. So,

$$\theta[\rho(g_w,g_{w+1})] \leq (\theta[\rho(g_{w-1},g_w)])^k$$

$$\theta[\rho(g_w,g_{w+1})] \leq (\theta[\rho(g_{w-1},g_w)])^k \leq (\theta[\rho(g_{w-2},g_{w-1})])^{k^2} \leq \cdots \leq (\theta[\rho(g,Tg)])^{k^w}$$

$$1 < \theta[\rho(g_w,g_{w+1})] \leq (\theta[\rho(g,Tg)])^{k^w} \quad for\ all \quad w \in \mathbb{N}$$

Letting $w \to \infty$ we get

$$\lim_{w\to\infty} \theta[(g_w,g_{w+1})] = \lim_{w\to\infty} \theta[\rho(T^w g,T^{w+1}g)] = 1$$

which implies

$$\lim_{w\to\infty} \rho(T^w g, T^{w+1}g) = 0$$

Now proceeding in similar way, such that there exists $w_1 \in \mathbb{N}$ we get

$$\rho(T^w g, T^{w+1}g) \leq \frac{1}{w^{\frac{1}{r}}} \quad for\ all \quad w > w_1$$

Now for $s > w$

$$\rho(T^w g, T^s g) \leq \rho(T^w g, T^{w+1}g) + \rho(T^{w+1}g, T^{w+2}g) + \rho(T^{w+2}g, T^{w+3}g) + \cdots + \rho(T^{s-1}g, T^s g)$$

$$\rho(T^w g, T^s g) \leq \frac{1}{w^{\frac{1}{r}}} + \frac{1}{(w+1)^{\left(\frac{1}{r}\right)}} + \frac{1}{(w+2)^{\left(\frac{1}{r}\right)}} + \cdots + \frac{1}{(s-1)^{\left(\frac{1}{r}\right)}}$$

$$\rho(T^w g, T^s g) \leq \sum_{i=w}^{s-1} \frac{1}{i^{\frac{1}{r}}}$$

$$\rho(T^w g, T^s g) \leq \sum_{i=w}^{s-1} \frac{1}{i^{\frac{1}{r}}} \leq \sum_{i=w}^{\infty} \frac{1}{i^{\frac{1}{r}}}$$

$$\rho(T^w g, T^s g) \leq \sum_{i=w}^{\infty} \frac{1}{i^{\frac{1}{r}}}$$

It is infinite geometric series and is convergent (as $0 < \frac{1}{i} < 1$). Therefore, $\{T^w g\}$ is C.S. As (S, ρ) is $T-$orbitally complete, so there exists $v \in S$ such that

$$v = \lim_{w\to\infty} T^w g$$

By orbital continuity of T

$$Tv = \lim_{w\to\infty} TT^w g = v$$

Now change some conditions in the above statement, and take a new theorem.

Theorem-2: Suppose S be non-empty set with bi-metrics ρ and δ on S. Suppose that (S, ρ) is $T-$orbitally complete and $T: S \to S$ be $T-$orbitally continuous. For $\epsilon > 0$ there exists $g_0 \in S$ such that $\rho(g_0, T^k g_0) < \epsilon$ for $k \in I^+$ and if T satisfies

(i) $\theta[\rho(Tg, Th)] \leq \left[\theta\left(\delta(g,h)\right)\right]^\alpha \quad for\ all \quad g, h \in S$ and $\alpha \in (0,1)$

(ii) $0 < \rho(g,h) < \epsilon$ implies that $\theta[min\{\rho(Tg,Th), \rho(g,Tg), \rho(h,Th)\}] \leq (\theta[\delta(g,h)])^c$ for $c \in (0,1)$

(iii) $\theta[\delta(g,h)] \leq (\theta[\rho(g,h)])^{c_1}$, $g \neq h$ and $c_1 \in (0,1)$

Then T has periodic point.

Proof: Let we define a subset $K = \left\{\rho\left(g, T^k g\right) < \epsilon \ for\ some \ g \in S\right\}$ of positive integer is non-empty. Let $m = \min K$ and suppose that

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$\rho(g, T^m g) < \epsilon \;\; for \;\; g \in S$

Case-1: If $m = \min K = 1$ then $0 < \rho(g, Tg) < \epsilon$ for g and Tg we have

$\theta[min\{\rho(Tg, TTg), \rho(g, Tg), \rho(Tg, TTg)\}] \leq [\theta(\delta(g, Tg))]^c$

$\theta[min\{\rho(g, Tg), \rho(Tg, T^2 g)\}] \leq [\theta(\delta(g, Tg))]^c$

By (iii)

$\theta[min\{\rho(g, Tg), \rho(Tg, T^2 g)\}] \leq [\theta(\rho(g, Tg))]^{cc_1}$

Let $cc_1 = k \in (0,1)$ and if $min\{\rho(g, Tg), \rho(Tg, T^2 g)\} = \rho(g, Tg)$

$\theta[\rho(g, Tg)] \leq [\theta(\rho(g, Tg))]^k$

$\theta[\rho(g, Tg)] \leq [\theta(\rho(g, Tg))]^k < \theta[\rho(g, Tg)]$

$\theta[\rho(g, Tg)] < \theta[\rho(g, Tg)]$

which is a contradiction. So,

$\theta[\rho(Tg, T^2 g)] \leq [\theta(\rho(g, Tg))]^k$

Similarly,

$\theta[\rho(T^2 g, T^3 g)] \leq [\theta(\rho(Tg, T^2 g))]^k \leq [\theta(\rho(g, Tg))]^{k^2}$

$\theta[\rho(T^w g, T^{w+1} g)] \leq \theta[\rho(T^{w-1} g, T^w g)]^k \leq \cdots \leq [\theta(\rho(g, Tg))]^{k^w}$

$1 < \theta[\rho(T^w g, T^{w+1} g)] \leq [\theta(\rho(g, Tg))]^{k^w}$

Applying limit $w \to \infty$ we get

$\lim_{w \to \infty} \theta[\rho(T^w g, T^{w+1} g)] = 1$

Implies

$\lim_{w \to \infty} \rho(T^w g, T^{w+1} g) = 0$

Now by same argument of Theorem 1 we get $Tv = v$

Case-2: If $\min K = m \geq 2$ and suppose

$\rho(h, Th) \geq \epsilon \quad for\, all \quad h \in S \quad \longrightarrow \quad (1)$

Then by our assumption $0 < \rho(g, T^m g) < \epsilon$

$\theta[min\{\rho(Tg, TT^m g), \rho(g, Tg), \rho(T^m g, T^{m+1} g)\}] \leq [\theta(\delta(g, T^m g))]^c$

$\theta[min\{\rho(Tg, T^{m+1} g), \rho(g, Tg), \rho(T^m g, T^{m+1} g)\}] \leq [\theta(\delta(g, T^m g))]^c$

By (iii)

$$\theta[min\{\rho(Tg,T^{m+1}g),\rho(g,Tg),\rho(T^m g,T^{m+1}g)\}] \leq \left[\theta\left(\rho(g,T^m g)\right)\right]^{cc_1}$$

Let $cc_1 = k \in (0,1)$ and if $min\{\rho(Tg,T^{m+1}g),\rho(g,Tg),\rho(T^m g,T^{m+1}g)\} = \rho(g,Tg)$

$$\theta[\rho(g,Tg)] \leq \left[\theta\left(\rho(g,T^m g)\right)\right]^k$$

$$\theta[\rho(g,Tg)] \leq \left[\theta\left(\rho(g,T^m g)\right)\right]^k < \theta\left(\rho(g,T^m g)\right)$$

$$\theta[\rho(g,Tg)] < \theta\left(\rho(g,T^m g)\right) < \theta(\epsilon)$$

By using (1)

$$\theta(\epsilon) \leq \theta[\rho(g,Tg)] < \theta(\epsilon)$$

$$\theta(\epsilon) < \theta(\epsilon)$$

which is not possible.

If

$$min\{\rho(Tg,T^{m+1}g),\rho(g,Tg),\rho(T^m g,T^{m+1}g)\} = \rho(T^m g,TT^m g)$$

Then by (1) it is again contradiction. So,

$$\theta[\rho(Tg,T^{m+1}g)] \leq \left[\theta\left(\rho(g,T^m g)\right)\right]^k$$

$$\theta[\rho(Tg,T^{m+1}g)] \leq \left[\theta\left(\rho(g,T^m g)\right)\right]^k \leq \left(\theta(\epsilon)\right)^k$$

Similarly,

$$\theta[\rho(T^2 g,T^{m+2}g)] \leq \left[\theta\left(\rho(Tg,T^{m+1}g)\right)\right]^k \leq \left[\theta\left(\rho(g,T^m g)\right)\right]^{k^2}$$

$$\theta[\rho(T^3 g,T^{m+3}g)] \leq \left[\theta\left(\rho(T^2 g,T^{m+2}g)\right)\right]^k \leq \left[\theta\left(\rho(Tg,T^{m+1}g)\right)\right]^{k^2} \leq \left[\theta\left(\rho(g,T^m g)\right)\right]^{k^3}$$

In general

$$\theta[\rho(T^w g,T^{m+w}g)] \leq \left[\theta\left(\rho(T^{w-1}g,T^{m+w-1}g)\right)\right]^k \leq \left[\theta\left(\rho(T^{w-2}g,T^{m+w-2}g)\right)\right]^{k^2} \leq \cdots \leq \left[\theta\left(\rho(g,T^m g)\right)\right]^k$$

$$1 < \theta[\rho(T^w g,T^{m+w}g)] \leq \left[\theta\left(\rho(g,T^m g)\right)\right]^k \leq \left(\theta(\epsilon)\right)^{k^w} \quad for\ all \quad w \in \mathbb{N}$$

$$\lim_{w\to\infty} \theta[\rho(T^w g,T^{m+w}g)] = 1$$

implies

$$\lim_{w\to\infty} \rho(T^w g,T^{m+w}g) = 0$$

Now proceeding in similar way such that there exists $w_2 \in \mathbb{N}$ we get

$$\rho(T^w g,T^{m+w}g) \leq \frac{1}{w^{\frac{1}{r}}} \quad for\ all \quad w > w_2$$

Now for $s > w$

$$\rho(T^w g, T^s g) \le \rho(T^w g, T^{w+1} g) + \rho(T^{w+1} g, T^{w+2} g) + \rho(T^{w+2} g, T^{w+3} g) + \cdots + \rho(T^{s-1} g, T^s g)$$

$$\rho(T^w g, T^s g) \le \frac{1}{w^{\frac{1}{r}}} + \frac{1}{(w+1)^{\frac{1}{r}}} + \frac{1}{(w+2)^{\frac{1}{r}}} + \cdots + \frac{1}{(s-1)^{\frac{1}{r}}}$$

$$\rho(T^w g, T^s g) \le \sum_{i=w}^{s-1} \frac{1}{i^{\frac{1}{r}}}$$

$$\rho(T^w g, T^s g) \le \sum_{i=w}^{s-1} \frac{1}{i^{\frac{1}{r}}} \le \sum_{i=w}^{\infty} \frac{1}{i^{\frac{1}{r}}}$$

$$\rho(T^w g, T^s g) \le \sum_{i=w}^{\infty} \frac{1}{i^{\frac{1}{r}}}$$

It is infinite geometric series and is convergent (as $0 < \frac{1}{i} < 1$). Therefore, $\{T^m g\}$ is a Cauchy sequence. As (S, ρ) is $T-$orbitally complete, so there exists $v \in S$ such that

$$\lim_{w\to\infty} T^w g = v$$

Now by orbital continuity of T

$$T^m v = \lim_{w\to\infty} T^w T^w g = v$$

Theorem-3: Suppose S be non-empty set with bi-metrics ρ and δ on S and $T : S \to S$ be $T-$orbitally continuous. If T satisfies

(i) $\theta\big(\rho(Tg, Th)\big) \le \big[\theta\big(\delta(g,h)\big)\big]^q \quad for\ all \quad g, h \in S\ and\ q \in (0,1)$

(ii) $\theta[\delta(g,h)] \le (\delta[\rho(g,h)])^k \ , \ g \ne h \quad and \quad k \in (0,1)$

(iii) $\frac{\theta[min\{\rho(Tg,Th), \rho(g,Tg), \rho(h,Th)\}]}{\theta[min\{\rho(g,Th), \rho(h,Tg)\}]} < \theta\big(\delta(g,h)\big) \ , \ g \ne h \ , \ for\ each \ \ g_0 \in S$

Then the sequence $\{T^w g_0\}$ has cluster point $v \in S$ then v is fixed point of T

Proof: Let $v \in S$ be a cluster point then

$$\lim_{w\to\infty} T^w g_0 = v$$

If $T^{w-1} g_0 = T^w g_0$ for some $w \in I^+$, then v is a fixed point of T which is our required result.

If $T^{w-1} g_0 \ne T^w g_0$ for all $w \in I^+$ then for $T^{w-1} g_0, T^w g_0 \in S$ we get

$$\frac{\theta[min\{\rho(TT^{w-1}g_0, TT^w g_0), \rho(T^{w-1}g_0, TT^{w-1}g_0), \rho(T^w g_0, TT^w g_0)\}]}{\theta[min\{\rho(T^{w-1}g_0, TT^w g_0), \rho(T^w g_0, TT^{w-1}g_0)\}]} < \theta\big(\delta(T^{w-1}g_0, T^w g_0)\big)$$

$$\frac{\theta[min\{\rho(T^w g_0, T^{w+1}g_0), \rho(T^{w-1}g_0, T^w g_0), \rho(T^w g_0, T^{w+1}g_0)\}]}{\theta[min\{\rho(T^{w-1}g_0, T^{w+1}g_0), \rho(T^w g_0, T^w g_0)\}]} < \theta\big(\delta(T^{w-1}g_0, T^w g_0)\big)$$

$$\frac{\theta[min\{\rho(T^w g_0, T^{w+1} g_0), \rho(T^{w-1} g_0, T^w g_0)\}]}{\theta[min\{\rho(T^{w-1} g_0, T^{w+1} g_0), 0\}]} < \theta\big(\delta(T^{w-1} g_0, T^w g_0)\big)$$

$$\frac{\theta[min\{\rho(T^w g_0, T^{w+1} g_0), \rho(T^{w-1} g_0, T^w g_0)\}]}{\theta[0]} < \theta\big(\delta(T^{w-1} g_0, T^w g_0)\big)$$

$$\theta[min\{\rho(T^w g_0, T^{w+1} g_0), \rho(T^{w-1} g_0, T^w g_0)\}] < \theta\big(\delta(T^{w-1} g_0, T^w g_0)\big)$$

By using (ii)

$$\theta[min\{\rho(T^w g_0, T^{w+1} g_0), \rho(T^{w-1} g_0, T^w g_0)\}] < \left[\theta\big(\rho(T^{w-1} g_0, T^w g_0)\big)\right]^k$$

If $n\{\rho(T^w g_0, T^{w+1} g_0), \rho(T^{w-1} g_0, T^w g_0)\} = \rho(T^{w-1} g_0, T^w g_0)$, then

$$\theta[\rho(T^{w-1} g_0, T^w g_0)] < \left[\theta\big(\rho(T^{w-1} g_0, T^w g_0)\big)\right]^k$$

$$\theta[\rho(T^{w-1} g_0, T^w g_0)] < \left[\theta\big(\rho(T^{w-1} g_0, T^w g_0)\big)\right]^k < \theta[\rho(T^{w-1} g_0, T^w g_0)]$$

$$\theta[\rho(T^{w-1} g_0, T^w g_0)] < \theta[\rho(T^{w-1} g_0, T^w g_0)]$$

which is not possible. So,

$$\theta[\rho(T^w g_0, T^{w+1} g_0)] < \left[\theta\big(\rho(T^{w-1} g_0, T^w g_0)\big)\right]^k$$

$$\theta[\rho(T^w g_0, T^{w+1} g_0)] < \left[\theta\big(\rho(T^{w-1} g_0, T^w g_0)\big)\right]^k < \left[\theta\big(\rho(T^{w-2} g_0, T^{w-1} g_0)\big)\right]^{k^2} < \cdots < \left[\theta\big(\rho(g_0, T g_0)\big)\right]^{k^w}$$

$$1 < \theta[\rho(T^w g_0, T^{w+1} g_0)] < \left[\theta\big(\rho(g_0, T g_0)\big)\right]^{k^w}$$

Applying limit $w \to \infty$ we have

$$\lim_{w\to\infty} \theta[\rho(T^w g_0, T^{w+1} g_0)] = 1$$

Implies

$$\lim_{w\to\infty} \rho(T^w g_0, T^{w+1} g_0) = 0$$

Now proceeding in similar way such that there exists $w_1 \in \mathbb{N}$ we get

$$\rho(T^w g_0, T^{w+1} g_0) \leq \frac{1}{w^{\frac{1}{r}}} \quad for\ all \quad w > w_1$$

Now for $s > w$

$$\rho(T^w g_0, T^s g_0) \leq \rho(T^w g_0, T^{w+1} g_0) + \rho(T^{w+1} g_0, T^{w+2} g_0) + \rho(T^{w+2} g_0, T^{w+3} g_0) + \cdots + \rho(T^{s-1} g_0, T^s g_0)$$

$$\rho(T^w g_0, T^s g_0) \leq \frac{1}{w^{\frac{1}{r}}} + \frac{1}{(w+1)^{\frac{1}{r}}} + \frac{1}{(w+2)^{\frac{1}{r}}} + \cdots + \frac{1}{(s-1)^{\frac{1}{r}}}$$

$$\rho(T^w g_0, T^s g_0) \leq \sum_{i=w}^{s-1} \frac{1}{i^{\frac{1}{r}}} = w$$

Hence, $\rho(T^w g_0, T^s g_0) \to 0$ as $w, s \to \infty$. So $\{T^w g_0\}$ is Cauchy sequence. So there exists $v \in S$ such that

$$\lim_{w\to\infty} T^w g_0 = v$$

Now by orbitally continuity of T

$$Tv = \lim_{w\to\infty} T^{w+1} g_0 = v$$

Now change some conditions in the theorem-3, and take a new theorem.

Theorem-4: Suppose S be a non-empty set with bi-metrics ρ and δ on S and $T : S \to S$ be T – orbitally continuous. If T satisfies

(i) $\theta\big(\rho(Tg, Th)\big) \leq \big[\theta\big(\delta(g,h)\big)\big]^q \quad for\ all \quad g,h \in S \quad and \quad q \in (0,1)$

(ii) $\theta[\delta(g,h)] \leq (\theta[\rho(g,h)])^k \ , \ g \neq h \quad and \quad k \in (0,1)$

(iii) $0 < \rho(g,h) < \epsilon$ implies that $\theta[min\{\rho(Tg,Th), \rho(g,Tg), \rho(h,Th)\}] < \theta\big(\delta(h,Th)\big)$, for each $g_0 \in S$ the sequence $\{T^w g_0\}$ has cluster point $v \in S$

Then v is a fixed point of T.

Proof: Let $v \in S$ be a cluster point i.e. $\lim_{w\to\infty} T^w g_0 = v$.

Then there exists $w_0 \in I^+$ such that

$$\rho(T^w g_0, v) < \frac{\epsilon}{2} \ , \ w > w_0$$

which gives

$$\rho(T^m g_0, T^{w+1} g_0) < \rho(T^m g_0, v) + \rho(v, T^{w+1} g_0) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Now, we define a set

$$K = \{\, k \in I^+ \ , \rho\big(T^r g_0 \, , T^{r+k} g_0\big) < \epsilon \ \ for\ some \ \ r \in I^+\}$$

is non-empty. Suppose that $\min K = m$

$$\min K = \{\, m \in I^+ \ \ , \rho(T^r g_0, T^{r+k} g_0) < \epsilon \ \ for\ some \ \ r \in I^+\}$$

If $T^r g_0 = T^{r+m} g_0$ for some $r \in I^+$ then $v = T^r g_0 = T^m v$ which is our required result. If $T^r g_0 \neq T^{r+m} g_0$ for all $r \in I^+$ then by our assumption

$$0 < \rho(T^r g_0, T^{r+m} g_0) < \epsilon$$

Case-1: If $\min K = 1$ then $0 < \rho(T^r g_0, T^{r+1} g_0) < \epsilon$ and for $T^{w-1} g_0, T^w g_0 \in S$ we have

$$\theta[min\{\rho(T^w g_0, T^{w+1} g_0), \rho(T^{w-1} g_0, T^w g_0), \rho(T^w g_0, T^{w+1} g_0)\}] < \big[\theta\big(\delta(T^{w-1} g_0, T^w g_0)\big)\big]$$

By (ii)

$$\theta[min\{\rho(T^{w-1} g_0, T^w g_0), \rho(T^w g_0, T^{w+1} g_0)\}] < (\theta[\rho(T^{w-1} g_0, T^w g_0)])^k$$

If $min\{\rho(T^{w-1} g_0, T^w g_0), \rho(T^w g_0, T^{w+1} g_0)\} = \rho(T^{w-1} g_0, T^w g_0)$, then

$\theta[\rho(T^{w-1}g_0,T^w g_0)] < (\theta[\rho(T^{w-1}g_0,T^w g_0)])^k$

$\theta[\rho(T^{w-1}g_0,T^w g_0)] < (\theta[\rho(T^{w-1}g_0,T^w g_0)])^k < \theta[\rho(T^{w-1}g_0,T^w g_0)]$

$\theta[\rho(T^{w-1}g_0,T^w g_0)] < \theta[\rho(T^{w-1}g_0,T^w g_0)]$

which is not possible. So,

$\theta[\rho(T^w g_0,T^{w+1}g_0)] < (\theta[\rho(T^{w-1}g_0,T^w g_0)])^k$

$1 < \theta[\rho(T^w g_0,T^{w+1}g_0)] < (\theta[\rho(T^{w-1}g_0,T^w g_0)])^k < \cdots < (\theta[\rho(g_0,Tg_0)])^{k^w}$

$1 < \theta[\rho(T^w g_0,T^{w+1}g_0)] < (\theta[\rho(g_0,Tg_0)])^{k^w}$

Applying limit $w \to \infty$ we get

$\lim_{w\to\infty} \theta[\rho(T^w g_0,T^{w+1}g_0)] = 1$

Implies

$\lim_{w\to\infty} \rho(T^w g_0,T^{w+1}g_0) = 0$

Now, we can say that

$Tv = v$

Case-2: If $m \geq 2$ suppose that

$\rho(T^w g_0,T^{w+1}g_0) \geq \epsilon \quad -\to \quad (2)$

for all $w \in I^+$ and by our assumption $0 < \rho(T^w g_0,T^{w+1}g_0) < \epsilon$ we have

$\theta[min\{\rho(T^{r+1}g_0,T^{r+1+m}g_0),\rho(T^r g_0,T^{r+1}g_0),\rho(T^{r+m}g_0,T^{r+m+1}g_0)\}] \leq \theta[\delta(T^r g_0,T^{r+m}g_0)]$

By (ii)

$\theta[min\{\rho(T^{r+1}g_0,T^{r+m+1}g_0),\rho(T^r g_0,T^{r+1}g_0),\rho(T^{r+m}g_0,T^{r+m+1}g_0)\}] \leq (\theta[\rho(T^r g_0,T^{r+m}g_0)])^k$

If $min\{\rho(T^{r+1}g_0,T^{r+m+1}g_0),\rho(T^r g_0,T^{r+1}g_0),\rho(T^{r+m}g_0,T^{r+m+1}g_0)\} = \rho(T^r g_0,T^{r+1}g_0)$ then

$\theta[\rho(T^r g_0,T^{r+1}g_0)] \leq (\theta[\rho(T^r g_0,T^{r+m}g_0)])^k$

$\theta[\rho(T^r g_0,T^{r+1}g_0)] \leq (\theta[\rho(T^r g_0,T^{r+m}g_0)])^k < \theta\big(\rho(T^r g_0,T^{r+m}g_0)\big)$

$\theta[\rho(T^r g_0,T^{r+1}g_0)] < \theta\big(\rho(T^r g_0,T^{r+m}g_0)\big)$

$\theta[\rho(T^r g_0,T^{r+1}g_0)] < \theta\big(\rho(T^r g_0,T^{r+m}g_0)\big) < \theta(\epsilon)$

$\theta[\rho(T^r g_0,T^{r+1}g_0)] < \theta(\epsilon)$

Using (2) we obtain

$\theta(\epsilon) \leq \theta[\rho(T^r g_0,T^{r+1}g_0)]$

From above both equations

$\theta(\epsilon) < \theta(\epsilon)$

which is not possible. If

$$min\{\rho(T^{r+1}g_0,T^{r+m+1}g_0),\rho(T^r g_0,T^{r+1}g_0),\rho(T^{r+m}g_0,T^{r+m+1}g_0)\}=\rho(T^{r+m}g_0,T^{r+m+1}g_0)$$

$$\theta[\rho(T^{r+m}g_0,T^{r+m+1}g_0)]<(\theta[\rho(T^{r+m}g_0,T^{r+m+1}g_0)])^k$$

$$\theta[\rho(T^{r+m}g_0,T^{r+m+1}g_0)]<(\theta[\rho(T^r g_0,T^{r+m}g_0)])^k<\theta[\rho(T^r g_0,T^{r+m}g_0)]$$

$$\theta[\rho(T^{r+m}g_0,T^{r+m+1}g_0)]<\theta[\rho(T^r g_0,T^{r+m}g_0)]$$

$$\theta[\rho(T^{r+m}g_0,T^{r+m+1}g_0)]<\theta[\rho(T^r g_0,T^{r+m}g_0)]<\theta(\epsilon)$$

$$\theta[\rho(T^{r+m}g_0,T^{r+m+1}g_0)]<\theta(\epsilon)$$

By (2) we have

$$\theta(\epsilon)<\theta(\epsilon)$$

Again it is not possible. So,

$$[\rho(T^{r+1}g_0,T^{r+m+1}g_0)]<(\theta[\rho(T^r g_0,T^{r+m}g_0)])^k$$

$$1<\theta[\rho(T^w g_0,T^{w+m}g_0)]<(\theta[\rho(T^{w-1}g_0,T^{w-1+m}g_0)])^k<\cdots<(\theta[\rho(T^{r+1}g_0,T^{r+m+1}g_0)])^{k^{w-1}}<(\theta[\rho(T^r g_0,T^{r+m}g_0)])^{k^w}$$

$$1<\theta[\rho(T^w g_0,T^{w+m}g_0)]<(\theta[\rho(T^r g_0,T^{r+m}g_0)])^{k^w}$$

Applying limit $w\to\infty$ we get

$$\lim_{w\to\infty}[\rho(T^w g_0,T^{w+m}g_0)]=1$$

implies

$$\lim_{w\to\infty}\rho(T^w g_0,T^{w+m}g_0)=0$$

Now proceeding in similar way such that there exists $w_2\in\mathbb{N}$ we have

$$(T^w g_0,T^{w+m}g_0)\le\frac{1}{w^{\frac{1}{r}}}\quad for\ all\quad w\ge w_2$$

Now for $s>w$

$$\rho(T^w g_0,T^s g_0)\le\rho(T^w g_0,T^{w+1}g_0)+\rho(T^{w+1}g_0,T^{w+2}g_0)+\rho(T^{w+2}g_0,T^{w+3}g_0)+\cdots+\rho(T^{s-1}g_0,T^s g_0)$$

$$\rho(T^w g_0,T^s g_0)\le\frac{1}{w^{\frac{1}{r}}}+\frac{1}{(w+1)^{\frac{1}{r}}}+\frac{1}{(w+2)^{\frac{1}{r}}}+\cdots+\frac{1}{(s-1)^{\frac{1}{r}}}$$

$$\rho(T^w g_0,T^s g_0)\le\sum_{i=w}^{s-1}\frac{1}{i^{\frac{1}{r}}}$$

As $(T^w g_0,T^s g_0)\to 0$ as $w,s\to\infty$. So $\{T^w g_0\}$ is a Cauchy sequence. Since $\lim_{w\to\infty}T^w g_0=v$. Now by orbital continuity of T

$$T^m v = \lim_{w\to\infty} T^{w+m} g_0 = v$$

Now, consider the example, in which we apply the new conditions, made in the above theorems. And prove that the following example satisfies the new conditions, but didn't satisfy the other conditions.

Example-1: Let $S = \{\, g_w = 2w - 1 \ \ ; \ for\ all \ \ w \in \mathbb{N} \,\}$ we define bi-metrics ρ and δ such that $\rho(g,h) = 3|g-h|$ and $\delta(g,h) = |g-h|$ for all $g,h \in S$. Let $T : S \to S$ be a mapping define by $Tg_w = g_{w-1}$ for all $w \geq 2$ and $Tg_1 = g_1$. Let $g = g_w$ and $h = g_1$

$$\rho(Tg_w . Tg_1) = 3|2w-4| = 6|w-2|$$

$$\rho(g_w, Tg_w) = 3|2| = 6$$

$$\rho(g_1, Tg_1) = 3|1-1| = 0$$

$$\rho(g_w, Tg_1) = 3|2w-2| = 6|w-1|$$

$$\rho(g_1, Tg_w) = 3|2(w-2)| = 6|w-2|$$

$$\delta(g_w, g_1) = |2w-2| = 6|w-1|$$

Now,

$$\lim_{w\to\infty} \frac{min\{\rho(Tg_w,Tg_1), \rho(g_w,Tg_w), \rho(g_1,Tg_1)\} - min\{\rho(g_w,Tg_1), \rho(g_1,Tg_w)\}}{\delta(g_w,g_1)}$$

$$= \lim_{w\to\infty} \frac{min\{6|w-2|, 6,0\} - min\{6|w-1|, 6|w-2|\}}{2|w-1|}$$

$$= \lim_{w\to\infty} \frac{0 - 6|w-2|}{2|w-1|}$$

$$= -3 \lim_{w\to\infty} \frac{w-2}{w-1}$$

$$= -3$$

Now, we consider $\theta(g) = e^g \in \Theta$.

$$\frac{\theta[min\{\rho(Tg_w,Tg_1), \rho(g_w,Tg_w), \rho(g_1,Tg_1)\}]}{\theta[min\{\rho(g_w,Tg_1), \rho(g_1,Tg_w)\}]} \leq (\theta[\delta(g_w,g_1)])^k$$

$$\frac{\theta[min\{6|w-2|, 6\ ,0\ \}]}{\theta[min\{6|w-1|, 6|w-2|\}]} \leq (\theta[2|w-1|])^k$$

$$\frac{1}{\theta[6(w-2)]} \leq (\theta[2(w-1)])^k$$

$$\frac{1}{e^{6(w-2)}} \leq e^{2k(w-1)}$$

which satisfy for all $w \geq 2$ and $k \in (0,1)$.

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