A Generalization of a Theorem of Pittie

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Abstract

For a compact Lie group G the complex representation ring RH of a closed subgroup H can be equipped with the structure of an RG-module using the ring homomorphism induced by the inclusion $H \subseteq G$. Pittie and Steinberg have shown that for certain compact connected Lie groups G the representation ring RT of a maximal torus T in G is a free RG-module. In particular, it is flat over RG. In this thesis, we take a look at closed connected subgroups H_1, H_2 of these types of Lie groups and study the vanishing of $\mathrm{Tor}_i^{RG}(RH_1, RH_2)$.

Contents

Introduction

This thesis deals with complex representation rings of compact connected Lie groups under an algebraic point of view.

Let G be a compact Lie group. The complex representation ring RG of G results from the semiring of equivalence classes of complex, finite-dimensional, linear representations of G. By the Closed Subgroup Theorem, every closed subgroup H of G is an embedded Lie subgroup. In particular, the inclusion $H \hookrightarrow G$ is a morphism of Lie groups. We are interested in the representation rings RH of closed subgroups H of G. More precisely, we will regard RH as an RG-module using the ring homomorphism $RG \to RH$ induced by the inclusion $H \subseteq G$. In this thesis, we will mainly ask ourselves the following question: If we take two closed connected subgroups H_1, H_2 of a compact connected Lie group G, then what can be said about the vanishing of $\mathrm{Tor}_i^{RG}(RH_1, RH_2)$? Our aim is to prove the following conjecture:

Conjecture (see Conjecture 100)**.** *Let G be a compact connected Lie group with RG isomorphic to a tensor product of a polynomial algebra and a Laurent algebra. In addition let $H_1, H_2 \subseteq G$ be two closed connected subgroups that satisfy the strict double coset condition. Then:*

$$\mathrm{Tor}_i^{RG}(RH_1, RH_2) = 0 \text{ for all } i > \operatorname{rank} G - (\operatorname{rank} H_1 + \operatorname{rank} H_2).$$

Here, the rank of a compact connected Lie group is the dimension of a maximal torus contained in the group. The additional conditions that appear in the conjecture are motivated by the following results of Singhof, Pittie and Steinberg:

- The *strict double coset condition* is a condition for a pair of subgroups that naturally appears when studying *biquotients manifolds*. We will introduce these terms in detail in the first two sections of Chapter 3. The motivation to demand this condition was a work of Singhof [Sin93], in which he studied these biquotients and managed to show a result similar to the conjecture for cohomology with coefficients in $\mathbb{Q}$ (see Theorem 101).

- In compact connected Lie groups G, a maximal torus $T \subseteq G$ always exists. Let T be such a maximal torus. The Theorem of Pittie yields that RT is free as an RG-module if the fundamental group $\pi_1(G)$ is free. This was generalized by Steinberg who introduced equivalent conditions for when RT is free as an RG-module. One of these equivalent conditions is that *RG is isomorphic to the tensor product of a polynomial and a Laurent algebra*. If RT is free, we have in particular:

  $$\mathrm{Tor}_i^{RG}(RT, \mathbb{Z}) = 0 \text{ for all } i > 0.$$

  This can be seen as a special case of the conjecture, because for the trivial subgroup $H = \{1\}$, we have $RH = \mathbb{Z}$. So assuming that the equivalent conditions of Steinberg are fulfilled is a reasonable condition.

We approach this problem by studying the primes in the support and the associated primes of the RG-modules RH_1 and RH_2. For this, results of Segal [Seg68] were crucial. Using them, we were able to connect the strict double coset condition with the occurring supports. If the strict double coset condition is fulfilled, we are able to describe the intersection of the supports of the representation rings:

Proposition (see Proposition 113)**.** *Let G be a compact Lie group and $H_1, H_2 \subseteq G$ two closed subgroups that satisfy the strict double coset condition. Then*

$$\mathrm{Supp}_{RG}(RH_1) \cap \mathrm{Supp}_{RG}(RH_2) = \{\mathfrak{I}\} \cup \{\mathfrak{I} + (p) \mid p \in \mathbb{Z} \textit{ prime } \}$$

where $\mathfrak{I}$ denotes the augmentation ideal given by the kernel of the rank map $RG \to \mathbb{Z}$.

In our setting, the ideals in this intersection do not appear as associated primes:

Proposition (see Proposition 114)**.** *Let G be a compact connected Lie group and $H \subseteq G$ a closed connected subgroup of G with $\mathrm{rank}(H) \geq 1$. Then the augmentation ideal $\mathfrak{I}$ of RG is not an associated prime of RH, nor is $\mathfrak{I} + (p)$ for any prime $p \in \mathbb{Z}$*

Using these results, we are able to prove the conjecture in the following case:

Theorem (see Theorem 115)**.** *Let G, H_1 and H_2 be as in Conjecture 100. In addition, assume that $\mathrm{rank}(H_1) \leq 1$ or $\mathrm{rank}(H_2) \leq 1$. Then Conjecture 100 holds.*

We also present an approach for the general case:

Proposition (see Proposition 117)**.** *Let G be a compact Lie group and H_1, H_2 two closed subgroups of G satisfying the strict double coset condition. In addition, let $\mathfrak{I} \subseteq RG$ be the augmentation ideal of RG and $i \in \mathbb{N}_{>0}$. If there exists an element $a \in \mathfrak{I}$ for which the multiplication*

$$\mathrm{Tor}_i^{RG}(RH_1, RH_2) \xrightarrow{\cdot a} \mathrm{Tor}_i^{RG}(RH_1, RH_2)$$

is injective, then $\mathrm{Tor}_i^{RG}(RH_1, RH_2) = 0$.

For compact connected Lie groups G, we propose a choice for a concrete element $a \in \mathfrak{I} \subseteq RG$. With this choice of element, we are able to prove the following:

Proposition (see Proposition 124)**.** *Conjecture 100 holds if $G = T$ is a torus.*

Structure of the Thesis:

This thesis is structured as follows: The purpose of the first two chapters is to outline the fundamentals in the two mathematical fields with which we will work in this thesis: *representation theory* and *commutative algebra*.

The first chapter starts with a brief introduction of Lie groups in which we also introduce maximal tori and the Weyl group of compact connected Lie groups. In the next section we deal with representation theory starting a bit more general with arbitrary topological groups and finally introducing the representation ring of compact Lie groups. Afterwards, we take a closer look at the representation rings of tori. The next section deals with the augmentation ideal of the representation rings of compact connected Lie groups. Finally, we take a closer look at the previously introduced concepts by studying classical examples of Lie groups.

The second chapter's purpose is to introduce some tools from commutative algebra which we will use when studying the representation rings as modules. After the introduction of some basic terms in the first section, the support and associated primes of modules are studied. The third section deals with the Tor functors, the derived functors of the tensor product. The last section introduces regular sequences and the Koszul complex.

The third chapter is the main part of this thesis. We start by introducing biquotient manifolds. In this context, we also consider some examples of biquotient manifolds that appear in Eschenburgs classification of biquotients. In the next section, we introduce the strict double coset condition and see how it is connected to biquotient manifolds. Afterwards, we again state the conjecture and take a look at the theorems of Singhof, Pittie and Steinberg. Then, we examine the primes that appear in $\mathrm{Supp}_{RG}(RH)$ and $\mathrm{Ass}_{RG}(RH)$. In the next section, this study of primes comes to use when proving our conjecture in case that one of the occurring subgroups is of rank 1 or lower. Finally, we conclude by presenting an approach for the general case and analyze which further problems may occur.

Afterwards, we conclude by a summary of the results.

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Chapter 1

Representation Theory of Compact Lie Groups

In this thesis the objects which we will study are representation rings of certain compact Lie groups. The idea behind representation theory is to describe the elements of an abstract group by mapping them to the automorphism group of a vector space. So after the choice of a basis, the group elements are mapped to (not necessarily distinct) matrices. Representations of groups are of great interest since they make it possible to examine abstract groups with tools used in Linear Algebra. In addition, they are of computational use since computers can handle the representing matrices more easily.

At the beginning of the first section, we will briefly introduce Lie groups. Then the focus will be set on maximal tori contained in Lie groups and the Weyl group. In the second section the representation ring of compact Lie groups will be introduced, mainly following the definitions in [Ada69]. Afterwards, the special case of representation rings of tori will be considered in more detail. This includes taking a look at the Weyl group action on the representation rings and how this connects the representation rings of a compact connected Lie group and a maximal torus of the group. In the fourth section, the augmentation ideal of representation rings will be defined and studied. Finally, we conclude by taking a look at the representation rings and the Weyl group action on it for some classical examples.

1.1 Basic Definitions

We will start by a quick reminder of the definition of Lie groups:

Definition 1 (Lie group)**.** *A* Lie group *G is a smooth manifold equipped with* *a group structure, such that the maps:*

$$\begin{aligned} G \times G &\longrightarrow G, \quad & (a,b) &\longmapsto ab \\ G &\longrightarrow G, \quad & a &\longmapsto a^{-1}, \end{aligned}$$

given by the product and inverse elements in the group, are smooth. A homomorphism of Lie groups *is a map $f: G \to H$ that is both a group homomorphism and smooth. An* isomorphism of Lie groups *is a homomorphism that has an inverse.*

Some common examples of Lie groups are:

- $\mathbb{R}^n$ for an $n \in \mathbb{N}$ as a group under the component-wise addition.
- S^1 as a multiplicative group.
- $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\dots$.

Next, we list some examples for Lie groups that are isomorphic to S^1:

Example 2.
- *S^1 is isomorphic to $\mathbb{R}/\mathbb{Z}$ as Lie groups. The isomorphism is given by the exponential map $\mathbb{R}/\mathbb{Z} \to S^1$, $a \mapsto e^{2\pi i a}$.*
- *$\mathbb{R}/\mathbb{Z}$ and $\mathrm{SO}(2)$ are isomorphic as Lie groups: Every element in $\mathrm{SO}(2)$ is of the form $B_\varphi = \begin{pmatrix} \cos(2\pi\varphi) & -\sin(2\pi\varphi) \\ \sin(2\pi\varphi) & \cos(2\pi\varphi) \end{pmatrix}$ for a $\varphi \in [0,1)$. Multiplication of B_{φ_1} and B_{φ_2} results in $B_{\varphi_1+\varphi_2}$, an addition of the arguments. The map given by $\mathbb{R}/\mathbb{Z} \to \mathrm{SO}(2)$, $a \mapsto B_a$ yields an isomorphism of Lie groups.*

One very important Theorem when dealing with Lie groups is the Closed-Subgroup Theorem of Élie Cartan, sometimes also referred to as Cartan's theorem:

Theorem 3 (Closed-Subgroup Theorem)**.** *Let G be a Lie group and H a closed subgroup of G. Then H is an embedded Lie subgroup. An embedded Lie subgroup H of G is a subgroup equipped with the subspace topology and a smooth structure so that it is a Lie group and an embedded submanifold, i.e. the inclusion map $H \hookrightarrow G$ is a smooth embedding with respect to the topology and the smooth structure.*

In particular, the inclusion $H \hookrightarrow G$ is a homomorphism of Lie groups.

Proof. See [Lee13, Chapter 20, Theorem 20.12]. □

So every closed subgroup of a Lie group is a Lie group as well. We will frequently use this theorem implicitly.

In this thesis, we will work a lot with tori. These are defined as follows:

Definition 4 (torus)**.** *A* torus *is a Lie group that is isomorphic to* $T^k = \underbrace{S^1 \times \cdots \times S^1}_{k}$ *for a* $k \in \mathbb{N}$.

Tori can be classified in the following way:

Proposition 5. *Let T be a Lie group. The following two statements are equivalent:*

1. *T is a torus.*

2. *T is compact, connected and abelian.*

Proof. See [Ada69, Chapter 2, Corollary 2.20]. □

Tori often occur as (closed) subgroups of Lie groups. For compact, connected Lie groups G, there are maximal subtori:

Definition 6. *Let G be a compact connected Lie group. A* maximal torus *of G is a subgroup $T \subseteq G$ which satisfies the following conditions:*

1. *T is a torus.*

2. *If T' is a torus with $T \subseteq T' \subseteq G$ then $T = T'$.*

These maximal tori are a powerful tool to study the respective group.

Proposition 7. *Let G be a compact connected Lie group and $T' \subseteq G$ a subtorus. Then there exists a maximal torus T containing T'.*

The following proof is based on [BtD85, Chapter IV.1, paragraph after (1.1) Definition]:

Proof. We will prove this using Zorn's lemma. Consider the set

$$\mathfrak{J} = \{T \subseteq G \mid T \text{ is a subtorus containing } T'\}.$$

It is not empty since $T' \in \mathfrak{J}$. Together with the inclusion $\subseteq$ the set $\mathfrak{J}$ is partially ordered.

Every Lie group G is a manifold and thus a Hausdorff space. As a compact subspace of a Hausdorff space, every subtorus $T \subseteq G$ is closed in G. By the Closed-Subgroup Theorem T is a submanifold of G. In particular if we have two subtori $T_1 \subsetneq T_2 \subseteq G$ then T_1 is a closed submanifold of T_2. Since tori are connected it follows that $\dim T_1 < \dim T_2$. Otherwise, T_1 would be an open subset of T_2 and since T_1 is closed and T_2 is connected we would have $T_1 = T_2$.

The connected manifold G is of finite dimension. This yields that every strict chain of the form $T' \subsetneq T_1 \subsetneq T_2 \subsetneq \cdots \subsetneq G$ has to be finite. So every arbitrary chain in $\mathfrak{J}$ stabilizes and thus has an upper bound in $\mathfrak{J}$. Zorn's lemma yields the existence of a maximal element T which is a maximal torus in G containing T'. □

Maximal tori fulfill the following properties:

Proposition 8. *Let G be a compact connected Lie group and $T \subseteq G$ a maximal torus. Then the following statements are true:*

1. *Every element $g \in G$ is conjugate to an element $t \in T$.*

2. *Every element g of G lies in a maximal torus.*

3. *Any two maximal tori are conjugated.*

4. *Any two maximal tori have the same dimension.*

Proof. See [Ada69, Chapter 4, Theorem 4.21, Corollary 4.22, Corollary 4.23, Definition 4.24]. □

So maximal tori are unique up to conjugation. Using Proposition 8.4, we can define the rank of a compact connected Lie group:

Definition 9 (rank)**.** *Let G be a compact connected Lie group and T a maximal torus. Then we define the* rank *of G as the dimension of the maximal torus T and write* $\mathrm{rank}(G) = \dim T$.

Finally, we will define the Weyl group, a tool that frequently appears in the study of Lie groups and their Lie algebras. In this thesis, we will need the Weyl group of compact connected Lie groups to connect the representation ring of a Lie group with the representation ring of a maximal torus which we understand better.

Definition 10 (Weyl group)**.** *Let G be a compact connected Lie group and T a maximal torus of G. Then the* Weyl group *W_T of G (with respect to T) is the quotient N/T where $N = \{g \in G \mid gTg^{-1} = T\}$ is the normalizer of T in G.*

As a group, this does not depend on the choice of a maximal torus [BtD85, Chapter IV.1, paragraph after (1.3) Definition]:

Proposition 11. *The Weyl group of a compact connected Lie group G is unique up to isomorphism.*

Proof. Let T, T' be two maximal tori of G. By Proposition 8 there exists an element $g \in G$ so that $gTg^{-1} = T'$. Let W, W' be the Weyl groups corresponding to T and T', respectively. Then $W = N(T)/T$ and $W' = N(T')/T'$.

We start by showing that $N(T)$ and $N(T')$ are isomorphic as groups: Consider the group homomorphism

$$N(T) \longrightarrow G, \quad n \longmapsto gwg^{-1}$$

given by conjugation with g. For every $n \in N(T)$, the element gng^{-1} is in the normalizer of T':

$$\begin{aligned}(gng^{-1})T'(gng^{-1})^{-1} &= (gng^{-1})T'(gn^{-1}g^{-1}) \\ &= (gn)\underbrace{(g^{-1}T'g)}_{=T}(n^{-1}g^{-1}) \\ &= g\underbrace{(nTn^{-1})}_{=T}g^{-1} = gTg^{-1} = T'.\end{aligned}$$

So conjugation with g yields a group homomorphism $N(T) \to N(T')$.

On the other hand, if we conjugate with g^{-1}, we get a group homomorphism $N(T') \to N(T)$ by similar arguments. As both group homomorphisms are clearly inverse to each other, it follows that $N(T) \cong N(T')$ as groups.

Concatenation with the projection map $\pi\colon N(T') \to T'$ yields the following group homomorphism:

$$N(T) \quad \overset{g()g^{-1}}{\longrightarrow} N(T') \overset{\pi}{\longrightarrow} N(T')/T'.$$

The kernel of this surjective map is T. By the fundamental homomorphism theorem it follows that $N(T)/T \cong N(T')/T'$. $\square$

1.2 Representation Rings of Compact Lie Groups

In this subsection, we will introduce the representation rings of compact Lie groups. These will later become our main object of interest. The initial definitions are kept more general. For these we will assume that G is a *topological group*:

Definition 12 (topological group). *A* topological group *is a group G equipped with a topology so that the maps:*

$$\begin{aligned} G \times G \longrightarrow G, \quad (a, b) &\longmapsto ab \\ G \longrightarrow G, \quad a &\longmapsto a^{-1}, \end{aligned}$$

given by the product and inverse elements in the group are continuous.

As every smooth manifold is in particular a topological space and smooth maps are continuous (see [Lee13, Chapter 2, Proposition 2.4]), every Lie group is a topological group.

Definition 13. *Let G be a topological group. A (complex)* representation *of G is a pair (V, ϕ) consisting of a finite-dimensional complex vector space V and a continuous action*

$$\phi \colon G \times V \to V$$

such that for each $g \in G$ the map $v \mapsto \phi(g, v)$ is $\mathbb{C}$-linear.

It is also common to use the following definition to define complex representations of topological groups:

Definition 14. *Let G be a topological group. Then a* (complex) representation *is a finite-dimensional vector space V over $\mathbb{C}$ together with a continuous homomorphism $\psi \colon G \to \mathrm{Aut}(V)$.*

Firstly, we will show that both definitions can be easily transferred to each other:

Proposition 15. *Let G be a topological group, V a finite-dimensional complex vector space and $\phi \colon G \times V \to V$ and $\psi \colon G \to \mathrm{Aut}(V)$ as in Definition 13 and Definition 14, respectively. Then the following statements are true:*

1. *$(V, \phi^{\#})$ with $\phi^{\#}(g) = (v \mapsto \phi(g, v))$ is a representation of G in terms of Definition 14.*
2. *$(V, \psi_{\#})$ with $\psi_{\#}(g, v) = (\psi(g))(v)$ is a representation of G in terms of Definition 13.*
3. *We have $(\phi^{\#})_{\#} = \phi$ and $(\psi_{\#})^{\#} = \psi$.*

Proof. Firstly we will show that $\phi^{\#}$ and $\psi_{\#}$ maintain their continuity. This is precisely the exponential law for locally compact spaces (see [LS15, Kapitel 4, Satz 4.21 (Exponentialgesetz)]). We can apply this since V is a finite dimensional complex vector space equipped with the norm-induced topology and thus a locally compact space.

Now, we start with $\phi^{\#}$. It is left to show that this is well-defined and indeed a homomorphism of groups. Since ϕ is a group action,we have $\phi(e, v) = v$ for the neutral element $e \in G$. Thus $\phi^{\#}(e)$ is the identity on V. In addition, ϕ being an action yields that for each $a, b \in G$ and each $v \in V$, $\phi(a, \phi(b, v)) = \phi(ab, v)$. In terms of $\phi^{\#}$ this is translated to $\phi^{\#}(a)(\phi^{\#}(b)(v)) = \phi^{\#}(ab)(v)$ and it follows that $\phi^{\#}$ is a homomorphism of groups. By choosing $b = a^{-1}$ we also directly see that every element in the image of $\phi^{\#}$ is invertible and thus in $\mathrm{Aut}(V)$.

For $\psi_{\#}$ we still have to show that it is a group action. Since ψ is a group homomorphism, $\psi(e) = \mathrm{id}_V$ for the neutral element $e \in G$ and thus $\psi_{\#}(e, v) = (\psi(e))(v) = v$. Furthermore, for $a, b \in G$ and $v \in V$ we have $\psi_{\#}(ab, v) = (\psi(ab))(v) = \psi(a)(\psi(b)(v)) = \psi_{\#}(a, \psi_{\#}(b, v))$. So $\psi_{\#}$ satisfies all necessary axioms for group actions.

Finally, the third statement directly follows from the definitions. □

Note that it is also common to replace the field $\mathbb{C}$ with the real numbers $\mathbb{R}$ or the quaternions $\mathbb{H}$. However, in this thesis we will only study complex representations. So when talking about representations, we will always mean complex representations.

Furthermore, we will often denote a representation only by its underlying vector space V. The omitted maps in the sense of both definitions will then be denoted as ϕ_V and ψ_V, respectively. In addition, the expression $\phi_V(g, v) = (\psi_V(g))(v)$ will be shortened to gv.

Definition 16 (rank of representations)**.** *Let G be a topological group and V a representation of G. The rank of V is* $\mathrm{rank}(V) = \dim_{\mathbb{C}}(V)$*, the dimension of V as a complex vector space.*

By choosing a basis of V we can view the map ϕ as a homomorphism $\phi\colon G \to \mathrm{GL}_n(\mathbb{C})$ with $n = \dim_{\mathbb{C}} V$. When such a basis is given, we will also call the representation of G a *matrix representation*. A matrix representation that only takes values in $\mathrm{U}(n)$ is called *unitary representation*.

The easiest way to gain a representation of G is to let G act trivially:

Definition 17. *A representation V of G is called* trivial *if $gv = v$ for all $g \in G$.*

Next, we take a look at more examples:

Example 18. *Here, we give some examples of matrix representations of the n-dimensional torus* $\mathrm{T}^n = (S^1)^n$*:*

1. *Let k be a natural number. The trivial representation of rank k is given by the homomorphism:*

$$T^n \to \mathrm{GL}_k(\mathbb{C}), \qquad t \mapsto \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} = I_k$$

that sends every element of the torus to the unit matrix I_k of rank k.

2. *For every $1 \leq j \leq n$ the projection p_j onto the j-th entry given by:*

$$p_j \colon T^n \to S^1 \subseteq \mathrm{GL}_1(\mathbb{C}), \qquad (d_1, \ldots, d_n) \mapsto d_j$$

is a representation of rank 1*.*

3. *Let $U = \{u_1, \ldots, u_k\}$ be a subset of $\{1, \ldots, n\}$ consisting of k pairwise disjoint elements. Then mapping to the diagonal matrix whose entries consists of the respective projections is a representation of rank k:*

$$p_U \colon T^n \to \mathrm{GL}_k(\mathbb{C}), \qquad (d_1, \ldots, d_n) \mapsto \begin{pmatrix} d_{u_1} & & \\ & \ddots & \\ & & d_{u_k} \end{pmatrix}.$$

For $|U| = 1$ this is precisely the second example above.

4. *For two subsets U_1, U_2 of $\{1, \ldots, n\}$ of the same cardinality, the product $p_{U_1} \cdot p_{U_2}$ that sends $t \in T^n$ to $p_{U_1}(t) \cdot p_{U_2}(t)$ is a representation of T^n.*

5. *For $U \subseteq \{1, \ldots, n\}$ the map p_U^{-1} that sends $t \in T^k$ to the inverse matrix $(p_U(t))^{-1}$ is a representation of G.*

In some proofs of the main part of this thesis, we will need the term *faithful* representation:

Definition 19. *A representation V of G is called* faithful *if ψ_V is injective.*

In our setting, faithful representations always exist:

Theorem 20. *Every compact Lie group G has a faithful representation.*

Proof. See [BtD85, Chapter III.4, (4.1) Theorem]. □

If we consider the representations of a group G as the objects of our study, the next step is to introduce the respective morphisms:

Definition 21. 1. *Let V, W be two representations of G. A* morphism of representations *of G is a $\mathbb{C}$-linear map $f \colon V \to W$ satisfying $f(gv) = gf(v)$ for all $g \in G$, $v \in V$, i.e. a map that commutes with the action of G.*

2. *The set of all such morphisms is denoted as* $\mathrm{Hom}_G(V, W)$*.*

3. *An* isomorphism of representations *of G is a homomorphism that has an inverse.*

4. *Two representations which only differ by an isomorphism are called* equivalent*.*

Let (V, ϕ_V), (W, ϕ_W) be two equivalent representations of G. In particular, V and W are isomorphic as vector spaces over $\mathbb{C}$. Thus by choosing bases, we can substitute V and W by $\mathbb{C}^n$ and take a look at the corresponding matrix representations ψ_V and ψ_W. An isomorphism $f: V \to W$ can be seen as an invertible matrix $A \in \mathrm{GL}_n(\mathbb{C})$ satisfying $A \cdot \psi_V(g) = \psi_W(g) \cdot A$ (or equivalently $A \cdot \psi_V(g) \cdot A^{-1} = \psi_W(g)$) for all $g \in G$.

There are some parallels between ordinary $\mathbb{C}$-linear maps and morphisms of representations:

Proposition 22. *Let $f: V \to W$ be a morphism of representations. Then the following statements are equivalent:*

1. *f is an isomorphism.*

2. $\mathrm{Ker}(f) = \{0\}$ *and* $\mathrm{Im}(f) = W$*.*

Proof. The first implication is trivial since every morphism of representations is especially a $\mathbb{C}$-linear map. So we have to show that $\mathrm{Ker}(f) = \{0\}$ and $\mathrm{Im}(f) = W$ imply that there exists an inverse morphism of representations. From linear algebra it is well known that for a linear map these conditions imply that the inverse map is a linear map as well. Let f^{-1} be this inverse linear map. We have to show that f^{-1} respects the action of G. For $w \in W$ there exists a unique $v \in V$ so that $f(v) = w$. Then $gf^{-1}(w) = gv = f^{-1}(f(gv)) = f^{-1}(gf(v)) = f^{-1}(gw)$ and thus f^{-1} is a morphism of representations. □

If we have two representations V and W of G, it is possible to construct a new representation using them. This will be done by using methods of constructing vector spaces: *direct sums* and *tensor products*. We will start with direct sums:

Definition 23 (direct sums of representations)**.** *Let G be a topological group and V, W two representations of G. The direct sum of V and W is given by the direct sum $V \oplus W$ as a complex vector space together with the continuous action given by $g(v, w) = (gv, gw)$ for all $g \in G$, $v \in V$, $w \in W$.*

The constructed map in the definition above is clearly continuous since both ϕ_V and ϕ_W are continuous. The conditions for being an action are directly transferred as well. The rank of $V \oplus W$ is given by: $\operatorname{rank}(V \oplus W) = \operatorname{rank}(V) + \operatorname{rank}(W)$. By considering the maps that permute the respective entries, one can see that taking direct sums is associative and commutative up to equivalence.

If we choose two bases $v_1, \ldots, v_n \in V$ and $w_1, \ldots, w_m \in W$, we can combine them to a basis $(v_1, 0), \ldots, (v_n, 0), (0, w_1), \ldots, (0, w_m)$ of $V \oplus W$. Considering the corresponding matrix representations, the direct sum of the representations $\psi_1 \colon G \to GL_{n_1}(\mathbb{C})$ and $\psi_2 \colon G \to GL_{n_2}(\mathbb{C})$ is given by:

$$g \mapsto \left(\begin{smallmatrix} \psi_1(g) & 0 \\ 0 & \psi_2(g) \end{smallmatrix}\right).$$

Example 24. *Again, consider the n-dimensional torus $G = T^n$. Let $1 \leq i < j \leq n$. The direct sum of the projections p_i and p_j is of the form:*

$$p_i \oplus p_j \colon T^n \longrightarrow \mathrm{GL}_2(\mathbb{C}), \qquad (d_1, \ldots, d_n) \longmapsto \left(\begin{smallmatrix} d_i & \\ & d_j \end{smallmatrix}\right)$$

which coincides with $p_{\{i,j\}}$.

Next, we construct new representations using the tensor product of vector spaces:

Definition 25 (tensor products of representations)**.** *Let G be a topological group and V, W two representations of G. The tensor product of the two representations is given by the tensor product $V \otimes_{\mathbb{C}} W$ of the two complex vector spaces together with the continuous action given on pure tensors by $g(v \otimes w) = gv \otimes gw$. For other elements in $V \otimes W$ the action of g is linearly extended.*

One can easily see that tensor products of representations are representations as well. The rank of $V \otimes W$ is the product of the respective ranks of V and W, i.e. $\operatorname{rank}(V \otimes W) = \operatorname{rank}(V) \operatorname{rank}(W)$. Again, the tensor product of representations is commutative and associative up to equivalence.

For $V = \mathbb{C}^n$ and $W = \mathbb{C}^m$, we can choose the standard bases $v_1, \ldots, v_n \in V$ and $w_1, \ldots, w_m \in W$. Then $v_1 \otimes w_1, \ldots, v_1 \otimes w_m, v_2 \otimes w_1, \ldots, v_n \otimes w_m$ is a basis for $V \otimes W$. Then for each $g \in G$, $\psi_{V \otimes W}(g)$ is the Kronecker product of $\psi_V(g)$ and $\psi_W(g)$.

Example 26. *Again, we take a look at the case $G = T^n$. The tensor product of two projections p_i and p_j is of the form:*

$$p_i \otimes p_j \colon T^n \longrightarrow GL_1(\mathbb{C}), \qquad (d_1, \ldots, d_n) \longmapsto d_i d_j$$

Additionally, one can show that up to equivalence, the direct sum (as addition) together with the tensor product (as multiplication) fulfill the distributive law.

In the setting of compact Lie groups, instead of taking values in $\mathrm{GL}_n(\mathbb{C})$ it is also possible to restrict ourselves to $\mathrm{U}(n)$. For this we need to equip the complex vector space V with an appropriate positive definite Hermitian form:

Proposition 27. *Let G be a compact Lie group and V a representation of G. Then there exists a positive definite Hermitian form H on V which is invariant under G, i.e. $H(v, w) = H(gv, gw)$ for all $g \in G$.*

The proof sketch underneath is from [Ada69, Chapter 3, 3.14 – 3.16]:

Proof sketch. The idea behind this is to choose an arbitrary positive definite Hermitian form H' on V and integrate $H'(g^{-1}v_1, g^{-1}v_2)$ over G, i.e.:

$$H(v, w) = \int_{g \in G} H'(g^{-1}v, g^{-1}w) dg.$$

It uses the *Haar integral*, in the special case of compact Lie groups also known as *invariant integral*. We will leave the concrete definition of this integral as a black box and refer to [Nac65] for more information. Using the properties of this integral, one can show that H is an invariant, positive definite Hermitian form on V. □

By equipping V with a positive definite Hermitian form, we can show that V is equivalent to an unitary representation:

Proposition 28. *Every representation of a compact Lie group is equivalent to an unitary representation.*

The following proof is based on [Ada69, Chapter 3, remark after 3.16 Proposition]:

Proof. Let G be a compact Lie group and V a representation of G. By Proposition 27, we can equip V with a positive definite Hermitian form H. Using Gram-Schmidt, we can find an orthonormal basis $a_1, \ldots, a_n$ of V with respect to H, i.e. $H(a_i, a_i) = 1$ and $H(a_i, a_j) = 0$ for all $1 \leq i, j \leq n$, $i \neq j$. As H is invariant under the action of G, we have $H(ga_i, ga_i) = 1$ and $H(ga_i, ga_j) = 0$ for all $1 \leq i, j \leq n$, $i \neq j$ and all $g \in G$.

We equip V with the basis $a_1, \ldots, a_n$ and thus can regard it as $V \cong \bigoplus_{i=1}^n \mathbb{C}a_i \cong \mathbb{C}^n$. Elements of V are then written as vectors $(c_1, \ldots, c_n) \in \mathbb{C}^n$

which correspond to the sum $\sum_{i=1}^{n} c_i a_i \in V$. The positive definite Hermitian form H is the standard scalar product on this complex vector space: For two vectors $c = (c_1, \dots, c_n), d = (d_1, \dots, d_n) \in \mathbb{C}$ their standard scalar product is given by $< c, d > = \sum_{i=1}^{n} \overline{c_i} d_i$. Applying H yields:

$$H(c,d) = H\left(\sum_{i=1}^{n} c_i a_i, \sum_{i=1}^{n} d_i a_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{c_i} d_j H(a_i, a_j) = \sum_{i=1}^{n} \overline{c_i} d_i.$$

With respect to our choice of basis, $\psi_V : G \to \mathrm{Aut}(V)$ sends every element $g \in G$ to a matrix $A_g \in \mathrm{GL}_n(\mathbb{C}^n)$. We show that A_g is unitary: A matrix A_g is unitary if and only if the columns of A_g are an orthonormal basis with respect to the standard complex scalar product. Here, the columns are given by $ga_1, \dots, ga_n$ which are orthonormal with respect to H, the standard scalar product.

For a matrix representation with a different choice of basis, one may take the respective change-of-basis matrix which yields an isomorphism of representations in the sense of the remark after Definition 21. □

Using the previous proposition, from now on, we will assume that all matrix representations of compact Lie groups are unitary.

Later, we will construct the representation ring of a compact Lie group using equivalence classes of *irreducible* representations of G. To define irreducibility, we have to start with the term of *subrepresentations*:

Definition 29. *Let G be a topological group and V a representation of G. A* subrepresentation *of V is a subspace $U \subseteq V$ which is invariant under G, i.e. $gu \in U$ for all $u \in U$.*

A common way to get a subrepresentation of a given representation is to take a look at the images and kernels of morphisms:

Proposition 30. *Let G be a topological group, V and W two representations of G and $f : V \to W$ a morphism of representations. Then $\mathrm{Ker}(f) \subseteq V$ and $\mathrm{Im}(f) \subseteq W$ are subrepresentations of V and W, respectively.*

Proof. The kernel and image are vector subspaces of V and W, respectively. It is left to show that they are invariant under G. For an element $v \in \mathrm{Ker}(f)$ and a group element $g \in G$ we get that $f(gv) = gf(v) = g0 = 0$ and thus gv is in the kernel as well. Similarly, for a $w \in \mathrm{Im}(f)$ there exists a $v \in V$ with $f(v) = w$ and for every $g \in G$ we find that $gw = gf(v) = f(gv) \in \mathrm{Im}(f)$. □

Irreducible representations are representations that only have trivial subrepresentations:

Definition 31. *A representation V of G is called* irreducible *if it is nonzero and if its only submodules are $\{0\}$ and V. Otherwise, we will call V* reducible.

In particular, representations of rank 1 are irreducible for dimensional reasons.

For compact Lie groups, to understand representations it suffices to study the irreducible ones:

Proposition 32. *Let G be a compact Lie group. Then every representation is a direct sum of irreducible representations.*

The following proof is taken from [Ada69, Chapter 3, Theorem 3.20]:

Proof. Let V be a representation. If V is reducible, there exists a subrepresentation U of V that is not trivial. Using Proposition 27 we can choose an invariant positive definite Hermitian form H on V. Now we define $W = \{w \in V \mid H(u,w) = 0 \text{ for all } u \in U\}$ as the orthogonal complement of U in V. Since H is a positive definite Hermitian form, the vector space V is the direct sum of U and its orthogonal complement, i.e. $V = U \oplus W$ (see [Bos14, Kapitel 7.2, Korollar 8]). It is left to show that W is invariant under the action of G, i.e. $gw \in W$ for all $g \in G$ and $w \in W$. We know that H is invariant under G, i.e. $0 = H(u,w) = H(gu, gw)$ for all $u \in U$, $w \in W$, $g \in G$. The element $g^{-1}u$ is in U because U is a subrepresentation of V. If we substitute u with $g^{-1}u$ in the previous equation we get that $0 = H(u, gw)$ for all $g \in G$, $u \in U$, $v \in W$. This means that gw is in the orthogonal complement of U and thus W is invariant under G.

If U and W are irreducible, we are finished. Otherwise we can continue inductively on V and W. Note that this induction will stop at some point because of the decreasing ranks. $\square$

The following theorem is extremely useful when studying morphisms of representations:

Theorem 33 (Schur's Lemma)**.** *Let G be a topological group and V, W two irreducible representations of G. Then the following statements are true:*

1. *Every morphism $f\colon V \to W$ is either zero or an isomorphism.*
2. *Every morphism $h\colon V \to V$ is of the form $h(v) = \lambda v$ for some constant $\lambda \in \mathbb{C}$.*

The following proof is taken from [BtD85, Chapter II.1, (1.10) Theorem]:

Proof. In Proposition 30 we have seen that the kernel and the image of a homomorphism of representations are subrepresentations. If both V and W are irreducible then $\mathrm{Ker}(f) \in \{\{0\}, V\}$ and $\mathrm{Im}(f) \in \{\{0\}, W\}$. The case $\mathrm{Ker}(f) = V$ is equivalent to the case $\mathrm{Im}(f) = \{0\}$ and is precisely the case when f is zero. By Proposition 22 the remaining case yields that f is an isomorphism. This proves the first statement.

For the second statement we choose an eigenvalue λ of the linear map h. Let $U = \{v \in W \mid h(v) = \lambda v\}$ be the eigenspace of λ. The vector subspace $U \subseteq V$ is a subrepresentation of V since for $u \in U$ we have $h(gu) = gh(u) = g(\lambda u) = \lambda gu$. By definition U cannot be zero. So V being irreducible yields that $U = V$ and thus $h(v) = \lambda v$ for all $v \in V$. □

For two representations V, W of G, the usual complex vector space structure on the set of $\mathbb{C}$-linear maps between V and W yields a complex vector space structure on $\mathrm{Hom}_G(V, W)$. Using Schur's Lemma, the following can be concluded [BtD85, Chapter II.1, (1.10) Theorem (iii)]:

Corollary 34. *Let V and W be two irreducible representations of G. Then the following statements are true:*

1. *If V and W are not equivalent then* $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, W) = 0$.
2. *If V and W are equivalent then* $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, W) = 1$.

Proof. By Schur's Lemma, every morphism between irreducible representations is either an isomorphism or zero. If V and W are not equivalent then the zero map is the only morphism and thus $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, W) = 0$.

Now assume that $V \cong W$. If we fix an isomorphism $g\colon W \to V$ then every morphism $f \in \mathrm{Hom}_G(V, W)$ can be seen as a morphism $V \to V$ by taking $g \circ f$. In Schur's Lemma these are characterized as linear maps of the form $f(v) = \lambda v$ for $\lambda \in \mathbb{C}$. Thus it follows that $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, W) = 1$. □

In Proposition 32, we have already seen that any representation of G can be expressed as a direct sum of irreducible representations. Next, we will see that these irreducible summands are unique up to equivalence:

Proposition 35. *Let G be a topological group and $V_1, \dots, V_k$, $k \in \mathbb{N}$, pairwise non-isomorphic, irreducible representations of G. In addition, let $m_i, n_i \in \mathbb{N}$, $1 \le i \le k$, be natural numbers. If the representations $\bigoplus_{i=1}^k m_i V_i$ and $\bigoplus_{i=1}^k n_i V_i$ are equivalent then $m_i = n_i$ for all $1 \le i \le n$.*

The following proof is taken from [Ada69, Chapter 3, 3.24 Theorem]:

Proof. Let $\bigoplus_{i=1}^k m_iV_i$ and $\bigoplus_{i=1}^k n_iV_i$ be equivalent representations. We fix an index $1 \leq j \leq k$. Since the representations above only differ by an isomorphism, we have: $\mathrm{Hom}_G(V_j, \bigoplus_{i=1}^k m_iV_i) \cong \mathrm{Hom}_G(V_j, \bigoplus_{i=1}^k n_iV_i)$ for all $1 \leq j \leq k$. Since morphisms of representations are in particular linear maps between vector spaces, we can use the universal property of the product in the category of vector spaces and gain that: $\bigoplus_{i=1}^k m_i \mathrm{Hom}_G(V_j, V_i) \cong \bigoplus_{i=1}^k n_i \mathrm{Hom}_G(V_j, V_i)$ as complex vector spaces. By Corollary 34(1) we have $\mathrm{Hom}_G(V_j, V_i) = 0$ for $i \neq j$ since V_j and V_i are not equivalent. So it follows that $m_i \mathrm{Hom}_G(V_j, V_j) \cong n_i \mathrm{Hom}_G(V_j, V_j)$. By taking the dimension as a complex vector space and using Corollary 34(2), it follows that:

$$m_i = \dim_{\mathbb{C}}(m_i \mathrm{Hom}_G(V_j, V_j)) = \dim_{\mathbb{C}}(n_i \mathrm{Hom}_G(V_j, V_j)) = n_i.$$

□

Now let G be a compact topological group. Using the irreducible representations of G we will construct an invariant of G. Let $H_G = \{[V] \mid V \text{ is an irreducible representation of } G\}$ be the set of equivalence classes of irreducible representations of G. When the context is clear, we will usually omit the brackets and just write $V \in H_G$ for the respective equivalence class of V. We define $R(G)$ as the free abelian group generated by the elements of H_G, i.e.

$$R(G) = \left\{ \sum_{V \in H_G} \lambda_V V \;\middle|\; \lambda_V \in \mathbb{Z},\, \lambda_V = 0 \text{ for all but finitely many } V \in H_G \right\}.$$

By Proposition 32 every representation is a direct sum of irreducible representations. So for every equivalence class of an arbitrary representation V of G there is an element $\sum_{i=1}^n \lambda_i V_i \in R(G)$ with $\lambda_i \geq 0$ so that $V \cong \bigoplus_{i=1}^n \lambda_i V_i$. Proposition 35 yields that these elements are uniquely determined. In addition, by taking the direct sum of the representations, every element in $R(G)$ containing only positive coefficients yields a representation of G. So for compact groups G the subset

$$R(G)^+ = \left\{ \sum_{V \in H_G} \lambda_V V \;\middle|\; \lambda_V \in \mathbb{N},\, \lambda_V = 0 \text{ for all but finitely many } V \in H_G \right\}$$

is in a one-to-one correspondence with the equivalence classes of representations of G. The identity element of addition is the formal sum in which every coefficient vanishes. It corresponds to the unique representation of rank zero. So the addition in $R(G)$ behaves in the same way as the direct sum of representations of G. Furthermore, the elements in $R(G)$ can (uniquely) be written as formal differences of elements in $R(G)^+$ by sorting by the sign of the coefficients:

$$\sum_{V \in H_G} \lambda_V V = \left(\sum_{V \in H_G} \max\{\lambda_V, 0\} V \right) - \left(\sum_{V \in H_G} \max\{-\lambda_V, 0\} V \right).$$

Direct sums together with tensor products satisfy the distributive law (up to equivalence). So we can equip $R(G)$ with a multiplication using the tensor product. This makes $R(G)$ into a ring with the trivial representation of rank 1 as the identity element of multiplication:

Definition 36 (representation ring)**.** *Let G be a compact topological group. We call $R(G)$ the* (complex) representation ring *of G.*

We will sometimes omit the brackets and only write RG instead of $R(G)$. Another, more formal approach to this is the concept of *Grothendieck groups*. It is a method to construct a group $\mathcal{G}(A)$ out of an abelian monoid A so that a certain universal property is satisfied. For the formal construction, see [Wei13, Chapter II.1]. In addition, if A is a semiring, $\mathcal{G}(A)$ yields a ring structure. Here, $R(G)^+$ together with the direct sum and the tensor product is a semiring and $R(G)$ is the Grothendieck group of $R(G)^+$ (see [BtD85, beginning of Chapter II.7]).

Now, let G, H be two compact Lie groups and $f\colon H \to G$ a morphism of Lie groups. If V is a representation of G with $\psi\colon G \to \mathrm{Aut}(V)$ we can consider the concatenation $\psi \circ f\colon H \to \mathrm{Aut}(V)$. As both maps are group homomorphisms, so is $\psi \circ f$. It is continuous as well, since f is smooth and thus in particular continuous. So $\psi \circ f\colon H \to \mathrm{Aut}(V)$ is a representation of H.

This construction respects direct sums and tensor products of representations: Let $V \oplus W$ be a direct sum of representations of G with $g \in G$ acting by $g(v,w) = (gv, gw)$. Then H acts on $V \oplus W$ by $f(h)(v,w) = (f(h)v, f(h)w)$ for all $h \in H$. This is the direct sum of the representation of H induced by the respective representations of G using f. A similar argument works for tensor products. In addition, the rank of a representation does not change and trivial representations of G are sent to the respective trivial representations of H.

Furthermore, if V_1 and V_2 are two equivalent representations of G, the respective representations of H are equivalent as well: If $l\colon V_1 \to V_2$ is an morphism of representations, we have $l(gv) = gl(v)$ for all $v \in V$ and $g \in G$. For $h \in H$ we have $hv = f(h)v$ and so $l(hv) = l(f(h)v) = f(h)l(v) = hl(v)$. Thus l is a morphism of representation for the respective representations of H. If l is an isomorphism, we can apply the same argument for the inverse.

So we have a map $f^*\colon R(G)^+ \to R(H)^+$ respecting the addition and multiplication, i.e. a semiring homomorphism. We can extends this to the formal sums $V_1 - V_2$ with $V_1, V_2 \in R(G)^+$ in $R(G)$ by $f^*(V_1) - f^*(V_2)$ and thus get a ring homomorphism $R(G) \to R(H)$. Another way to see this is to use the contravariant functorial properties of $\mathcal{G}(-)$. This sums up to:

Proposition 37. *Let G, H be two compact Lie groups. Then every homomorphism of Lie groups $f\colon H \to G$ induces a ring homomorphism $f^*\colon RG \to RH$.*

1.3 Representation Rings of Tori

After introducing representation rings of arbitrary compact topological groups, we will take a closer look at tori. We will see that their representation rings are always Laurent rings generated by the representations given by projecting to the respective components of $T^k = (S^1)^k$. In addition, we will see that the representation ring of a maximal torus of compact connected Lie groups and the representation ring of the Lie group are related. The elements in RG are exactly the elements of RT which are invariant under the action of the Weyl group.

Since the representation ring is generated by the irreducible representation, we will start by taking a look at these:

Proposition 38. *Let T be a torus. Then the irreducible representations of T are of rank* 1.

The following proof is from [Ada69, Chapter 3, 3.71 Proposition]:

Proof. Let V be an irreducible representation of T with $\psi\colon G \to \mathrm{Aut}(V)$. If we fix an element $g \in G$ we can consider the linear map $\psi(g)\colon V \to V$. This yields a morphism of representations $\psi(g)\colon V \to V$ since for all $g' \in T$, $v \in V$ we have:

$$\begin{aligned}\psi(g)(g'v) &= \psi(g)(\psi(g')(v)) = \psi(gg')(v)\\ &\overset{T \text{ is abelian}}{=} \psi(g'g)(v) = \psi(g')(\psi(g)(v)) = g'\psi(g)(v).\end{aligned}$$

By Schur's Lemma (Theorem 33), we know that for an irreducible V, morphisms of the form $V \to V$ are scalar multiplications by a $\lambda \in \mathbb{C}$.

So for every $g \in T$ the linear map $\psi(g)\colon V \to V$ is a scalar multiplication with a scalar $\lambda(g) \in T$. Every subspace of a complex vector space V is invariant under scalar multiplication and thus invariant under the action of T. So every subspace of V yields a subrepresentation of V. Since V is irreducible, there are only trivial subrepresentations and so it follows that V is one-dimensional. $\square$

Note that the previous proof works the same for arbitrary abelian topological groups G. Proposition 28 allowed us to assume that all matrix representations of compact Lie groups are unitary, thus the scalars $\lambda(g)$, $g \in T$, in the previous proposition are in S^1.

We can equip the irreducible representations of T with a multiplication given by the pointwise multiplication of the respective maps $T \to S^1$. The trivial representation of rank 1 acts as the neutral element. There are also inverse elements since for a representation $\psi \colon T \to S^1$ the map $\psi' \colon T \to S^1$ sending t to $\psi(t)^{-1}$ is a representation as well. This yields a structure of an abelian group on the set of irreducible representation of T:

Definition 39 (character group)**.** *We call the set of irreducible representation of a torus T the* character group $\chi(T) = \mathrm{Hom}_T(T, S^1)$ *and an element of $\chi(T)$ a* character *of T.*

Every character is a product of characters given by the projections:

Proposition 40. *The characters of a k-dimensional torus T^k are all of the following form:*

$$\begin{aligned} S^1 \times \cdots \times S^1 &\longrightarrow S^1 \\ (x_1, \ldots, x_k) &\longmapsto x_1^{a_1} \cdot \ldots \cdot x_k^{a_k} \end{aligned}$$

for $a_1, \ldots, a_k \in \mathbb{Z}$.

Proof. See [BtD85, Chapter II.8, (8.1) Proposition]. $\square$

If we remember Example 18, we see that every character of a torus can be expressed as a combination of the second, forth and fifth subitem of the example, i.e. every character can be obtained by using projections, their inverses and products.

The proposition yields that $\chi(T^k)$ is isomorphic to $\mathbb{Z}^k$ as an abelian group. The generators are given by the projections $\pi_j \colon T^k \to S^1$ sending $(x_1, \ldots, x_k) \in (S^1)^k$ to $x_j \in S^1$.

The representation ring of T can be expressed using the character group:

Proposition 41. *Let T be a torus. Then the representation ring of T is canonically isomorph to the group ring $\mathbb{Z}[\chi(T)]$ over the abelian group $\chi(T)$.*

Proof. See [BtD85, Chapter II.8, (8.3) Proposition]. $\square$

As for a k-dimensional torus T the character group $\chi(T)$ is a free abelian group of rank k, the representation ring RT is of the form:

$$RT = \mathbb{Z}[x_1^{\pm 1}, \dots, x_k^{\pm 1}],$$

the ring of Laurent polynomials over k variables where x_j represents the projection to the j-th factor. It directly follows that:

Corollary 42. *The representation ring RT of a torus T is an integral domain.*

For a subtorus $T' \subseteq T$, we may describe the representation rings of T and T' in an intuitive way:

Proposition 43. *Let T be a torus and $T' \subseteq T$ a subtorus. Then the representation rings of T and T' are of the following form:*

$$RT = \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$
$$RT' = \mathbb{Z}[x_1^{\pm 1}, \dots, x_k^{\pm 1}]$$

with $n = \operatorname{rank}(T)$ and $k = \operatorname{rank}(T') \leq n$. The ring homomorphism $RT \to RT'$ induced by the inclusion is given by the map that sends x_j to x_j for $j \leq k$ and to 1 for $j > k$.

The following proof is based on notes in [Zib]:

Proof. As shown in [HN12, Chapter 15.3.2, Lemma 15.3.2] we can assume that T' is a direct factor of T, i.e. there exists a subtorus $S \subseteq T$ so that $T \cong T' \times S$ as a Lie group. The isomorphism is given by the multiplication map sending $(t, s) \in T' \times S$ to $ts \in T$.

So we have an isomorphism $T \cong (S^1)^{\operatorname{rank}(T')} \times (S^1)^{\operatorname{rank}(T) - \operatorname{rank}(T')}$ in which the first $\operatorname{rank}(T')$ factors belong to T' and the remaining factors to S. The representation ring of T is a Laurent ring generated by the projection to the respective S^1-factors. So we can write:

$$RT \cong \mathbb{Z}[x_1^{\pm 1}, \dots, x_{\operatorname{rank}(T')}^{\pm 1}, x_{\operatorname{rank}(T')+1}^{\pm 1}, \dots, x_{\operatorname{rank}(T)}^{\pm 1}]$$

in which x_i, with $i \leq \operatorname{rank}(T')$, are the projections to factors of T' and the remaining x_j, $j > \operatorname{rank}(T')$, the projections to factors of S. As T' is a torus, its representation ring is generated by the projections as well. The generators are the restrictions of the respective generators of RT and thus the representation ring is of the form:

$$RT' = \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_{\operatorname{rank}(T')}^{\pm 1}].$$

Restricted to T', the projections to factors of S become trivial. So the induced map by the inclusion $T' \subseteq T$ is of the form stated in the proposition. $\square$

Now, let G be a compact connected Lie group and T a maximal torus of G. Let W be the Weyl group of G (with respect to T). The Weyl group W acts on T by conjugation. This yields an action on the representations of T: Let $\psi\colon T \to \mathrm{Aut}(V)$ be a representation of T and $w \in W$. Then consider the following map:

$$\psi_w\colon T \longrightarrow \mathrm{Aut}(V), \quad t \mapsto \psi(wtw^{-1}).$$

This is well-defined since $wTw^{-1} = T$ for all elements w in the Weyl group. It is continuous since ψ is continuous. In addition, it fulfills:

$$\begin{aligned}\psi_w(tt') &= \psi(wtt'w^{-1}) = \psi(wtw^{-1}wtw^{-1}) \\ &= \psi(wtw^{-1})\psi(wt'w^{-1}) = \psi_w(t)\psi_w(t').\end{aligned}$$

So ψ_w is representation of T.

The action of W commutes with direct sums and tensor products of representations: Let $\psi_1\colon T \to \mathrm{Aut}(V_1)$, $\psi_1\colon T \to \mathrm{Aut}(V_2)$ be two representations of T and $w \in W$. The element w acts on the direct sum of ψ_1 and ψ_2 by:

$$\begin{aligned}(\psi_1 \oplus \psi_2)_w\colon T \longrightarrow \mathrm{Aut}(V_1 \oplus V_2), \quad t &\longmapsto (\psi_1 \oplus \psi_2)(wtw^{-1}) \\ &= (\psi_1(wtw^{-1}), \psi_2(wtw^{-1})) \\ &= (\psi_{1w} \oplus \psi_{2w})(t)\end{aligned}$$

and on the tensor product of ψ_1 and ψ_2 by:

$$\begin{aligned}(\psi_1 \otimes \psi_2)_w\colon T \longrightarrow \mathrm{Aut}(V_1 \otimes V_2), \quad t &\longmapsto (\psi_1 \otimes \psi_2)(wtw^{-1}) \\ &= \psi_1(wtw^{-1}) \otimes \psi_2(wtw^{-1}) \\ &= (\psi_{1w} \otimes \psi_{2w})(t).\end{aligned}$$

Furthermore, by setting $(-\psi)_w = -(\psi_w)$ for each representation $\psi\colon T \to \mathrm{Aut}(V)$ of T and each element $w \in W$, we gain an action of W on the representation ring RT. By construction, the rank of a representation is invariant under the action of W.

With a lot of additional theory, one is able to prove that RG can be described by RT using the Weyl group action:

Theorem 44. *Let G be a compact connected Lie group and T a maximal torus of G. Then the morphism $RG \to RT$ induced by the inclusion $T \subseteq G$ is injective and yields an isomorphism:*

$$RG \xrightarrow{\cong} (RT)^W.$$

Here $(RT)^W$ denotes the elements of RT that are invariant under the action of the Weyl group W of G.

Proof. See [BtD85, Chapter IV.2, (2.8) Corollary and Chapter VI.2, (2.1) Proposition]. ☐

By Corollary 42, representation rings of tori are integral domains. So using Theorem 44, one can directly conclude:

Corollary 45. *Let G be a compact connected Lie group. Then RG is an integral domain.*

1.4 The Augmentation Ideal

By the Closed Subgroup Theorem, for every closed subgroup $H \subseteq G$ of a compact Lie group G, the inclusion $i\colon H \hookrightarrow G$ is a morphism of Lie groups. So by Proposition 37, it induces a ring homomorphism $i^*\colon RG \to RH$. On representations $\psi \in RG^+$, $i^*(\psi)$ is the restriction of $\psi\colon G \to \mathrm{Aut}(V)$ to H.

In this subsection, we take a look at the case in which H is the subgroup $\{1\} \subseteq G$. It yields a ring homomorphism

$$RG \longrightarrow R(\{1\}) \cong \mathbb{Z}$$

with $z \in \mathbb{Z}_{\geq 0}$ corresponding to the trivial representation of rank z. When restricted to $\{1\} \subseteq G$ any representation $\psi\colon G \to \mathrm{Aut}(V)$ is trivial since $\psi(1) = \mathrm{id}_V$. Thus $\psi \in RG^+$ is sent to $\mathrm{rank}(\psi) \in \mathbb{Z}$. We refer to this homomorphism as *rank map* (of RG) and write $\mathrm{rank}\colon RG \to \mathbb{Z}$.

Definition 46. *Let G be a compact Lie group. The* augmentation ideal $\mathfrak{I}$ *of RG is the kernel of the rank map.*

Now assume that G is connected. Let $T \subseteq G$ be a maximal torus of G. By Theorem 44, RG is the subring of RT consisting of all elements that are invariant under the action of the Weyl group. By using the inclusions $\{1\} \hookrightarrow T \hookrightarrow G$, the rank map of RG factors over RT:

$$RG \longleftrightarrow RT \longrightarrow \mathbb{Z}$$

with $RT \longrightarrow \mathbb{Z}$ being the respective rank map of RT. By Proposition 41 we know that RT is of the form $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ with x_j corresponding to the projection to the j-th factor of $T \cong (S^1)^n$. As every x_j is of rank 1, the rank map of RT is given by sending each variable x_j, $1 \leq j \leq n$, to $1 \in \mathbb{Z}$. So on a Laurent polynomial $f \in RT$, the rank map is of the following form:

$$f = \sum_{i \in I} \alpha_i \prod_{j=1}^{n} x_j^{r_{i,j}} \quad \longmapsto \quad \sum_{i \in I} \alpha_i \in \mathbb{Z}$$

with I being a finite index set and $\alpha_i, r_{i,j} \in \mathbb{Z}$.

Lastly, we will take a look at the ideals of the form $\mathfrak{I} + (p)$ for a prime $p \in \mathbb{Z}$. Here, $(p) \subseteq RG$ is the ideal generated by the equivalence class of the trivial representation of rank p. Thus, the elements in $\mathfrak{I} + (p)$ are of the form: $x + p \cdot y$ with $x, y \in RG$ and $\operatorname{rank}(x) = 0$. Applying the rank map on an element $x + p \cdot y \in \mathfrak{I} + (p)$ yields:

$$\operatorname{rank}(x + p \cdot y) = \underbrace{\operatorname{rank}(x)}_{=0} + \operatorname{rank}(p \cdot y) = p \cdot \operatorname{rank}(y).$$

Thus $p \in \mathbb{Z}$ divides the rank of every element in $\mathfrak{I} + (p)$. On the other hand, if we take an element $z \in RG$ with $rank(z) = p \cdot u$ for $u \in \mathbb{Z}$, we can write it as $(z - p \cdot u) + p \cdot u \in RG$ where p and u are the trivial representations of the respective rank (or their additive inverse in case of a negative $u \in \mathbb{Z}$). This shows that z is in $\mathfrak{I} + (p)$, because $z - p \cdot u$ fulfills $\operatorname{rank}(z - p \cdot u) = \operatorname{rank}(z) - p \cdot u = 0$. So we can conclude:

Proposition 47. *Let G be a compact Lie group and $\mathfrak{I} \subseteq RG$ the augmentation ideal of RG. For each prime $p \in \mathbb{Z}$, we have:*

$$\mathfrak{I} + (p) = \{z \in RG \mid p \textit{ divides } \operatorname{rank}(z)\}.$$

1.5 Some Examples

In this subsection, the maximal tori and the Weyl groups of some well-known (classical) Lie groups will be discussed. All of this is based on Chapter IV.3 of [BtD85]. We will omit most of the proofs and focus on illustrating how the respective Weyl group acts on the maximal torus and its representation ring. For the abstract Weyl groups, we will state representing elements for the cosets in $W = N(T)/T$. In our examples, we will do so by giving an embedding i of the Weyl group W in the normalizer $N(T)$ in G, so that $p \circ i = \mathrm{id}$ for $p \colon N(T) \to N(T)/T = W$.

U(n):

The *unitary group* $\mathrm{U}(n)$ consists of all complex valued $n \times n$ matrices for which its conjugate transpose is its inverse, i.e.

$$\mathrm{U}(n) = \left\{ U \in GL_n(\mathbb{C}) \mid U^H U = I \right\}.$$

One maximal torus of $\mathrm{U}(n)$ is given by the subgroup of diagonal matrices in $\mathrm{U}(n)$:

$$T_{\mathrm{U}(n)} = \left\{ \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_n \end{pmatrix} \;\middle|\; \alpha_j \in S^1 \text{ for } 1 \leq j \leq n \right\}.$$

The Weyl group of $\mathrm{U}(n)$ is $W_{\mathrm{U}(n)} = S_n$, the symmetric group on n elements. As representing elements in $N(T)$, one may choose the permutation matrices in $\mathrm{U}(n)$, i.e. matrices whose columns are permutations of the standard basis vectors $e_1, \ldots, e_n$. An element $\pi \in S_n$ is then sent to the permutation matrix in which the k-th column is $e_{\pi(k)}$ for $1 \leq k \leq n$. The Weyl group $W_{\mathrm{U}(n)}$ acts on the maximal torus $T_{\mathrm{U}(n)}$ by permuting the diagonal entries.

In Proposition 41, we have seen that the representation ring of the maximal torus $T_{\mathrm{U}(n)}$ is given by $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ where x_j corresponds to the projection to the j-th diagonal entry. So the Weyl group $W_{\mathrm{U}(n)}$ acts on $RT_{\mathrm{U}(n)}$ by permuting the indices of x_j, $1 \leq j \leq n$. In Theorem 44 we have seen that $R(\mathrm{U}(n))$ is the subset of $RT_{\mathrm{U}(n)}$ consisting of all elements that are invariant under the action of the Weyl group $W_{\mathrm{U}(n)}$.

Proposition 48. *The representation ring of* $\mathrm{U}(n)$ *is* $\mathbb{Z}[\sigma_1, \ldots, \sigma_n, \sigma_n^{-1}]$ *where* σ_j *is the* j*-th elementary symmetric polynomial in* $x_1, \ldots, x_n$.

The following proof is from [BtD85, Chapter IV, (3.13) Application and the remarks before]:

Proof. Let $f \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be invariant under permutation of the indices. By multiplying with $\sigma_n^k = (x_1 \cdot \ldots \cdot x_n)^k$ for a sufficiently large $k \in \mathbb{N}$, we may assume that $f \cdot \sigma_n^k$ is in $\mathbb{Z}[x_1, \ldots, x_n]$. The element $f \cdot \sigma_n^k$ is still invariant under the Weyl group action, because both factors are invariant. It is known that the symmetric polynomials in $\mathbb{Z}[x_1, \ldots, x_n]$ form the polynomial ring $\mathbb{Z}[\sigma_1, \ldots, \sigma_n]$ (see [Mac95, Chapter I.2, (2.4)]). Thus $f \cdot \sigma_n^k \in \mathbb{Z}[\sigma_1, \ldots, \sigma_n]$ which implies that f is in $\mathbb{Z}[\sigma_1, \ldots, \sigma_n, \sigma_n^{-1}]$. In addition, one may easily check that every element of $\mathbb{Z}[\sigma_1, \ldots, \sigma_n, \sigma_n^{-1}]$ is invariant under permutation of indices. □

SU(n):

The *special unitary group* $\mathrm{SU}(n)$ consists of all unitary matrices of degree n with determinant 1:

$$\mathrm{SU}(n) = \{U \in \mathrm{U}(n) \mid \det(U) = 1\}.$$

A maximal torus of $\mathrm{SU}(n)$ is given by the intersection of $T_{\mathrm{U}(n)}$ and $\mathrm{SU}(n)$:

$$\begin{aligned} T_{\mathrm{SU}(n)} &= T_{\mathrm{U}(n)} \cap \mathrm{SU}(n) \\ &= \left\{ \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_n \end{pmatrix} \,\middle|\, \alpha_j \in S^1 \text{ for } 1 \leq j \leq n \text{ and } \prod_{j=1}^{n} \alpha_j = 1 \right\}. \end{aligned}$$

As any $n-1$ diagonal entries determine the remaining entry, this torus is of dimension $n-1$ and thus $\operatorname{rank}(\mathrm{SU}(n)) = n-1$. The Weyl group of $\mathrm{SU}(n)$ coincides with the Weyl group of $\mathrm{U}(n)$, thus $W_{\mathrm{SU}(n)} = S_n$. The same holds for the action of the Weyl group: $W_{\mathrm{SU}(n)}$ can be represented by the permutation matrices and acts on $T_{\mathrm{SU}(n)}$ by permuting the diagonal entries. To understand the action of the Weyl group $W_{\mathrm{SU}(n)}$ on the representation ring $RT_{\mathrm{SU}(n)}$, it is useful to write it as the quotient $RT_{\mathrm{SU}(n)} = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]/(x_1 \cdot \ldots \cdot x_n - 1)$ with x_j corresponding to the projection to the j-th diagonal entry. Then $W_{\mathrm{SU}(n)}$ acts by permuting the indices. When writing it as a Laurent ring in $n-1$ variables, one has to be more cautious when considering the Weyl group action.

Sp(n):

The *symplectic group* $\mathrm{Sp}(n)$ consists of matrices of $\mathrm{U}(2n)$ that are of a special form:

$$\mathrm{Sp}(n) = \left\{ U \in \mathrm{U}(2n) \,\middle|\, U = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \text{ for } A, B \in \mathbb{C}^{n \times n} \right\}$$

There is a natural inclusion of $\mathrm{U}(n) \subseteq \mathrm{Sp}(n)$ given by:

$$\mathrm{U}(n) \hookrightarrow \mathrm{SU}(n), \quad A \longmapsto \begin{pmatrix} A & \\ & \overline{A} \end{pmatrix}.$$

The image of the maximal torus $T_{\mathrm{U}(n)}$ of $\mathrm{U}(n)$ yields a maximal torus of $\mathrm{SU}(n)$ of the form:

$$T_{\mathrm{Sp}(n)} = \left\{ \begin{pmatrix} \alpha_1 & & & & & \\ & \ddots & & & & \\ & & \alpha_n & & & \\ & & & \overline{\alpha}_1 & & \\ & & & & \ddots & \\ & & & & & \overline{\alpha}_n \end{pmatrix} \,\middle|\, \alpha_j \in S^1 \text{ for } 1 \leq j \leq n \right\}.$$

So the first n diagonal entries determine the remaining n entries in the second block. Thus the torus is of dimension n. The Weyl group of $\mathrm{Sp}(n)$ is the group $\mathrm{G}(n)$. It consists of all permutations π of the set $\{1, \ldots, n\} \cup \{-1, \ldots, -n\}$

satisfying $\pi(-k) = -\pi(k)$ for all $k \in \{1, \dots, n\}$. It acts on $T_{\mathrm{Sp}(n)}$ by permuting the diagonal entries while preserving the form of the torus. This means that if α_j is sent to the k-th diagonal entry, $1 \leq k \leq 2n$, then $\overline{\alpha}_j$ is sent to the entry on position $(k+n) \bmod 2n$.

The embedding $i\colon W \to N(T)/T$ is given by sending $\pi \in \mathrm{G}(n)$ to a $2n \times 2n$ permutation matrix in the following way: We consider the the image under π of $k \in \{1, \dots, n\}$. If $\pi(k) = t > 0$, then the k-th column of the permutation matrix is the t-th standard basis vector e_t and the $(k+n)$-th column is e_{t+n}. If on the other hand $\pi(k) = -t < 0$, then the k-th column of the matrix is e_{t+n} and the $(k+n)$-th column is e_t. So the $2n \times 2n$-permutation matrices fulfill that for all $1 \leq t \leq n$ the standard basis vectors e_t and e_{t+n} appear in columns of the matrix that are n indices apart. By symmetry the same holds for the rows.

The group $\mathrm{G}(n)$ can also be described as a semi-direct product $(\mathbb{Z}/2\mathbb{Z}) \rtimes S_n$.

SO(2n + 1):

The *special orthogonal group* $\mathrm{SO}(n)$ consists of all orthogonal matrices of order n with determinant 1, i.e.:

$$\mathrm{SO}(n) = \left\{ Q \in GL_n(\mathbb{C}) \mid Q^T Q = I \text{ and } \det(Q) = 1 \right\}$$

For this group we have to distinguish between the cases of odd and even order since the respective maximal tori and Weyl groups differ. We start by considering the matrices of odd order. The following subgroup yields a maximal torus of $\mathrm{SO}(2n+1)$:

$$T_{\mathrm{SO}(2n+1)} = \left\{ \begin{pmatrix} B_{\vartheta_1} & & & \\ & \ddots & & \\ & & B_{\vartheta_n} & \\ & & & 1 \end{pmatrix} \in \mathrm{GL}_{2n+1}(\mathbb{R}) \;\middle|\; \vartheta_j \in [0,1) \right\}$$

with block matrices $B_\vartheta = \left(\begin{smallmatrix} \cos(2\pi\vartheta) & -\sin(2\pi\vartheta) \\ \sin(2\pi\vartheta) & \cos(2\pi\vartheta) \end{smallmatrix}\right) \in \mathrm{SO}(2)$. $\mathrm{SO}(2)$ is isomorphic to S^1 (see Example 2) and thus $T_{\mathrm{SO}(2n+1)}$ is a torus of dimension n.

The Weyl group of $\mathrm{SO}(2n+1)$ is $W_{\mathrm{SO}(2n+1)} = \mathrm{G}(n)$. On $T_{\mathrm{SO}(2n+1)}$ it acts by permuting the B_ϑ-blocks and possibly changing signs of ϑ. More precisely, if under $\pi \in \mathrm{G}(n)$ the element k is sent to $t > 0$, then the B_{ϑ_k} is sent to the position of the t-th diagonal block while preserving the sign of ϑ_k. If it is sent to $-t < 0$ then B_{ϑ_k} is sent to the t-th block on the diagonal and in addition, ϑ_k becomes $-\vartheta_k$.

It can be embedded into $N(T_{\mathrm{SO}(2n+1)})$ in the following way: An element $\pi \in \mathrm{G}(n)$ is sent to the following matrix:

$$\left(\begin{array}{c|c} A & \\ \hline & \det(A) \end{array}\right) \in \mathrm{SO}(2n+1)$$

in which A is obtained from a $n \times n$ permutation matrix by substituting each 0 with $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and each 1 with either $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The $n \times n$ permutation matrix is obtained in the following way: Consider the image of $k \in \{1, \ldots, n\}$. If $\pi(k) = \pm t$, the k-th column is the standard basis vector e_t. If $\pi(k)$ is positive, this entry is later substituted by $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and otherwise we choose $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

If no $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$-block is used, the matrix A has determinant 1. In this case A itself is a $2n \times 2n$ permutation matrix which results from an even number of switching rows of the identity matrix. Every $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$-block used in A yields a change of sign of $\det(A)$. So if there is an odd number of these blocks used, we need to compensate this using $\det(A)$ as an offset. It does not have an effect on $T_{\mathrm{SO}(2n+1)}$ and only serves the purpose of controlling the sign.

SO(2n):

Now we take a closer look at the special orthogonal matrices of even degree. A maximal torus of $\mathrm{SO}(2n)$ is given by the following subgroup:

$$T_{\mathrm{SO}(2n)} = \left\{ \begin{pmatrix} B_{\vartheta_1} & & \\ & \ddots & \\ & & B_{\vartheta_n} \end{pmatrix} \in \mathrm{GL}_{2n}(\mathbb{R}) \;\middle|\; \vartheta_j \in \mathbb{R}/\mathbb{Z} \right\}.$$

The B_ϑ are of the same form as in the previous example. The Weyl group $W_{\mathrm{SO}(2n)}$ can be represented by matrices $A \in \mathrm{SO}(2n)$ which result from permutation matrices in the same way as in $W_{\mathrm{SO}(2n+1)}$. However, since we are missing an offset to control the determinant of A, only an even number of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$-blocks can be used. We can describe $W_{\mathrm{SO}(2n)}$ as $\mathrm{SG}(n) \subseteq \mathrm{G}(n)$, the subgroup consisting of all even permutations in $\mathrm{G}(n)$.

Chapter 2

Commutative Algebra

In the main part of this thesis, we will interpret representation rings RH of closed subgroups H of compact Lie groups G as RG-modules. In this chapter, we will introduce the algebraic means to study these RG-modules. We start by quickly summing up some of the basic terms in commutative algebra, including the Zariski spectrum $\mathrm{Spec}(R)$ of a ring R and localization of R-modules at prime ideals. Then we take a closer look at the support and the associated primes of R-modules. Next, we will briefly introduce the Tor functors stating some properties which we will later use. Lastly, there is a short section dealing with Koszul complexes.

For us a ring will always be commutative with unit if not mentioned otherwise.

2.1 Basic Commutative Algebra

In commutative algebra, the prime ideals of rings are of great importance:

Definition 49 (spectra of rings). *The* spectrum *of a ring R is the set of all prime ideals of R. We note it as* $\mathrm{Spec}(R)$.

Proposition 50. *Let R, S be two rings and let $f\colon R \to S$ be a ring homomorphism. Then for each prime $\mathfrak{p} \in \mathrm{Spec}(S)$, its inverse image $f^{-1}(\mathfrak{p}) \subseteq R$ is in* $\mathrm{Spec}(R)$.

Proof. Consider the following ring homomorphism given by concatenation:

$$R \xrightarrow{f} S \xrightarrow{\pi} S/\mathfrak{p}.$$

As the kernel of this homomorphism, $f^{-1}(\mathfrak{p})$ is an ideal. By the fundamental homomorphism theorem for rings, we get an injective ring homomorphism $R/f^{-1}(\mathfrak{p}) \hookrightarrow S/\mathfrak{p}$. Since $\mathfrak{p}$ is prime, $S/\mathfrak{p}$ is an integral domain.

Thus $R/f^{-1}(\mathfrak{p})$ is an integral domain as well which implies that $f^{-1}(\mathfrak{p})$ is prime. ☐

The previous proposition yields that for every ring homomorphism $f: R \to S$ we get a map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ by sending $\mathfrak{p} \in \mathrm{Spec}(S)$ to $f^{-1}(\mathfrak{p})$. This makes Spec into a contravariant functor:

Corollary 51. $\mathrm{Spec}\colon \mathrm{CRing} \to \mathrm{Set}$ *is a contravariant functor from the category* CRing *of commutative rings to the category* Set *of sets.*

Actually, one can even equip $\mathrm{Spec}(R)$ with a topology so that Spec becomes a functor into the category Top of topological spaces. This topology is called *Zariski topology.*

In addition to the basic functorial properties, we will need the following proposition:

Proposition 52. *Let $r: R \to A$ be an injective ring homomorphism and let A be integral as an R-module. Then the induced map* $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ *is surjective.*

Proof. See [Bou06, Chapitre V, §2, n^o 1, Théoréme 1]. ☐

We will see that representation rings of compact Lie groups belong to the following large class of rings:

Definition 53 (Noetherian ring)**.** *A ring R is called* Noetherian *if it satisfies the* ascending chain condition*. This means that for every sequence of ideals of the form*

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq I_{n+1} \subseteq \ldots$$

there exists an $N \in \mathbb{N}$ so that $I_N = I_n$ for all $n \geq N$.

In many propositions in the following section we will need the assumption that the respective ring is Noetherian.

A fundamental theorem concerning Noetherian rings is Hilbert's Basis Theorem:

Theorem 54 (Hilbert's Basis Theorem)**.** *Let R be a Noetherian ring and $R[x_1, \ldots, x_n]$ a polynomial ring over R. Then $R[x_1, \ldots, x_n]$ is Noetherian.*

Proof. See [Eis04, Chapter 1.4, Theorem 1.2]. ☐

One may expand the term Noetherian to R-modules:

Definition 55 (Noetherian module)**.** *A module M over a ring R is called* Noetherian*, if for every sequence of submodules of the form*

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq M_{n+1} \subseteq \ldots$$

there exists an $N \in \mathbb{N}$ so that $M_N = M_n$ for all $n \geq N$.

If we regard a ring R as a module over itself, this definition coincides with the definition of Noetherian rings.

Proposition 56. *Every finitely generated module over a Noetherian ring is Noetherian.*

Proof. See [Eis04, Chapter 1.4, Proposition 1.4]. □

Another term which we will frequently need is *localization*:

Definition 57 (localization of modules)**.** *Let M be an R-module and $S \subseteq R$ a multiplicatively closed subset. Then the* localization *of M at S is defined as $S^{-1}M = (M \times S)/_{\approx}$ with $\approx$ being the following equivalence relation: Two elements $(m, u), (n, v) \in M \times S$ are equivalent if and only if there exists an $s \in S$ so that $s(vm - un) = 0$.*

For $M = R$ we can interpret R as a module over itself. Then the localization $S^{-1}R$ of R at a multiplicatively closed subset $S \subseteq R$ is a ring in which the addition and multiplication are given by:

$$\begin{aligned}(r_1, s_1) + (r_2, s_2) &= (s_2 r_1 + s_1 r_2, s_1 s_2) \quad \text{and} \\ (r_1, s_1) \cdot (r_2, s_2) &= (r_1 r_2, s_1 s_2).\end{aligned}$$

The neutral elements of addition and multiplication are $(0, 1)$ and $(1, 1)$, respectively.

For an R-module M, the localization $S^{-1}M$ has the structure of an $S^{-1}R$-module: The addition is obtained in a similar way as above. The scalar multiplication with elements of $S^{-1}R$ is given by

$$(r, s_1) \cdot (m, s_2) = (rm, s_1 s_2)$$

where rm is the scalar multiplication of M. To obtain an R-module structure on $S^{-1}M$, map $r \in R$ to $(r, 1) \in S^{-1}R$ and use the previous scalar multiplication.

A common choice for S is the set $R \setminus \mathfrak{p}$ for a prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$. We also refer to this as the *localization at the prime ideal* $\mathfrak{p}$ and write $M_{\mathfrak{p}}$ instead of $(R \setminus \mathfrak{p})^{-1}M$.

Proposition 58. *Any localization of a Noetherian ring is Noetherian.*

Proof. See [Eis04, Chapter 2.1, Corollary 2.3] □

We can interpret the localization of R-modules as a tensor product:

Proposition 59. *Let M be an R-module and $S \subseteq R$ multiplicatively closed. Then $S^{-1}M$ is isomorphic to $M \otimes_R S^{-1}R$. An isomorphism is defined by the map given on pure tensors by:*

$$M \otimes_R S^{-1}R \longrightarrow S^{-1}M, \qquad (r, s) \otimes m \longmapsto (rm, s).$$

Proof. See [Eis04, Chapter 2.2, Lemma 2.4]. □

As the tensor product $- \otimes S^{-1}R$ is a functor, we see in particular that every homomorphism of R-modules $f\colon M \to N$ induces a map $S^{-1}f\colon S^{-1}M \to S^{-1}N$.

Proposition 60. *Localization is exact. This means that for every short exact sequence of R-modules:*

$$0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$$

the sequence localized at a multiplicatively closed set $S \subseteq R$:

$$0 \to S^{-1}A \xrightarrow{S^{-1}\phi} S^{-1}B \xrightarrow{S^{-1}\psi} S^{-1}C \to 0$$

is exact as well.

The proof is taken from [Eis04, Chapter 2.2, Proposition 2.5]:

Proof. Let $0 \to A \to B \to C \to 0$ be exact. By Proposition 59 localization can be interpreted as tensoring with $S^{-1}R$. It is well-known that tensor products are right exact. Thus the sequence $S^{-1}A \to S^{-1}B \to S^{-1}C \to 0$ is exact.

So we have to show that $S^{-1}\phi\colon S^{-1}A \to S^{-1}B$ is injective. By assuming otherwise let $(a, s) \in S^{-1}A$ be in the kernel of $S^{-1}\phi$. Thus $(\phi(a), s) = (0, 1)$ in $S^{-1}B$. That is the case if and only if there exists a $t \in S$ so that $t(1 \cdot \phi(a) - s \cdot 0) = t\phi(a)$ is zero in B. Since ϕ is an injective homomorphism $t\phi(a) = \phi(ta)$ is zero if and only if ta is zero in A. However, if $ta = 0 \in A$ then $(a, s) = (0, 1)$ in $S^{-1}A$, because $t(1 \cdot a - s \cdot 0) = ta = 0$. □

From the exactness of localization, we can conclude the following:

Corollary 61. *Let $\phi\colon M \to N$ be a homomorphism of R-modules and $S \subseteq R$ a multiplicatively closed subset. Then $S^{-1}\operatorname{Ker}(M \to N) \cong \operatorname{Ker}(S^{-1}M \to S^{-1}N)$ and $S^{-1}\operatorname{Im}(M \to N) \cong \operatorname{Im}(S^{-1}M \to S^{-1}N)$.*

Proof. The homomorphism ϕ yields an exact sequence of the form:

$$0 \to \operatorname{Ker}(\phi) \to M \xrightarrow{\phi} \operatorname{Im}(\phi) \to 0$$

After localization at S the sequence:

$$0 \to S^{-1}\operatorname{Ker}(\phi) \to S^{-1}M \xrightarrow{S^{-1}\phi} S^{-1}\operatorname{Im}(\phi) \to 0$$

is exact as well. The map $S^{-1}\phi$ comes from the localization $S^{-1}\phi\colon S^{-1}M \to S^{-1}N$. By the exactness of the second sequence it follows that $S^{-1}\operatorname{Im}(\phi) \cong \operatorname{Im}(S^{-1}\phi)$ and $S^{-1}\ker(\phi) \cong \ker(S^{-1}\phi)$. □

2.2 Support and Associated Primes

In this subsection, we will introduce the *support* and the *associated primes* of R-modules. They will prove to be a very useful tool when showing that certain modules vanish.

As we ended the previous subsection with the localization of modules, we will start with the definition of the support:

Definition 62 (support of a module)**.** *Let M be an R-module. The* support *of M is the set* $\operatorname{Supp}_R(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid M_\mathfrak{p} \neq \{0\}\}$ *of all primes in* $\operatorname{Spec}(R)$ *at which the localized module $M_\mathfrak{p}$ does not vanish.*

The support of an R-module is an upward closed subset of $\operatorname{Spec}(R)$, in the following sense:

Proposition 63. *Let M be an R-module and $\mathfrak{p}' \in \operatorname{Supp}(M)$. If a prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$ contains $\mathfrak{p}'$, then $\mathfrak{p} \in \operatorname{Supp}(M)$.*

Proof. If $\mathfrak{p}'$ is in the support of M, then $M_{\mathfrak{p}'} \neq 0$, i.e. there exists an element $(m, s) \in M_{\mathfrak{p}'}$ that is not equivalent to $(0, 1)$. This means that for all $s' \in R \smallsetminus \mathfrak{p}'$ we have $s'(1 \cdot m - s \cdot 0) = s' \cdot m \neq 0$. In particular, the element $(m, 1) \in M_{\mathfrak{p}'}$ is not zero.

For a prime $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\mathfrak{p}' \subseteq \mathfrak{p}$, we have $R \smallsetminus \mathfrak{p} \subseteq R \smallsetminus \mathfrak{p}'$. Thus $(m, 1) \in M_\mathfrak{p}$ is not equivalent to zero as $s' \cdot m \neq 0$ holds for all $R \smallsetminus \mathfrak{p}'$. □

For introducing associated primes, we need *annihilators*:

Definition 64. *Let M be an R-module and $S \subseteq M$. The* annihilator *of S is the set* $\operatorname{ann}_R(S) = \{r \in R \mid rs = 0 \text{ for all } s \in S\}$ *consisting of all elements in $r \in R$ for which rs vanishes for all $s \in S$.*

If the set S is a singleton $S = \{a\}$ we usually write $\operatorname{ann}_R(a)$ instead of $\operatorname{ann}_R(\{a\})$. When the context is clear, we will sometimes omit R.

For finitely generated R-modules M, the support is characterized by the annihilator of M:

Proposition 65. *Let M be a finitely generated R-module and $\mathfrak{p} \in \operatorname{Spec}(R)$. Then the following two statements are equivalent:*

1. $\mathfrak{p} \in \operatorname{Supp}(M)$.

2. $\mathfrak{p}$ *contains* $\operatorname{ann}(M)$.

Proof. See [Eis04, Chapter 2.2, Corollary 2.7]. □

Proposition 66. *Let M be an R-module and $S \subseteq M$. Then* $\operatorname{ann}(S)$ *is an ideal.*

Proof. We have to show that $\operatorname{ann}(S)$ is an additive subgroup of R and closed under multiplication with arbitrary elements of R. Obviously, the zero-element of R is also in $\operatorname{ann}(S)$. Now let a, b be two elements in $\operatorname{ann}(S)$ and $r \in R$. We have to show that $ra - b$ is in $\operatorname{ann}(S)$. For $s \in S$ we have $(ra - b)s = (ra)s - bs = r(as) - bs = 0$ since a and b both annihilate $s \in S$. □

However, in general not every annihilator is prime:

Example 67. *Consider $R = M = \mathbb{Z}/6\mathbb{Z}$ with the canonical module structure. The annihilator of the element $[1] \in M$ is the zero ideal. This is not prime, since $[2] \cdot [3] = [0]$ and $[2], [3] \neq [0]$.*

Now, we can define the associated primes of an R-module M:

Definition 68. *Let M be an R-module. A prime $\mathfrak{p} \in \operatorname{Spec}(R)$ is called an* associated prime *if there is a non-zero element $m \in M$ so that $\mathfrak{p}$ annihilates m. The set of all associated primes is written as:*

$$\operatorname{Ass}_R(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \exists m \in M \setminus \{0\} \text{ with } \mathfrak{p} = \operatorname{ann}(m)\}.$$

There is an equivalent characterization which we will also be using [Eis04, Chapter 3.1, remark after first definition]:

Proposition 69. *Let M be an R-module and $\mathfrak{p} \in \operatorname{Spec}(R)$. Then the following two statements are equivalent:*

1. $\mathfrak{p} \in \operatorname{Ass}(M)$.

2. *$R/\mathfrak{p}$ is isomorphic to a submodule of M (as R-module).*

Proof. For the first implication, let $\mathfrak{p} = \operatorname{ann}(a)$ be in $\operatorname{Ass}(M)$. Then the submodule generated by a is isomorphic to $R/\mathfrak{p}$.

If on the other hand $R/\mathfrak{p}$ is isomorphic to a submodule of M, let $a \in M$ be the image of the equivalence class of 1 in $R/\mathfrak{p}$ under the inclusion $R/\mathfrak{p} \hookrightarrow M$. Then $\mathfrak{p} = \operatorname{ann}(a)$. $\square$

Between $\operatorname{Ass}(M)$ and $\operatorname{Supp}(M)$, the following relation holds:

Proposition 70. *Let R be a ring and M an R-module. Then* $\operatorname{Ass}(M) \subseteq \operatorname{Supp}(M)$.

Proof. If M is finitely generated, this follows directly by using Proposition 65 and noting that $\operatorname{ann}(M) \subseteq \operatorname{ann}(m)$ for all $m \in M \setminus \{0\}$.

For the general case, take a prime $\mathfrak{a} \in \operatorname{Ass}(M)$ with $\mathfrak{a} = \operatorname{ann}(a)$ for an $a \in M$. In $M_\mathfrak{a}$ the element $(a, 1)$ is not zero: If $(a, 1) \approx (0, 1)$ in $M_\mathfrak{a}$ then there exists an $s \in R \setminus \mathfrak{a}$ so that $s(1 \cdot a - 1 \cdot 0) = 0$ in M. So $sa = 0$ and s annihilates the element a. Thus $s \in \operatorname{ann}(a)$ and s cannot be contained in $R \setminus \operatorname{ann}(a)$. It follows that $M_\mathfrak{a} \neq 0$. $\square$

We can ask ourselves which annihilators are prime and thus associated, and whether there even exist associated primes. Maximal elements among the annihilators of elements in M are prime:

Proposition 71. *Let R be a ring and M an R-module. Then every maximal element of the set* $\{\operatorname{ann}(x) \mid x \in M \setminus \{0\}\}$ *is prime.*

The proof is from [Eis04, Chapter 3.2, Proposition 3.4]:

Proof. Let $\mathfrak{p} = \operatorname{ann}(x)$ be a maximal element. To show that $\mathfrak{p}$ is prime we have to prove that if $ab \in \mathfrak{p}$ then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

So take $a, b \in R$ with $ab \in \mathfrak{p}$. Without loss of generality we can assume that $b \notin \mathfrak{p}$. So by definition $abx = 0$ while $bx \neq 0$. So a is in the annihilator of $bx \in M$. Let $c \in \mathfrak{p}$ be an arbitrary annihilator of x. Then $c(bx) = b(cx) = 0$ since R is commutative. So $\mathfrak{p} \subseteq \operatorname{ann}(bx)$. If a is not in $\mathfrak{p}$ this inclusion is strict. This would be a contradiction to $\mathfrak{p}$ being a maximal element. $\square$

In Noetherian rings, the respective maximal elements exist:

Proposition 72. *Let R be a Noetherian ring and M an R-module. Then for every $m \in M \setminus \{0\}$ there exists an associated prime $\mathfrak{p} \in \mathrm{Ass}(M)$ with $\mathrm{ann}(m) \subseteq \mathfrak{p}$.*

Proof. Let $m \neq 0$ be an element in M. Consider the set $\Phi = \{\mathfrak{a} \subseteq R \mid \mathrm{ann}(m) \subseteq \mathfrak{a} \text{ and } \mathfrak{a} = \mathrm{ann}(a) \text{ for an } a \in M\}$. Together with the inclusion $\subseteq$ the set Φ is partially ordered. Let $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots$ be a chain in Φ. Since R is Noetherian there exists an $N \in \mathbb{N}$ at which the chain stabilizes. Then $\mathfrak{a}_N \in \Phi$ is an upper bound of the given chain. So we can apply Zorn's lemma which yields that there is a maximal element $\mathfrak{p}$ in Φ. It is clear that such an element is maximal among the set of all annihilators of non-zero elements of M as well. Together with Proposition 71 we get that $\mathfrak{p}$ is prime and thus in $\mathrm{Ass}(M)$. □

In particular, for non-zero modules over Noetherian rings it follows that associated primes exist:

Corollary 73. *Let R be a Noetherian ring and M an R-module. Then the following statements are equivalent:*

1. $M = 0$.
2. $\mathrm{Ass}_R(M) = \emptyset$.

Proof. If $M = 0$, it is trivial that the set of associated primes is empty. If $M \neq 0$, we can apply Proposition 72 for an element $m \neq 0$ in M and get an associated prime $\mathfrak{p} \in \mathrm{Ass}(M)$ containing $\mathrm{ann}(m)$. In particular, $\mathrm{Ass}(M)$ is not empty. □

One result which we will need is that injectivity can be checked at associated primes [Eis04, Chapter 3.2, Corollary 3.5 (c)]:

Proposition 74. *Let R be a Noetherian ring and M, N two R-modules. Let $\phi\colon M \to N$ be an R-module homomorphism. The following two statements are equivalent:*

(1) $\phi\colon M \to N$ is injective.

(2) The localization $\phi_{\mathfrak{p}}\colon M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is injective for all $\mathfrak{p} \in \mathrm{Ass}(M)$.

Proof. The implication (1) $\Rightarrow$ (2) follows directly by the exactness of localization shown in Proposition 60.

We will show the contraposition of (2) $\Rightarrow$ (1). Assume that ϕ is not injective. We have to show that there exists an associated prime in $\mathrm{Spec}(R)$ for which the localized map is not injective as well. Then there exists an

$m \in M$ with $m \neq 0$ and $\phi(m) = 0$. By Proposition 72 one can find a prime $\mathfrak{b} \in \mathrm{Ass}(M)$ that contains $\mathrm{ann}(m)$.

We can take a look at the localized map $M_{\mathfrak{b}} \to N_{\mathfrak{b}}$. The element $m = (m, 1) \in M_{\mathfrak{b}}$ is still sent to 0. If we show that $(m, 1) \neq (0, 1)$ in $M_{\mathfrak{b}}$, it follows that the localized map is not injective as well. Assume that $(m, 1) \approx (0, 1) \in M_{\mathfrak{b}}$. Then there exists an $s \in R \setminus \mathfrak{b}$ so that:

$$s \cdot (m \cdot 1 - 0 \cdot 1) = s \cdot m = 0.$$

Then s has to be in the annihilator of m. However, we chose $\mathfrak{b}$ so that $\mathrm{Ann}(m) \subseteq \mathfrak{b}$ and thus $s \notin R \setminus \mathfrak{b}$ which is a contradiction. □

Short exact sequences of R-modules contain some information about the associated primes of the respective modules:

Proposition 75. *Let R be a ring and M, M' and M'' R-modules. In addition let*

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence of R-modules. Then the following statements are true:

1. $\mathrm{Ass}(M') \subseteq \mathrm{Ass}(M)$ *and*
2. $\mathrm{Ass}(M) \subseteq \mathrm{Ass}(M') \cup \mathrm{Ass}(M'')$*.*

In particular, if the sequence is split exact, then $\mathrm{Ass}(M) = \mathrm{Ass}(M') \cup \mathrm{Ass}(M')$*.*

The proof is based on [Eis04, Chapter 3.2, Lemma 3.6]:

Proof. To prove this we will use the characterization of associated primes as submodules of the form $R/\mathfrak{p}$. Let $\mathfrak{p}$ be in $\mathrm{Ass}(M')$. Then there exists an inclusion $R/\mathfrak{p} \hookrightarrow M'$ of R-modules. Together with the inclusion $M' \hookrightarrow M$ given in the short exact sequence, we get an inclusion $R/\mathfrak{p} \hookrightarrow M' \hookrightarrow M$ of R-modules. Thus $\mathfrak{p}$ is in $\mathrm{Ass}(M)$ and $\mathrm{Ass}(M') \subseteq \mathrm{Ass}(M)$.

Now let $\mathfrak{p}$ be in $\mathrm{Ass}(M) \setminus \mathrm{Ass}(M')$. We have to show that $\mathfrak{p}$ is in $\mathrm{Ass}(M'')$. We have an inclusion of R-modules of the form $R/\mathfrak{p} \hookrightarrow M$. The images of M' and $R/\mathfrak{p}$ in M intersect trivially: Assume this is not the case. Then there exists an non-zero element $m \in M$ that is contained in both submodules. We consider the annihilator of m. As m is in the image of $R/\mathfrak{p}$, we have $\mathfrak{p} \subseteq \mathrm{ann}(m)$. Since $\mathfrak{p}$ is prime, there are no zero-divisors in $R/\mathfrak{p}$ and thus $rm \neq 0$ for all $r \in R \setminus \mathfrak{p}$. So $\mathrm{ann}(m) = \mathfrak{p}$. However, as m is in $M' \subseteq M$, it follows that $\mathfrak{p}$ is an associated prime of M'. This is a contradiction.

As the images of M' and $R/\mathfrak{p}$ intersect trivially, the homomorphism of R-modules given by $R/\mathfrak{p} \hookrightarrow M \twoheadrightarrow M''$ is injective.

If the sequence is split exact then $M \cong M' \oplus M''$. Using the map that sends M'' to the respective summand of $M \cong M' \oplus M''$ yields another short exact sequence of the form:

$$0 \longrightarrow M'' \longrightarrow M \longrightarrow M' \longrightarrow 0.$$

Together with the previous short exact sequence, we get:

$$\left.\begin{matrix} \operatorname{Ass}(M') \\ \operatorname{Ass}(M'') \end{matrix}\right\} \subseteq \operatorname{Ass}(M) \subseteq \operatorname{Ass}(M') \cup \operatorname{Ass}(M'')$$

and thus $\operatorname{Ass}(M) = \operatorname{Ass}(M') \cup \operatorname{Ass}(M'')$. □

Using this we can directly conclude:

Corollary 76. *Let R be a ring and $N_1, \dots, N_k$ a finite number of R-modules. Then $\operatorname{Ass}(\bigoplus_{i=1}^k N_i) = \bigcup_{i=1}^k \operatorname{Ass}(N_i)$.*

Proof. Use the remark at the end of Proposition 75 and do an induction over the number of summands in $\bigoplus_{i=1}^k N_i$. □

Again, we consider finitely generated modules over Noetherian rings:

Theorem 77. *Let R be a Noetherian ring and $M \neq 0$ a finitely generated R-module. Then each prime in $\operatorname{Ass}(M)$ contains $\operatorname{ann} M$ and all minimal primes among those that contain $\operatorname{ann} M$ are in $\operatorname{Ass}(M)$.*

Proof. It is clear that the annihilator of M is contained in each associated prime in $\operatorname{Ass}(M)$. For the remaining part of the statement, see [Eis04, Theorem 3.1]. □

For finitely generated R-modules, we can conclude the following:

Corollary 78. *Let R be a Noetherian ring and $M \neq 0$ a finitely generated R-module. Then the minimal primes in $\operatorname{Ass}(M)$ and $\operatorname{Supp}(M)$ coincide.*

Proof. By Proposition 65 a prime $\mathfrak{p} \in \operatorname{Spec}(R)$ is in the support of a finitely generated R-module M if and only if it contains $\operatorname{ann}(M)$. Together with Theorem 77 the statement follows. □

2.3 The Tor Functors

Let R be a ring and M an R-module. By fixing the first term in the tensor product of two R-modules, we gain a covariant functor of the form $M \otimes_R -$. It is right-exact, i.e. for a short exact sequence

$$0 \to A \to B \to C \to 0$$

of R-modules, applying the functor yields an exact sequence of the form:

$$M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0.$$

R-modules M for which $M \otimes_R -$ preserves injectivity are called *flat*. However in general, for an injective homomorphism $A \to B$, after applying $M \otimes_R -$ the homomorphism $M \otimes_R A \to M \otimes_R B$ might not be injective. One of the simplest examples for this is:

Example 79. *Consider the following short exact sequence of $\mathbb{Z}$ modules:*

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

After tensoring with $\mathbb{Z}/2\mathbb{Z}$, it is of the form:

$$0 \to \underbrace{\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}}_{\cong \mathbb{Z}/2\mathbb{Z}} \xrightarrow{\cdot 2} \underbrace{\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}}_{\cong \mathbb{Z}/2\mathbb{Z}} \to \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to 0.$$

This is not a short exact sequence of $\mathbb{Z}$-modules, because $\mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/2\mathbb{Z}$ is the constant zero map and thus not injective.

Left derived functors yield a (canonical) way to make these types of sequences exact by adding terms on the left: Let $F\colon \mathcal{A} \to \mathcal{B}$ be a right exact functor between two abelian categories A, B. For certain $\mathcal{A}$ we can construct functors $L_iF\colon \mathcal{A} \to \mathcal{B}$ with $i \in \mathbb{N}$ with the property that a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ yields a long exact sequence of the form:

$$\begin{tikzcd}
 & \dots \arrow[r] & L_2F(C) \arrow[dll, bend left] \\
L_1F(A) \arrow[r] & L_1F(B) \arrow[r] & L_1F(C) \arrow[dll, bend left] \\
\underbrace{L_0F(A)}_{\cong F(A)} \arrow[r] & \underbrace{L_0F(B)}_{\cong F(B)} \arrow[r] & \underbrace{L_0F(C)}_{\cong F(C)} \arrow[r] & 0
\end{tikzcd}$$

More details on derived functors and how they are constructed can be found in [Wei94].

$\mathrm{Tor}_i^R(M,N)$ are the left derived functors of $M\otimes_R-\colon \mathrm{Mod}_R \to \mathrm{Mod}_R$ evaluated at N. Equivalently, one can also view $\mathrm{Tor}_i^R(M,N)$ as the left derived functors of $-\otimes_R N$ evaluated at M (see [Wei94, Chapter 2.7, Theorem 2.7.2]). One can also show that Tor is symmetric, i.e. $\mathrm{Tor}_*^R(M,N) \cong \mathrm{Tor}_*^R(N,M)$ (see [Wei94, Chapter 3.1, Remark (Balancing Tor)]).

A free resolution F of N is an exact sequence of R-modules of the form:

$$F\colon \cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0$$

in which every F_j, $j \in \mathbb{N}$, is a free R-module. For an R-module N, we can compute $\mathrm{Tor}_i^R(M,N)$ by using a free resolution of N:

Proposition 80. *Let M, N be two R-modules and let*

$$F\colon \cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0$$

be an arbitrary free resolution of N. Then $\mathrm{Tor}_*^R(M,N)$ is given by the homology of the following complex:

$$\cdots \to M \otimes_R F_n \to M \otimes_R F_{n-1} \to \cdots \to M \otimes_R F_1 \to M \otimes_R F_0 \to 0.$$

Proof. See [Wei94, Chapter 3.2, Flat Resolution Lemma 3.2.8] and note that free R-modules are in particular flat. □

Tor commutes with flat base change:

Proposition 81. *Let R be a ring, let T be a flat R-algebra, and let A, B be two R-modules. Then*

$$T \otimes_R \mathrm{Tor}_i^R(A,B) \cong \mathrm{Tor}_i^T(A \otimes_R T, T \otimes_R B).$$

Proof. See [Wei94, Chapter 3.2, Corollary 3.2.10]. □

Corollary 82. *Let R be a ring and let $S \subseteq R$ be multiplicatively closed. Then* Tor *commutes with localization, i.e.*

$$S^{-1}\,\mathrm{Tor}_i^R(A,B) \cong \mathrm{Tor}_i^{S^{-1}R}(S^{-1}A, S^{-1}B)$$

for all R-modules A and B.

Proof. In Proposition 59 we have seen that we can interpret localizing R-modules at S as a tensoring over R with $S^{-1}R$. By Proposition 60 localization preserves exactness and thus $S^{-1}R$ is flat over R. So we can use Proposition 81 and get:

$$\begin{aligned} S^{-1}\operatorname{Tor}_i^R(A,B) &\cong S^{-1}R \otimes_R \operatorname{Tor}_i^R(A,B) \\ &\cong \operatorname{Tor}_i^{S^{-1}R}(A \otimes_R S^{-1}R, S^{-1}R \otimes_R B) \\ &\cong \operatorname{Tor}_i^{S^{-1}R}(S^{-1}A, S^{-1}B). \end{aligned}$$

□

The scalar multiplication on the R-modules $\operatorname{Tor}_i^R(M,N)$ is induced by the respective multiplications on M and N:

Proposition 83. *Let R be a ring, $r \in R$ and A, B two R-modules. If $\mu\colon A \to A$ is multiplication with r, then for all $n \in \mathbb{N}$ the induced map $\mu_*\colon \operatorname{Tor}_i^R(A,B) \to \operatorname{Tor}_i^R(A,B)$ is multiplication with r on $\operatorname{Tor}_i^R(A,B)$ as well. The same holds for multiplication maps on B.*

Proof. See [Wei94, Chapter 3.2, Lemma 3.2.11]. □

The next Proposition will be crucial for the main part of this thesis:

Proposition 84. *Let R be a Noetherian ring, N an R-module with a free resolution of length r of the form:*

$$0 \longrightarrow R^{n_r} \longrightarrow R^{n_r} \longrightarrow \ldots \longrightarrow R^{n_0} \to N \longrightarrow 0$$

with $n_j \in \mathbb{N}$, $0 \leq j \leq r$ and $n_r \neq 0$ and let M be an arbitrary R-module. Then the following statements are true:

1. $\operatorname{Tor}_i^R(M,N) = 0$ *for all* $i > r$.
2. *If* $\operatorname{Supp}(N) \cap \operatorname{Ass}(M) = \emptyset$ *then* $\operatorname{Tor}_r^R(M,N) = 0$.

Proof. By assumption, N has a free resolution of the form

$$0 \longrightarrow R^{n_r} \longrightarrow R^{n_{r-1}} \longrightarrow \ldots \longrightarrow R^{n_0} \to N \longrightarrow 0$$

with $n_j \in \mathbb{N}$, $0 \leq j \leq r$ and $n_r \neq 0$. We use Proposition 80 to calculate $\operatorname{Tor}_i^R(M,N)$. For this we consider the complex

$$0 \longrightarrow \underbrace{R^{n_r} \otimes_R M}_{\cong M^{n_r}} \longrightarrow \underbrace{R^{n_r} \otimes_R M}_{M^{n_{r-1}}} \longrightarrow \ldots \longrightarrow \underbrace{R^{n_0} \otimes_R M}_{M^{n_0}} \longrightarrow 0$$

and calculate its homology:

$$\mathrm{Tor}_i^R(M,N) = \ker(M^{n_i} \to M^{n_{i-1}})/\operatorname{Im}(M^{n_{i+1}} \to M^{n_i}).$$

(To cover all cases, set $n_i = 0$ for $i > r$ and $i = -1$.)

The first statement follows directly, since for $i > r$ the kernel $\ker(M^{n_i} \to M^{n_{i-1}}) = \ker(\{0\} \to M^{n_{i-1}})$ always vanishes.

It is left to show that $\mathrm{Tor}_r^R(M,N) = 0$ if $\mathrm{Supp}(N) \cap \mathrm{Ass}(M) = \emptyset$. The image $\operatorname{Im}(M^{n_{r+1}} \to M^{n_r})$ is zero since $n_{r+1} = 0$. So $\mathrm{Tor}_r^R(M,N)$ is the kernel $\ker(M^{n_r} \to M^{n_{r-1}})$, and hence the following two statements are equivalent:

1. $\mathrm{Tor}_r^R(M,N) = 0$.
2. $M^{n_r} \to M^{n_{r-1}}$ is injective.

In Proposition 74 we have seen that injectivity can be checked at associated primes, i.e. $M^{n_r} \to M^{n_{r-1}}$ is injective if and only if the localization $(M^{n_r})_{\mathfrak{p}} \to (M^{n_{r-1}})_{\mathfrak{p}}$ is injective at every associated prime $\mathfrak{p}$ in $\mathrm{Ass}(M^{n_r})$. By Corollary 76, the set of associated primes of M^{n_r} equals $\mathrm{Ass}(M)$. Thus

3. $(M^{n_r})_{\mathfrak{p}} \to (M^{n_{r-1}})_{\mathfrak{p}}$ is injective for all $\mathfrak{p} \in \mathrm{Ass}(M)$.

is equivalent to the second statement.

For a fixed prime $\mathfrak{p} \in \mathrm{Ass}(M)$ the map $(M^{n_r})_{\mathfrak{p}} \to (M^{n_{r-1}})_{\mathfrak{p}}$ is injective if and only if its kernel is zero. As we have seen in Corollary 61, localization commutes with kernels, so $\ker((M^{n_r})_{\mathfrak{p}} \to (M^{n_{r-1}})_{\mathfrak{p}}) = 0$ if and only if $\ker(M^{n_r} \to M^{n_{r-1}})_{\mathfrak{p}}$ vanishes. Since $\mathrm{Tor}_r^R(M,N) = \ker(M^{n_r} \to M^{n_{r-1}})$ the following statement is equivalent to the third statement:

4. $\mathrm{Tor}_r^R(M,N)_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathrm{Ass}(M)$.

Now suppose $\mathrm{Supp}(N) \cap \mathrm{Ass}(M) = \emptyset$. Let $\mathfrak{p}$ be in $\mathrm{Ass}(M)$ and consider $\mathrm{Tor}_r^R(M,N)_{\mathfrak{p}}$. By Corollary 82, Tor commutes with localization, i.e. $\mathrm{Tor}_r^R(N,M)_{\mathfrak{p}} \cong \mathrm{Tor}_r^{R_{\mathfrak{p}}}(N_{\mathfrak{p}}, M_{\mathfrak{p}})$. The right side vanishes since $N_{\mathfrak{p}} = \{0\}$ by the previous assumption. Since this is the case for all $\mathfrak{p} \in \mathrm{Ass}(M)$ we can conclude that $\mathrm{Tor}_r^R(M,N) = 0$, which proves the second statement. $\square$

2.4 Regular Sequences and the Koszul Complex

The Koszul complex will be of great use when computing the Tor functors in our setting. We will introduce regular sequences and the Koszul complex using the corresponding chapter of Matsumura [Mat87, Chapter 6.16] as a reference.

Definition 85 (regular sequences)**.** *Let R be a ring and M an R-module. A sequence $a_1, a_2, \dots, a_n$ of elements in R is called M-*regular *if the following conditions are true:*

1. *a_1 is not a zero divisor in M, i.e. a_1 annihilates no element $m \in M \setminus \{0\}$.*
2. *a_i is a not a zero divisor in $M/(a_1, \dots, a_{i-1})M$ for all $2 \leq i \leq n$.*
3. *$M/(a_1, \dots, a_n)M \neq 0$.*

For $M = R$ we will call the sequence $a_1, \dots, a_n$ regular*.*

In this thesis, the regular sequences with which we work will appear as generators of kernels:

Proposition 86. *Let T be a torus and T' a subtorus of T. Then we can find an RT-regular sequence of the form $\lambda_1, \lambda_2, \dots, \lambda_{\mathrm{rank}(T)-\mathrm{rank}(T')} \in RT$ generating the kernel of the map $i^* \colon RT \to RT'$ induced by the inclusion $T' \subseteq T$.*

The following proof is based on the notes in [Zib]:

Proof. In Proposition 43 we have seen that the representation rings of RT and RT' are given by:

$$\begin{aligned} RT &= \mathbb{Z}[x_1^{\pm}, \dots, x_{\mathrm{rank}(T')}^{\pm 1}, y_1^{\pm 1}, \dots, y_{\mathrm{rank}(T)-\mathrm{rank}(T')}^{\pm 1}] \\ RT' &= \mathbb{Z}[x_1^{\pm}, \dots, x_{\mathrm{rank}(T')}^{\pm 1}] \end{aligned}$$

with i^* sending $x_i \in RT$ to $x_i \in RT'$ and $y_j \in RT$ to $1 \in RT'$. Then the sequence given by $\lambda_j = y_j - 1$ for $1 \leq j \leq \mathrm{rank}(T) - \mathrm{rank}(T')$ is an RT-regular sequence generating the kernel of i^*. The regularity can be seen in the following way: As a Laurent ring, the ring RT is an integral domain, i.e. RT has no zero divisors. So for λ_1 the first condition of Definition 85 is fulfilled. $RT/(\lambda_1)$ is isomorphic to the Laurent ring given by $\mathbb{Z}[x_1^{\pm}, \dots, x_{\mathrm{rank}(T')}^{\pm 1}, y_2^{\pm 1}, \dots, y_{\mathrm{rank}(T)-\mathrm{rank}(T')}^{\pm 1}]$. So one can inductively argue that the second condition of Definition 85 is true for all λ_i, $i > 1$. $\square$

Now, we define the Koszul complex:

Definition 87 (Koszul complex)**.** *Let R be a ring and $a_1, a_2, \dots, a_n$ a sequence of elements in R. The* Koszul complex *$K_\bullet(a_1, \dots, a_n)$, or $K_\bullet(\overline{a})$ for short, corresponding to this sequence is given by:*

1. *$K_0(\overline{a}) = R$*

2. $K_i(\overline{a}) = 0$ *for* $i < 0$ *or* $i > n$.

3. $K_k(\overline{a}) = \oplus Re_{i_1 \ldots i_k}$ *is the free* R*-module of rank* $\binom{n}{k}$ *with basis* $\{e_{i_1,\ldots,i_k} \mid 1 \leq i_1 < i_2 < \ldots < i_k \leq n\}$.

4. *Differentials* $d_k \colon K_k(\overline{a}) \to K_{k-1}(\overline{a})$, $1 \leq k \leq n$, *given on the basis by:*

$$d_k(e_{i_1,\ldots,i_k}) = \sum_{r=1}^{k} (-1)^{r-1} a_{i_r} e_{i_1,\ldots,i_{r-1},\hat{i_r},i_{r+1},\ldots,i_k}$$

   (The notation $\hat{i_r}$ *means that the* r*-th index is deleted. In addition, if there is no index left we set* $e = 1$*.)*

5. *For every remaining index* $d_k \colon K_k(\overline{a}) \to K_{k-1}(\overline{a})$ *is the constant zero map.*

This definition yields a chain complex:

Proposition 88. *The Koszul complex is a chain complex, i.e. the differential fulfills* $d_{k-1} \circ d_k = 0$ *for all* $k \in \mathbb{N}$.

Proof. For $k < 1$ and $k > 1$ this is clear since one of the occurring maps is zero. Now, let $1 \leq k \leq n$. It suffices to show that $d_{k-1} \circ d_k$ vanishes on all basis elements $e_{i_1,\ldots,i_k}$:

$$\begin{aligned}
d_{k-1} \circ d_k(e_{i_1,\ldots,i_k}) &= d_{k-1}\left(\sum_{r=1}^{k} (-1)^{r-1} a_{i_r} e_{i_1,\ldots,i_{r-1},\hat{i_r},i_{r+1},\ldots,i_k}\right) \\
&= \sum_{r=1}^{k} (-1)^{r-1} a_{i_r} d_{k-1}\left(e_{i_1,\ldots,i_{r-1},\hat{i_r},i_{r+1},\ldots,i_k}\right) \\
&= \sum_{r=1}^{k} (-1)^{r-1} a_{i_r} \left(\begin{array}{c} \sum_{s=1}^{r-1} (-1)^{s-1} a_{i_s} e_{i_1,\ldots,\hat{i_s},\ldots,\hat{i_r},\ldots,i_k} \\ + \sum_{t=r+1}^{k} (-1)^{t-2} a_{i_t} e_{i_1,\ldots,\hat{i_r},\ldots,\hat{i_t},\ldots,i_k} \end{array} \right) \\
&= \left(\begin{array}{c} \sum_{r=1}^{k} \sum_{s=1}^{r-1} (-1)^{r+s} a_{i_r} a_{i_s} e_{i_1,\ldots,\hat{i_s},\ldots,\hat{i_r},\ldots,i_k} \\ + \sum_{r=1}^{k} \sum_{t=r+1}^{k} (-1)^{r+t-1} a_{i_r} a_{i_t} e_{i_1,\ldots,\hat{i_r},\ldots,\hat{i_t},\ldots,i_k} \end{array} \right) \\
&= \left(\begin{array}{c} \sum_{1 \leq s < r \leq k} (-1)^{r+s} a_{i_r} a_{i_s} e_{i_1,\ldots,\hat{i_s},\ldots,\hat{i_r},\ldots,i_k} \\ + \sum_{1 \leq r < t \leq k} (-1)^{r+t-1} a_{i_r} a_{i_t} e_{i_1,\ldots,\hat{i_r},\ldots,\hat{i_t},\ldots,i_k} \end{array} \right) \\
&= 0 \text{ (by renaming the indices in the second sum).}
\end{aligned}$$

$\square$

It is also common to construct the Koszul complex $K_\bullet(\overline{a})$ as a tensor product of chain complexes. For this we take the Koszul complexes corresponding to the respective elements of the sequence $K_\bullet(a_i) = \left(0 \to R \xrightarrow{\cdot a_i} R \to 0\right)$ and set $K_\bullet(\overline{a}) = K_\bullet(a_1) \otimes K_\bullet(a_2) \otimes \ldots \otimes K_\bullet(a_n)$ as their tensor product. This definition yields the same complex as we defined before.

Definition 89. *For an R-module M and a sequence $a_1, \ldots, a_r \in R$ we set $K_\bullet(\overline{a}, M) = K_\bullet(\overline{a}) \otimes M$.*

When taking the homology of a Koszul complex $K_\bullet(\overline{a})$ or $K_\bullet(\overline{a}, M)$, we write $H_k(K_\bullet(\overline{a})) = H_k(\overline{a})$ and $H_k(K_\bullet(\overline{a}, M)) = H_k(\overline{a}, M)$, respectively.

Theorem 90. *Let R be a ring and M an R-module. Let $a_1, \ldots, a_n \in R$ be M-regular. Then:*

$$H_k(\overline{a}, M) = 0 \text{ for } k > 0 \quad \text{ and } H_0(\overline{a}, M) = M/\overline{a}M \ .$$

Proof. See [Mat87, Chapter 6.16, Theorem 16.5(i)]. □

Proposition 91. *Let R be a ring and $a_1, \ldots, a_n$ a regular sequence in R. Then the Koszul complex yields a free resolution of $R/(a_1, \ldots a_n)$ of length n as an R-module.*

Proof. Theorem 90 yields that the Koszul complex:

$$K_\bullet(\overline{a}) = (0 \to K_n(\overline{a}) \to K_{n-1}(\overline{a}) \to \cdots \to K_1(\overline{a}) \to K_0(\overline{a}) \to 0)$$

is exact everywhere except at $K_0(\overline{a})$. In addition, we know that $H_0(\overline{a}) = R/\overline{a}R = R/(a_1, \ldots, a_n)$. So by adding $H_0(\overline{a})$ at the left end of the complex, we gain a chain complex:

$$0 \to K_n(\overline{a}) \to K_{n-1}(\overline{a}) \to \cdots \to K_1(\overline{a}) \xrightarrow{d_1} K_0(\overline{a}) \xrightarrow{d_0} H_0(\overline{a}) \to 0$$

that is exact everywhere, because by definition $H_0(\overline{a}) = \ker(d_0)/\operatorname{Im}(d_1)$. Since every $K_i(\overline{a})$ is a free R-module, the chain complex above is a free resolution of $R/(a_1, \ldots, a_n)$. □

Chapter 3

A Generalization of a Theorem of Pittie

The first section of the third chapter, the main part of this thesis, deals with biquotient manifolds, a type of manifolds that appears as the orbit space of two closed subgroups acting freely on a compact Lie group. We will also consider some examples for these types of group actions appearing in the studies of Eschenburg [ZK09].

In the next section, the *strict double coset condition* is introduced. We will see that this condition is equivalent to the biquotient action of the previous section being free. In addition, we will consider some examples and check if they satisfy the strict double coset condition. To simplify the verification of the strict double coset condition, we will see that for closed connected subgroups of compact Lie groups, one may equivalently check if two maximal tori of the respective subgroups satisfy the condition.

In Section 3, we will again state the conjecture while taking a look at the theorems of Singhof, Pittie and Steinberg that served as motivation and evidence for the conjecture. In addition, we will prove that is suffices to only consider subtori as closed subgroups.

The fourth section deals with the study of primes in $\mathrm{Ass}_{RG}(RH)$ and $\mathrm{Supp}_{RG}(RH)$ for closed subgroups H of compact Lie groups G. We will present some of Segal's work in [Seg68] which allows us to build a bridge between the strict double coset condition and the supports of the respective representation rings. We will study the intersection $\mathrm{Supp}(RH_1) \cap \mathrm{Supp}(RH_2)$ for two subgroups H_1 and H_2 satisfying the strict double coset condition and see that none of the ideals appearing in this intersection are associated primes of nontrivial, closed connected subgroups of compact connected Lie groups.

Then, in Section 5 we prove the conjecture in the case that one of the subgroups is of rank 1 or lower using our results in the previous sections.

Finally, in the last section, we will conclude by discussing an approach for the general case.

3.1 Biquotient Manifolds

Let G be a compact connected Lie group and H_1, H_2 two closed subgroups of G. Then $H_1 \times H_2$ operates on G from the right by:

$$G \times (H_1 \times H_2) \longrightarrow G, \quad (g, (h_1, h_2)) \longmapsto h_1^{-1} g h_2.$$

We are interested in the orbit space $G/(H_1 \times H_2)$ of this action. The following theorem states under which conditions the orbit space admits the structure of a smooth manifold:

Theorem 92 (Quotient Manifold Theorem)**.** *Let M be a smooth manifold and G a Lie group acting smoothly, freely and properly on M (from left or right). Then the orbit space M/G admits the structure of a smooth topological manifold of dimension* $\dim M - \dim G$*. There is a unique smooth structure such that $G \to M/G$ is a smooth submersion.*

Proof. See [Lee13, Chapter 21, Theorem 21.10]. □

In our setting, the smooth manifold is the compact Lie group G and $H_1 \times H_2$ acts on it. The action is smooth since multiplication in Lie groups is a smooth map. As H_1 and H_2 are closed subspaces of the compact space G, they are *compact* as well. In particular, $G \times (H_1 \times H_2)$ is compact. So the action is proper because it is a continuous map from a compact space to a Hausdorff space. So if the action above is free, we get the desired smooth structure on the orbit space $G/(H_1 \times H_2)$.

The orbit space $G/(H_1 \times H_2)$ of this group action is also denoted as $H_1 \setminus G/H_2$. This notation is used, because the quotient can be seen in the following way: Consider the right action of H_2 on G given by $(g, h_2) \mapsto gh_2$. This action is clearly free and fulfills the remaining conditions of Theorem 92 because of similar reasons as above. So G/H_2 admits the structure of a smooth manifold. Then we can let H_1 act on the orbit space from the left by $h_1(gH_2)$ and the biquotient $H_1 \setminus G/H_2$ has the structure of a smooth manifold if this action is free. If the action is free, one also calls the resulting smooth manifold a *(strict) double coset manifold*.

Many examples for these types of manifolds can be extracted from Eschenburgs study and classification of biquotients [Esc84] that was summarized by Ziller and Kerin [ZK09]:

Example 93 ([ZK09], Section 5, Remark 2 after Theorem 5.1)**.** *Consider the group $G = \mathrm{SU}(2m)$ for $m \in \mathbb{N}_{>0}$. Then the subgroups*

$$H_1 = \left\{ \begin{pmatrix} z & & & & & \\ & \ddots & & & & \\ & & z & & & \\ & & & z^{-1} & & \\ & & & & \ddots & \\ & & & & & z^{-1} \end{pmatrix} \in \mathrm{SU}(2m) \;\middle|\; z \in S^1 \right\} \quad \textit{and}$$

$$H_2 = \left\{ \begin{pmatrix} (w_1 \ldots w_{2m-2})^{-1} & & & & \\ & w_1 & & & \\ & & \ddots & & \\ & & & w_{2m-2} & \\ & & & & 1 \end{pmatrix} \in \mathrm{SU}(2m) \;\middle|\; w_1, \ldots, w_{2m-1} \in S^1 \right\}$$

yield a free action on G and thus the orbit space $H_1 \setminus G/H_2$ admits the structure of a smooth manifold. The proof that this action is free will be given in the next section.

We state another example resulting from this classification:

Example 94 ([ZK09], Section 5, Theorem 5.2)**.** *Consider the group $G = \mathrm{Sp}(n)$ for $n \geq 3$. Then for the subgroups H_1, H_2 given by:*

$$H_1 = \left\{ \begin{pmatrix} A & \\ & \overline{A} \end{pmatrix} \;\middle|\; A = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & z \end{pmatrix} \quad \textit{with} \;\; z \in S^1 \right\} \quad \textit{and}$$

$$H_2 = \left\{ \begin{pmatrix} A & \\ & \overline{A} \end{pmatrix} \;\middle|\; A = \begin{pmatrix} w_1 & & & \\ & \ddots & & \\ & & w_{n-1} & \\ & & & (w_1 \ldots w_{n-1})^{-1} \end{pmatrix} \textit{with } w_1, \ldots, w_{n-1} \in S^1 \right\}$$

the orbit space $H_1 \setminus G/H_2$ has the structure of a smooth manifold.

More examples can be found in Eschenburg's classification of biquotient actions of maximal rank (see [ZK09, Section 6, Table A and Table B]). It is important to take into account that Eschenburg's definition of biquotients is a bit more general than ours: He takes a closed subgroup U of $G \times G$ together with the projections p_l and p_r to the left and right factor, respectively. The action of U on G is then given by:

$$G \times U \longrightarrow G, \quad (g, u) \longmapsto p_l(u)^{-1} g p_r(u).$$

If U is of the form $H_1 \times H_2$ for two closed subgroups H_1, H_2 of G, this coincides with our definition. However, in his classification there also occur groups $U \subseteq G \times G$ that are not of this form (see for example [ZK09, Section 6, Table A (1) and Table B (9)]).

3.2 Strict Double Coset Condition

The *strict double coset condition* arises from the double coset manifolds introduced in the previous section:

Definition 95 (strict double coset condition)**.** *Let G be a group and H_1, H_2 two subgroups of G. If H_1 intersects every conjugate of H_2 trivially we say that the pair of subgroups H_1, H_2 satisfies the* strict double coset condition*.*

If G is abelian, for instance $G = T^k$, this condition reduces to the requirement that H_1 and H_2 intersect trivially.

Demanding H_1 and H_2 to fulfill the strict double coset condition means demanding the action of $H_1 \times H_2$ on G to be free [Sin93, Section 1, (1.1)]:

Proposition 96. *Let G be a group and H_1, H_2 two subgroups of G. We consider the right group action of $H_1 \times H_2$ on G given by $(g, (h_1, h_2)) \mapsto h_1^{-1} g h_2$. This group action is free if and only if H_1 and H_2 satisfy the strict double coset condition.*

Proof. Assume that H_1 and H_2 fulfill the strict double coset condition. Let $(h_1, h_2), (h_1', h_2') \in H_1 \times H_2$ and $g \in G$ so that $h_1^{-1} g h_2 = h_1'^{-1} g h_2'$. To prove that the action is free we have to show that $h_1 = h_1'$ and $h_2 = h_2'$. We have:

$$\begin{aligned} h_1^{-1} g h_2 = h_1'^{-1} g h_2' \quad &\iff \quad h_1' h_1^{-1} g = g h_2' h_2^{-1} \\ &\iff \quad h_1' h_1^{-1} = g h_2' h_2^{-1} g^{-1}. \end{aligned}$$

The element $h_1' h_1^{-1}$ is in H_1 and $g h_2' h_2^{-1} g^{-1}$ is an element of H_2 conjugated with g. Because of the strict double coset condition, these elements have to be the neutral element $1 \in G$. It follows directly that $h_1 = h_1'$ and $h_2 = h_2'$.

Now, let the action of $H_1 \times H_2$ be free. Let $h_1 \in H_1$ and $h_2 \in H_2$ be conjugate. We have to show that $h_1 = h_2 = 1$. By definition there exists a $g \in G$ so that $g h_1 g^{-1} = h_2$. Equivalently $(h_1^{-1})^{-1} g^{-1} = g^{-1} h_2$. Thus $(h_1^{-1}, 1), (1, h_2) \in H_1 \times H_2$ have the same action on $g^{-1} \in G$. Since the action of $H_1 \times H_2$ is free, it follows that $h_1 = 1$ and $h_2 = 1$. □

So for compact Lie groups G and two closed subgroups H_1 and H_2 of G we are able to check whether the coset space $H_1 \setminus G / H_2$ admits the structure of a smooth manifold by checking if the strict double coset condition is fulfilled. We illustrate this by again considering Example 93:

Example 97. *Let $G = \mathrm{SU}(2m)$, $m \in \mathbb{N}_{>0}$, and let H_1 and H_2 be as in Example 93. We show that H_1 and H_2 satisfy the strict double coset condition: Let $A \in H_1$ and $B \in H_2$ and suppose that they are conjugate. We have to* *show that both A and B are the identity. The elements are of the following form:*

$$A = \begin{pmatrix} z & & & & & \\ & \ddots & & & & \\ & & z & & & \\ & & & z^{-1} & & \\ & & & & \ddots & \\ & & & & & z^{-1} \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} (w_1 \dots w_{2m-2})^{-1} & & & & \\ & w_1 & & & \\ & & \ddots & & \\ & & & w_{2m-2} & \\ & & & & 1 \end{pmatrix}$$

with $z, w_1, \dots, w_{2m-1} \in S^1$. The eigenvalues of a matrix are invariant under conjugation. As matrix B has 1 as an eigenvalue with multiplicity at least 1, matrix A has to fulfill the same. Thus $z = z^{-1} = 1$ and A is the identity matrix. The same follows for B.

To verify that two subgroups satisfy the strict double coset condition it often suffices to restrict oneself to tori [Sin93, Section 1, (1.4)]:

Proposition 98. *Let G be a compact Lie group and H_1, H_2 two closed connected subgroups of G. Let T_1 and T_2 be maximal tori of H_1 and H_2, respectively. Then the following statements are equivalent:*

1. *H_1 and H_2 satisfy the strict double coset condition.*
2. *T_1 and T_2 satisfy the strict double coset condition.*

Proof. The implication (1) $\Rightarrow$ (2) is trivial. Now assume that T_1 and T_2 satisfy the strict double coset condition. Let $h_1 \in H_1$, $h_2 \in H_2$ and $g \in G$ so that $h_1 = gh_2g^{-1}$. We have to show that $h_1 = h_2 = 1$. Since T_1 is a maximal torus of H_1 there exist elements $a \in H_1$ and $t_1 \in T_1$ so that $h_1 = at_1a^{-1}$ (see Proposition 8). Similarly there are $b \in H_2$ and $t_2 \in T_2$ satisfying $h_2 = bt_2b^{-1}$. Then we have:

$$at_1a^{-1} = h_1 = gh_2g^{-1} = gbt_2b^{-1}g^{-1} = (gb)t_2(gb)^{-1}$$

and thus t_1 and t_2 are conjugate in G. Since T_1 and T_2 satisfy the strict double coset condition it follows that $t_1 = t_2 = 1$ yielding $h_1 = h_2 = 1$. $\square$

So far, we only considered examples in which at least one of the occurring groups is of rank 1. However, there also exist examples, in which this is not the case. The following example was brought to our attention by Jason DeVito:

Example 99. *Let $G = \mathrm{SU}(6)$ and consider the subgroups $H_1 = H_2 = \mathrm{SU}(3)$ embedded into G in the following ways:*

$$H_1 = \mathrm{SU}(3) \xrightarrow{i_1} \mathrm{SU}(6), \quad A \longmapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

$$H_2 = \mathrm{SU}(3) \xrightarrow{i_2} \mathrm{SU}(6), \quad B \longmapsto \begin{pmatrix} B & 0 \\ 0 & I_3 \end{pmatrix}$$

Then the action is of the form:

$$G \times (H_1 \times H_2) \longrightarrow G, \quad (C, (A, B)) \longmapsto \begin{pmatrix} A^{-1} & 0 \\ 0 & A^{-1} \end{pmatrix} \ C \ \begin{pmatrix} B & 0 \\ 0 & I_3 \end{pmatrix}.$$

To show that this action is free, it suffices to consider maximal tori T_1 and T_2 of H_1 and H_2, respectively. For $\mathrm{SU}(3)$ we can choose the maximal torus consisting of elements of the form:

$$T = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \;\middle|\; a, b, c \in S^1 \text{ with } c = a^{-1}b^{-1} \right\}.$$

We will show that $i_1(T)$ and $i_2(T_2)$ satisfy the strict double coset condition. For this take two elements $A \in i_1(T)$ and $B \in i_2(T_2)$ which are conjugate. They have to be of the following form:

$$A = \begin{pmatrix} a &&&&& \\ & b &&&& \\ && c &&& \\ &&& a && \\ &&&& b & \\ &&&&& c \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} x &&&&& \\ & y &&&& \\ && z &&& \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}$$

with $a, b, c, x, y, z \in S^1$ such that $abc = 1$ and $xyz = 1$. The eigenvalues are invariant under conjugation, thus A and B share the same eigenvalues. For diagonal matrices, the eigenvalues are exactly the diagonal entries. Thus matrix B has the eigenvalue 1 with a multiplicity of at least 3. So at least three diagonal entries of matrix A have to be 1. Since we have at most three distinct entries $a, b, c \in S^1$ and each one occurs two times, we know that two of them have to be 1. Since the product abc is 1, we can conclude that $a = b = c = 1$. As both matrices share the same eigenvalues, it follows for B that $x = y = z = 1$. So H_1 and H_2 satisfy the strict double coset condition, and both of them are of rank 2.

3.3 Motivation and Evidence for the Conjecture

We start by outlining the setting of the conjecture. When introducing the representation ring, we have seen that it has some functorial properties: Any homomorphism $f\colon H \to G$ of Lie groups induces a ring homomorphism $f^*\colon RG \to RH$. In particular, for every closed subgroup $H \subseteq G$, the inclusion $i\colon H \hookrightarrow G$ induces a ring homomorphism $i^*\colon RG \to RH$ which restricts representations of G to H. Using this we can equip RH with an RG-module structure in the following way:

$$\begin{aligned} RG \times RH &\longrightarrow RH \\ (a, b) &\longmapsto i^*(a) \cdot b. \end{aligned}$$

Instead of $i^*(a) \cdot b$ we may also write $a \cdot b$ or ab.

In this thesis, our aim is to show the following conjecture proposed by Marcus Zibrowius in [Zib]:

Conjecture 100. *Let G be a compact connected Lie group with RG isomorphic to a tensor product of a polynomial algebra and a Laurent algebra. In addition let $H_1, H_2 \subseteq G$ be two closed connected subgroups that satisfy the strict double coset condition. Then:*

$$\operatorname{Tor}_i^{RG}(RH_1, RH_2) = 0 \text{ for all } i > \operatorname{rank} G - (\operatorname{rank} H_1 + \operatorname{rank} H_2).$$

As we already mentioned in the introduction, results from Singhof served as a major motivation and inspiration for the conjecture. In [Sin93], Singhof studied topological properties of double coset manifolds. One of his results was:

Theorem 101. *Let G be a compact connected Lie group and H_1, H_2 two closed connected subgroups of G satisfying the strict double coset condition. Let EG be the total space and $BG = (EG)/G$ the base space of a fixed universal G-bundle. Take $BH_1 = (EG)/H_1$ and $BH_2 = (EG)/H_2$ as the classifying spaces of H_1 and H_2, respectively. Then for cohomology with coefficients in $\mathbb{Q}$, we have*

$$\operatorname{Tor}^{s,*}_{H^*(BG)}(H^*(BH_1), H^*(BH_2)) = 0$$

for $s < \operatorname{rank} H_1 + \operatorname{rank} H_2 - \operatorname{rank} G$.

Proof. See [Sin93, Section 6, (6.4) Proposition] and note that the required conditions are fulfilled for $R = \mathbb{Q}$. □

While the theorem of Singhof served as a motivation for the conjecture, the theorems which we will now study can be considered as an evidence for it. For this, we consider the special case that $H_1 = T$ for a maximal torus T in G and $H_2 = \{1\}$. In Theorem 44, we have seen that in this case the induced map $RG \to RT$ is an inclusion and RG consists of all elements in RT that are invariant under the action of the Weyl group W in G. The Theorem of Pittie yields a condition under which the resulting RG-module RT is free over RG:

Theorem 102 (Theorem of Pittie)**.** *Let G be a compact connected Lie group with $\pi_1(G)$ free and T a maximal torus of G. Then RT is a free RG-module of rank equal to the order $|W|$ of the Weyl group.*

Proof. See [Pit72, Theorem 1]. □

However, in general this is not an equivalence. Steinberg generalized this theorem by giving a full characterization of the cases in which RT is free over RG:

Theorem 103. *Let G be a compact connected Lie group and S its semisimple component. Then the following conditions are equivalent:*

1. *RG' is free over RG for every connected subgroup G' of maximal rank.*
2. *RT is free over RG for some maximal torus T.*
3. *RG is the tensor product of a polynomial algebra and a Laurent algebra.*
4. *RS is a polynomial algebra.*
5. *S is a direct product of simple groups, each simply connected or of type $SO(2r+1)$.*

Proof. See [Ste75, Theorem 1.2]. □

In our conjecture, the third condition of the previous theorem appears as an assumption for G. So for a maximal torus T of G it follows that RT is free as an RG-module. As free RG-modules are in particular flat over RG, for all $i > 0$ it follows:

$$\mathrm{Tor}_i^{RG}(RT, R(\{1\})) \cong \mathrm{Tor}_i^{RG}(RG^m, \mathbb{Z}) = 0.$$

So the theorems of Pittie and Steinberg can be seen as a special case of the conjecture.

Next, we quickly list some groups for which the conditions of Theorem 103 are fulfilled: The classical groups which we considered in Subsection 1.5 have the following fundamental groups:

- $\pi_1(\mathrm{U}(n)) = \mathbb{Z}$,

- $\pi_1(\mathrm{SU}(n)) = 0$,
- $\pi_1(\mathrm{Sp}(n)) = 0$,
- $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2\mathbb{Z}$ for $n > 2$.

So we can apply the Theorem of Pittie for $G = \mathrm{U}(n), \mathrm{SU}(n)$ and $\mathrm{Sp}(n)$. For $\mathrm{SO}(2n+1)$ with $n > 0$ the fifth condition of Theorem 103 is true.

We will conclude the section by showing that one may assume that H_1 and H_2 are tori:

Proposition 104. *Let G be a compact connected Lie group with RG isomorphic to a tensor product of a polynomial algebra and a Laurent algebra. In addition let $H_1, H_2 \subseteq G$ be two closed connected subgroups that satisfy the strict double coset condition. Let $T_1 \subseteq H_1$ and $T_2 \subseteq H_2$ be two maximal tori of H_1 and H_2, respectively. Then the following implication holds:*

$$\mathrm{Tor}_i^{RG}(RT_1, RT_2) = 0 \quad \Longrightarrow \quad \mathrm{Tor}_i^{RG}(RH_1, RH_2) = 0.$$

The following proof is bases on notes in [Zib]:

Proof. The two inclusion maps of the maximal tori into the subgroups

$$i_1 \colon T_1 \hookrightarrow H_1 \qquad i_2 \colon T_1 \hookrightarrow H_2$$

induce the following ring homomorphisms:

$$i_1^* \colon RH_1 \hookrightarrow RT_1 \qquad i_2^* \colon RH_2 \hookrightarrow RT_2.$$

These ring homomorphisms can be seen as RG-linear maps, because the RG-module structure on RT_j comes from the ring homomorphism $RG \to RH_j \hookrightarrow RT_j$ factoring over RH_j.

Atiyah [Ati68, Proposition (4.9) and the following Remark (1)] proved that there exist homomorphisms $i_{j*} \colon RT_j \to RH_j$ of RH_j-modules splitting the restriction maps above, i.e. $i_{j*} \circ i_j^* = \mathrm{id}$ for $j = 1, 2$. The morphisms i_{1*} and i_{2*} are RG-linear as well: Consider $i_{1*} \colon RT_1 \to RH_1$. As an RH_1-linear map, we have $i_{1*}(h \cdot t) = h \cdot i_{1*}(t)$ for all $h \in RH_1$ and $t \in RT_1$. The RH_1-module structure comes from i_1^*, so $i_{1*}(h \cdot t) = i_{1*}(i_1^*(h)t)$. In RH_1, $h \cdot i_{1*}(t)$ is the multiplication of the respective elements. Write $i \colon H_1 \to G$ for the inclusion of H_1 in G. Then the inclusion of T_1 in G is given by the concatenation $i \circ i_1$. Then

$$i_{1*}(g \cdot r) = i_{1*}((i \circ i_1)^*(g)r) = i_{1*}(i_1^*(i^*(g))r) \underbrace{=}_{RH_1\text{-linear}} i^*(g) i_{1*}(r) = g \cdot i_{1*}(r)$$

for all $g \in RG$ and all $r \in RT_1$ and thus i_{1*} is RG-linear. The same argument can be used for i_{2*}.

Now assume $\mathrm{Tor}_i^{RG}(RT_1, RT_2) = 0$. As we have seen above there are RG-module homomorphisms so that the following diagram commutes:

$$\begin{tikzcd} RH_1 \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] & RT_1 \arrow[r] & RH_1 \end{tikzcd} .$$

If we apply the functor $\mathrm{Tor}_i^{RG}(-, RH_2)$ we get a diagram of the following form:

$$\begin{tikzcd} \mathrm{Tor}_i^{RG}(RH_1, RH_2) \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] & \mathrm{Tor}_i^{RG}(RT_1, RH_2) \arrow[r] & \mathrm{Tor}_i^{RG}(RH_1, RH_2) \end{tikzcd} .$$

The diagram shows that the vanishing of $\mathrm{Tor}_i^{RG}(RT_1, RH_2)$ implies that $\mathrm{Tor}_i^{RG}(RH_1, RH_2)$ is zero as well since its identity factors over zero. Now consider the corresponding diagram for RH_2:

$$\begin{tikzcd} RH_2 \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] & RT_2 \arrow[r] & RH_2 \end{tikzcd} .$$

Applying the functor $\mathrm{Tor}_i^{RG}(RT_1, -)$ yields:

$$\begin{tikzcd} \mathrm{Tor}_i^{RG}(RT_1, RH_2) \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] & \mathrm{Tor}_i^{RG}(RT_1, RT_2) \arrow[r] & \mathrm{Tor}_i^{RG}(RT_1, RH_2) \end{tikzcd} .$$

By assumption we have $\mathrm{Tor}_i^{RG}(RT_1, RT_2) = 0$. Similarly, this implies $\mathrm{Tor}_i^{RG}(RT_1, RH_2) = 0$. Together with the previous observation it follows that $\mathrm{Tor}_i^{RG}(RH_1, RH_2) = 0$, which proves the proposition.

□

3.4 Prime Ideals of Closed Subgroups

We want to prove Conjecture 100 using means of commutative algebra. More accurately, we want to use that if H_1, H_2 fulfill the strict double coset condition, we are able to understand the associated primes and the primes in the support better.

To build this bridge between prime ideals and the strict double coset condition, one of Segal's works [Seg68] is crucial. In this subsection we will start by outlining the results of Segal which will be used to further study the associated primes.

Proposition 105. *Let G be a compact Lie group and H a closed subgroup of G. Then the RG-module RH given by the restriction is finitely generated. In particular, RG is Noetherian.*

Proof. See [Seg68, Proposition (3.2) and Corollary (3.3)].

The second part can be concluded in the following way: By Theorem 20 and Proposition 28, we know that G has a faithful unitary representation $\psi\colon G \to \mathrm{U}(n)$ for an $n \in \mathbb{N}$. Since G is compact, its image $\psi(G) \subseteq \mathrm{U}(n)$ is compact as well. As a compact subspace of a Hausdorff space, the image $\psi(G)$ is closed in $\mathrm{U}(n)$. Using that ψ is faithful, i.e. injective, we get that G is a closed subgroup of $\mathrm{U}(n)$. So the first part of the proposition yields that RG is finitely generated as an $R(\mathrm{U}(n))$-module.

In Proposition 48 we have seen that $R(\mathrm{U}(n)) = \mathbb{Z}[\sigma_1, \ldots, \sigma_n, \sigma_n^{-1}]$ in which σ_j is the j-th elementary symmetric polynomial over n variables. The ring $\mathbb{Z}[\sigma_1, \ldots, \sigma_n]$ is a polynomial ring (see [Mac95, Chapter I.2, (2.4)]) over a Noetherian ring and thus by Hilbert's Basis Theorem Noetherian. $R(\mathrm{U}(n))$ is the localization of this ring at the multiplicatively closed subset given by the powers of σ_n and thus Noetherian by Proposition 58.

By the first part of the proposition, it follows that RG is a finitely generated $R(\mathrm{U}(n))$-module. As $R(\mathrm{U}(n))$ is Noetherian, Proposition 56 yields that RG is Noetherian as an $R(\mathrm{U}(n))$-module. Every ideal of RG is a submodule over $R(\mathrm{U}(n))$, because the scalar multiplication with elements in $R(\mathrm{U}(n))$ is given by multiplication with elements in RG. Thus RG being Noetherian over $R(\mathrm{U}(n))$ implies that RG is Noetherian as a ring. $\square$

Let $\mathfrak{p} \in \mathrm{Spec}(RG)$ be a prime in RG. Consider the following set of closed subgroups of G:

$$\mathfrak{J}_{\mathfrak{p}} = \left\{ H \subseteq G \;\middle|\; \begin{array}{l} H \text{ is a closed subgroup of } G \text{ with} \\ \mathfrak{p} \in \mathrm{Im}(\mathrm{Spec}(RH) \to \mathrm{Spec(RG)}) \end{array} \right\}.$$

This set is upward closed in the following sense:

Proposition 106. *If H is in $\mathfrak{J}_{\mathfrak{p}}$ and H' is a closed subgroup of G with $H \subseteq H'$ then $H' \in \mathfrak{J}_{\mathfrak{p}}$.*

Proof. Since the inclusion of $H \subseteq G$ factors over H' the restriction map $RG \to RH$ factors over RH'. After applying Spec we get $\mathrm{Spec}(RH) \to \mathrm{Spec}(RH') \to \mathrm{Spec}(G)$ and thus the image $\mathrm{Im}(\mathrm{Spec}(RH) \to \mathrm{Spec}(RG))$ is a subset of $\mathrm{Im}(\mathrm{Spec}(RH') \to \mathrm{Spec}(RG))$. $\square$

Proposition 107 ([Seg68]). *Let G be a compact Lie group. Then the set $\mathfrak{J}_\mathfrak{p}$ has minimal elements.*

Proof. Closed subgroups of compact Lie groups satisfy the descending chain condition, i.e. if there is a chain:

$$G \supseteq H_1 \supseteq H_2 \supseteq \ldots \supseteq H_i \supseteq \ldots$$

of closed subgroups H_i of G, then there exists an index $N \in \mathbb{N}$ so that $H_n = H_N$ for all $n \geq N$. The proof of this is based on the fact that a closed subgroup of a Lie group is an embedded submanifold. If we have a proper inclusion then either the dimension or the (finite) number of connected components decreases. A proof can be found in [AM07, Chapter 5.7, Lemma 5.7.9].

Now we can view $\mathfrak{J}_\mathfrak{p}$ as a partially ordered set with respect to the inclusion $\supseteq$. The set is not empty since for every $\mathfrak{p} \in \mathrm{Spec}(RG)$ it contains G. Because of the descending chain condition, every chain has a lower bound in $\mathfrak{J}_\mathfrak{p}$. Zorn's lemma yields the existence of a minimal element. □

Segal [Seg68] showed the following theorem:

Theorem 108. *Let G be a compact Lie group and $\mathfrak{p} \in \mathrm{Spec}(RG)$. Then any two minimal elements of $\mathfrak{J}_\mathfrak{p}$ are conjugate.*

Proof. See [Seg68, Proposition (3.7) (i)]. □

Based on this, Segal defined the *support* of a prime $\mathfrak{p} \in \mathrm{Spec}(RG)$:

Definition 109. *Let G be a compact Lie group and $\mathfrak{p} \in \mathrm{Spec}(RG)$. Then the* support *of $\mathfrak{p}$ is an arbitrary (but fixed) minimal element H in $\mathfrak{J}_\mathfrak{p}$.*

By Theorem 108, the support of $\mathfrak{p}$ is unique up to conjugation. As a consequence of Theorem 108, Segal concludes the following [Seg68, Proposition (3.7)(iv)]:

Proposition 110. *Let G be a compact Lie group, H a closed subgroup of G and $\mathfrak{p} \in \mathrm{Spec}(RG)$. Then the following statements are equivalent:*

1. $\mathfrak{p} \in \mathrm{Im}(\mathrm{Spec}(RH) \to \mathrm{Spec}(RG))$.
2. $\mathfrak{p}$ *contains* $\ker(RG \to RH)$.
3. $\mathfrak{p} \in \mathrm{Supp}_{RG}(RH)$.
4. *The support of $\mathfrak{p}$ is conjugate to a subgroup of H.*

As this Proposition will be crucial for studying the primes of subgroups fulfilling the strict double coset condition, we will prove this in detail:

Proof. **(1)** $\Leftrightarrow$ **(2)**: If $\mathfrak{p}$ is in $\mathrm{Im}(\mathrm{Spec}(RH) \to \mathrm{Spec}(RG))$, then there exists a prime $\mathfrak{a} \in \mathrm{Spec}(RH)$ so that $i^{*-1}(\mathfrak{a}) = \mathfrak{p}$ with $i^*\colon RG \to RH$ the map induced by the inclusion $H \subseteq G$. Since $0 \in \mathfrak{a}$ we have $\ker(RG \to RH) = i^{*-1}(0) \subseteq i^{*-1}(\mathfrak{a}) = \mathfrak{p}$.

Now assume that $\mathfrak{p} \in \mathrm{Spec}(RG)$ contains the kernel of $i^*\colon RG \to RH$. We have to show that $\mathfrak{p}$ is in the image of $\mathrm{Spec}(RH) \to \mathrm{Spec}(RG)$, i.e. we have to show that there exists an element $\mathfrak{a} \in \mathrm{Spec}(RH)$ so that $i^{*-1}(\mathfrak{a}) = \mathfrak{p}$. For this we consider the map:

$$RG \longrightarrow RG/\ker(i^*) \longrightarrow RH.$$

The contravariant functor Spec maps this to:

$$\mathrm{Spec}(RH) \longrightarrow \mathrm{Spec}(RG/\ker(i^*)) \longrightarrow \mathrm{Spec}(RG).$$

The primes in $RG/\ker(i^*)$ correspond to the primes in RG that contain $\ker(i^*)$ and the map $\mathrm{Spec}(RG/\ker(i^*)) \to \mathrm{Spec}(RG)$ is injective with image $\{\mathfrak{q} \in \mathrm{Spec}(RG) \mid \ker(i^*) \subseteq \mathfrak{q}\}$. In particular, $\mathfrak{p}$ is in the image. If we show that $\mathrm{Spec}(RH) \to \mathrm{Spec}(RG/\ker(i^*))$ is surjective we are finished. By Proposition 105, RH is finitely generated as an RG-module. The injective ring homomorphism $RG/\ker(i^*) \to RH$ makes RH into an $RG/\ker(i^*)$-module. It is still finitely generated since the images of RG and $RG/\ker(i^*)$ coincide and thus we can pick the same generating set as for RH as an RG-module. Since every finitely generated module is also integral, we can apply Proposition 52 which yields the surjectivity of $\mathrm{Spec}(RH) \to \mathrm{Spec}(RG/\ker(i^*))$.

(2) $\Leftrightarrow$ **(3)**: By Proposition 105, we know that RH is a finitely generated RG-module. Clearly, the kernel $\ker(RG \to RH)$ is contained in $\mathrm{ann}(RH)$. This is an equality, because the element $1 \in RH$ is not annihilated by any other element in $RG \setminus \ker(i^*)$. The equivalence of (2) and (3) is given by Proposition 65.

(1) $\Leftrightarrow$ **(4)**: If $\mathfrak{p}$ is in $\mathrm{Im}(\mathrm{Spec}(RH) \to \mathrm{Spec}(RG))$ we can consider the set $\mathfrak{J}_\mathfrak{p} \cap \{H' \mid H' \subseteq H\}$. Similarly to $\mathfrak{J}_\mathfrak{p}$ it can be shown that this set has a minimal element $H' \subseteq H$. Since H' is also minimal in $\mathfrak{J}_\mathfrak{p}$ it is conjugated to the support of $\mathfrak{p}$ and thus the fourth statement holds.

If on the other hand, the support of $\mathfrak{p}$ is conjugated to a subgroup $H' \subseteq H$ then H' is a minimal element in $\mathfrak{J}_\mathfrak{p}$. In particular we have $\mathfrak{p} \in \mathrm{Im}(\mathrm{Spec}(RH') \to \mathrm{Spec}(RG))$. This map comes from $RG \to RH'$ which factors over RH. Thus the respective map on the spectra factors over $\mathrm{Spec}(RH)$ and it follows that $\mathfrak{p} \in \mathrm{Im}(\mathrm{Spec}(RH) \to \mathrm{Spec}(RG))$. $\square$

If G is a compact connected Lie group and H a closed connected subgroup of G, the associated primes of the RG-module RH can be described very easily:

Proposition 111. *Let G be a compact connected Lie group and $H \subseteq G$ a closed connected subgroup. Then $\mathrm{Ass}_{RG}(RH)$ is a singleton, more precisely $\mathrm{Ass}_{RG}(RH) = \{\ker(RG \to RH)\}$.*

Proof. As a closed subgroup of a Lie group, H is a Lie group as well. In addition, H is compact since it is a closed subspace of a compact space. So H is a compact connected Lie group as well. Thus by Corollary 45 the ring RH is an integral domain.

The RG-module structure on RH is given by the ring homomorphism $i^*\colon RG \to RH$ induced from the inclusion $i\colon H \hookrightarrow G$. Now, let $a \in RH$ be non-zero. The annihilator $\mathrm{ann}_{RG}(a)$ consists of all elements $g \in RG$ so that $i^*(g) \cdot a = 0$. Since RH is an integral domain and a is non-zero it follows that $i^*(g) = 0$ and thus $g \in \ker(i^*)$. So for every non-zero element $a \in RH$ we have $\mathrm{ann}(a) = \ker(i^*)$.

As RH is an integral domain, the zero ideal (0) is prime. Thus the kernel $\ker(i^*) \subseteq RG$ is prime as it is the inverse image of a prime ideal. □

Now based on Segal's results, we will prove some statements that will help us to prove the conjecture. For this, we start by classifying the primes of RG which have trivial support:

Proposition 112. *Let G be a compact Lie group. The primes $\mathfrak{p} \in \mathrm{Spec}(RG)$ that have trivial support are given by:*

$$\{\mathfrak{p} \in \mathrm{Spec}(RG) \mid \mathrm{Supp}(\mathfrak{p}) = 1\} = \{\mathfrak{I}\} \cup \{\mathfrak{I} + (p) \mid p \in \mathbb{Z} \textit{ prime } \}$$

with $\mathfrak{I}$ being the augmentation ideal of RG.

Proof. By definition, the support H of $\mathfrak{p}$ satisfies $\mathfrak{p} \in \mathrm{Im}(\mathrm{Spec}(RH) \to \mathrm{Spec}(RG))$. For $H = \{1\}$ this map arises from the inclusion $\{1\} \hookrightarrow G$. On the representation rings this inclusion induces the rank map $RG \to \mathbb{Z}$. The map $\mathrm{Spec}(\mathbb{Z}) \to \mathrm{Spec}(RG)$ is then obtained by taking the inverse images of the primes of $\mathbb{Z}$ under the previous map. The spectrum of $\mathbb{Z}$ consists of the following ideals $\mathrm{Spec}(\mathbb{Z}) = \{(0)\} \cup \{(p) \mid p \in \mathbb{Z} \text{ prime}\}$. The inverse image of the zero-ideal is the kernel of the map $RG \to \mathbb{Z}$, the augmentation ideal $\mathfrak{I}$ of RG. By Proposition 47 the inverse image of $(p) \in \mathrm{Spec}(\mathbb{Z})$ under the rank map is given by $\mathfrak{I} + (p)$.

So a prime with trivial support has to be of the form $\mathfrak{I}$ or $\mathfrak{I} + (p)$. As $H = 1$ is the smallest subgroup of G, both $\mathfrak{I}$ and $\mathfrak{I} + (p)$, $p \in \mathbb{Z}$ prime, have trivial support. □

Using this, we can link the strict double coset condition with the supports of the respective representation rings of the subgroups:

Proposition 113. *Let G be a compact Lie group and $H_1, H_2 \subseteq G$ two closed subgroups that satisfy the strict double coset condition. Then*

$$\mathrm{Supp}_{RG}(RH_1) \cap \mathrm{Supp}_{RG}(RH_2) = \{\mathfrak{I}\} \cup \{\mathfrak{I} + (p) \mid p \in \mathbb{Z} \textit{ prime } \}$$

Proof. We use Proposition 110: The third statement is fulfilled for H_1 and H_2 since $\mathfrak{p} \in \mathrm{Supp}(RT_1) \cap \mathrm{Supp}(RT_2)$. It follows that there exist $g_1, g_2 \in G$ and subgroups H_1' and H_2' of H_1 and H_2, respectively, so that

$$g_1 \,\mathrm{Supp}(\mathfrak{p}) g_1^{-1} = H_1' \subseteq H_1 \quad \text{and} \quad g_2 \,\mathrm{Supp}(\mathfrak{p}) g_2^{-1} = H_2' \subseteq H_2.$$

This yields that $g_2 g_1^{-1} H_1' (g_2 g_1^{-1})^{-1} = H_2'$ and thus $H_1' \subseteq H_1$ and $H_2' \subseteq H_2$ are conjugate. We assumed that H_1 and H_2 satisfy the strict double coset condition. Thus $H_1' = H_2' = \{1\}$ and it follows that the support of $\mathfrak{p}$ is trivial. Applying Proposition 112 yields:

$$\mathrm{Supp}_{RG}(RT_1) \cap \mathrm{Supp}_{RG}(RT_1) \subseteq \{\mathfrak{I}\} \cup \{\mathfrak{I} + (p) \mid p \in \mathbb{Z} \text{ prime } \}.$$

For the remaining inclusion, use implication (4) $\Rightarrow$ (3) from Proposition 110 and Proposition 112: As $\mathfrak{I}$ and $\mathfrak{I} + (p)$ have trivial support, it in particular a subgroup of H_1 and H_2. Thus they are in both $\mathrm{Supp}(RH_1)$ and $\mathrm{Supp}(RH_2)$. □

We conclude this subsection by showing, that none of the ideals that appeared in the prior propositions are associated primes of the representation rings of closed connected subgroups of compact connected Lie groups:

Proposition 114. *Let G be a compact connected Lie group and $H \subseteq G$ a closed connected subgroup of G with $\mathrm{rank}(H) \geq 1$. Then the augmentation ideal $\mathfrak{I}$ of RG is not an associated prime of RH, nor is $\mathfrak{I} + (p)$ for any prime $p \in \mathbb{Z}$*

Proof. We write $i \colon H \hookrightarrow G$ for the inclusion $H \subseteq G$. In Proposition 111 we have seen that $\mathrm{Ass}(RH)$ is a singleton consisting only of the ideal $\ker(i^* \colon RG \to RH)$.

Now assume that one of the ideals stated above is an associated prime of RH. Since every ideal above has the augmentation ideal $\mathfrak{I}$ as a subset we can conclude that $i^*(a) = 0$ for all $a \in \mathfrak{I}$ and thus $\mathfrak{I} \subseteq \mathrm{Ker}(i^*)$.

Since G is a compact Lie group, the equivalence classes of representations of G correspond to the elements in $R(G)^+$. We have seen this when introducing the representation ring in Definition 36. Under the restriction i^* the trivial representations of G in RG are sent to the respective trivial representations of H in RH' preserving their rank. So for an element $\rho \in RG^+$ corresponding to a representation of G the element $\rho - \mathrm{rank}(\rho) \in \mathfrak{I}$ is sent to $0 = i^*(\rho - \mathrm{rank}(\rho)) = i^*(\rho) - \mathrm{rank}(\rho)$. So in RH we have $i^*(\rho) = \mathrm{rank}(\rho)$. Because of the 1-1 correspondence of equivalence classes of representations of H and elements in RH this means precisely that restricting an arbitrary representation of G to H results in a representation that is isomorphic to a trivial representation.

Two representations are isomorphic if and only if after choosing respective bases, their matrix representations are conjugate. So it follows that there exists an $A \in GL_{rank(\rho)}(\mathbb{C})$ so that for all $h \in H$: $AI_{rank(\rho)}A^{-1} = (\rho \circ i)(h)$. The left hand side of this equation clearly can be simplified to $I_{rank(\rho)}$. So on H the map $\rho \circ i$ is constant. As G is a compact Lie group, by Theorem 20 there exists a faithful representation $\rho\colon G \to \mathrm{Aut}(V)$ for a finite-dimensional vector space V. As a composition of injective maps, the map $\rho \circ i$ is injective. This is a contradiction to $\rho \circ i$ being a constant map. □

3.5 The Rank 1 Case

In this section we will prove Conjecture 100 in the case that one of the occurring subgroups is of rank 1 or lower.

Theorem 115. *Let G be a compact connected Lie group satisfying the equivalent conditions in Theorem 103. Let H_1, $H_2 \subseteq G$ be two closed subgroups of G which satisfy the strict double coset condition. In addition, assume that* $\mathrm{rank}(H_1) \leq 1$ *or* $\mathrm{rank}(H_2) \leq 1$. *Then Conjecture 100 holds, i.e.*

$$\mathrm{Tor}_i^{RG}(RH_1, RH_2) = 0 \textit{ for all } i > \mathrm{rank}\, G - (\mathrm{rank}\, H_1 + \mathrm{rank}\, H_2).$$

Proof. By Proposition 104 it suffices to consider maximal tori T_1, T_2 of H_1 and H_2, respectively. Without loss of generality, assume that $\mathrm{rank}(T_2) \leq 1$. By Proposition 7 there exists a maximal torus T of G containing T_1. Proposition 86 yields that we can choose an RT-regular sequence $\lambda_1, \lambda_2, \dots, \lambda_{n-n_1}$ with $n = \mathrm{rank}(T)$ and $n_1 = \mathrm{rank}(T_1)$ which generates the kernel of the projection $RT \to RT_1$. Since the sequence is regular, by Proposition 91 we get a free resolution for $RT/(\lambda_1, \dots \lambda_r)$ over RT from the corresponding Koszul complex which has the form:

$$0 \to RT^{m_{n-n_1}} \to RT^{m_{n-n_1-1}} \to \dots \to RT^{m_0} \to RT/(\lambda_1, \dots, \lambda_{n-n_1}) \to 0.$$

We want to use this sequence to calculate the Tor_i^{RG} terms. We can consider this sequence of RT-modules as a sequence of RG-modules. This can be done by restricting the existing RT-module structure to RG which is embedded in RT by the inclusion $RG \hookrightarrow RT$. By doing this, every RT-linear map is RG-linear as well.

By the Theorem of Pittie, the module RT is free over RG and thus $RT = RG^m$ for an $m \in \mathbb{N}$. In addition, we have $RT/(\lambda_1, \dots, \lambda_{n-n_1}) \cong RT_1$. So the previous resolution yields a free resolution of RT_1 as an RG-module of the form:

$$0 \to RG^{m \cdot m_{n-n_1}} \to RG^{m \cdot m_{n-n_1-1}} \to \cdots \to RG^{m \cdot m_0} \to RT_1 \to 0.$$

Proposition 84 (1) yields that $\mathrm{Tor}_i^{RG}(RT_1, RT_2) = 0$ for $i > n - n_1$. So if $\mathrm{rank}(H_2) = 0$ we are already finished. Now assume that $\mathrm{rank}(H_2) = 1$. In this case it remains to show that $\mathrm{Tor}_{n-n_1}^{RG}(RT_1, RT_2)$ vanishes as well. This follows from Proposition 84 (2) if $\mathrm{Supp}(RT_1) \cap \mathrm{Ass}(RT_2) = \emptyset$.

By assuming otherwise, take $\mathfrak{p} \in \mathrm{Supp}(RT_1) \cap \mathrm{Ass}(RT_2)$. Since $\mathrm{Ass}(RT_2) \subseteq \mathrm{Supp}(RT_2)$ the prime $\mathfrak{p}$ is contained in the support of both RG-modules. T_1 and T_2 satisfy the strict double coset condition, so by Proposition 113 the ideal $\mathfrak{p}$ is either the augmentation ideal $\mathfrak{I} \subseteq RG$ or of the form $\mathfrak{I} + (p)$ for $p \in \mathbb{Z}$ prime. In Proposition 114 we have seen that neither $\mathfrak{I}$ nor $\mathfrak{I} + (p)$ are in $\mathrm{Ass}(RT_2)$. So $\mathfrak{p} \notin \mathrm{Supp}(RT_1) \cap \mathrm{Ass}(RT_2)$ which is a contradiction. $\square$

In the first part of the proof, we did not use any assumption on the rank of H_1. So the following corollary can be directly concluded:

Corollary 116. *Let G be a compact connected Lie group satisfying the equivalent conditions of Theorem 103. Let H_1 and H_2 be two closed connected subgroups of G satisfying the strict double coset condition. Then:*

$$\mathrm{Tor}_i^{RG}(RH_1, RH_2) = 0 \textit{ for all } i > \mathrm{rank}\, G - \max(\mathrm{rank}\, H_1, \mathrm{rank}\, H_2).$$

3.6 Ideas for the General Case

We were able to prove the conjecture in case that one of the occurring subgroups is of rank ≤ 1. In this section, we present some attempts to prove the general case. The idea is to inductively find a proof for higher ranks using the rank 1 case as the base case. It is based on the following proposition:

Proposition 117. *Let G be a compact Lie group and H_1, H_2 two closed subgroups of G satisfying the strict double coset condition. In addition, let $\mathfrak{I} \subseteq RG$ be the augmentation ideal of RG and $i \in \mathbb{N}_{>0}$. If there exists an element $a \in \mathfrak{I}$ for which the multiplication*

$$\mathrm{Tor}_i^{RG}(RH_1, RH_2) \xrightarrow{\cdot a} \mathrm{Tor}_i^{RG}(RH_1, RH_2)$$

is injective, then $\mathrm{Tor}_i^{RG}(RH_1, RH_2) = 0$.

Proof. In Corollary 73 we have seen that modules over Noetherian rings vanish if and only if they have no associated primes. This can be applied here, because we have seen in Proposition 105 that RG is Noetherian.

So we need to show that $\mathrm{Ass}(\mathrm{Tor}_i^{RG}(RH_1, RH_2))$ is empty. By Proposition 70 we know that the set of associated primes $\mathrm{Ass}(\mathrm{Tor}_i^{RG}(RH_1, RH_2))$ is a subset of $\mathrm{Supp}(\mathrm{Tor}_i^{RG}(RH_1, RH_2))$. In Corollary 82 we have seen that Tor commutes with localization, i.e. for all primes $\mathfrak{p} \in \mathrm{Spec}(RG)$ we have

$$\mathrm{Tor}_i^{RG}(RH_1, RH_2)_{\mathfrak{p}} \cong \mathrm{Tor}_i^{RG_{\mathfrak{p}}}(RH_{1\mathfrak{p}}, RH_{2\mathfrak{p}}).$$

In particular, if $RH_{1\mathfrak{p}} = 0$ or $RH_{2\mathfrak{p}} = 0$, it follows that $\mathrm{Tor}_i^{RG}(RH_1, RH_2)_{\mathfrak{p}} = 0$. So we can conclude that:

$$\begin{aligned}
\mathrm{Ass}(\mathrm{Tor}_i^{RG}(RH_1, RH_2)) \quad &\subseteq \quad \mathrm{Supp}(\mathrm{Tor}_i^{RG}(RH_1, RH_2)) \\
&\subseteq \quad \mathrm{Supp}(RH_1) \cap \mathrm{Supp}(RH_2) \\
&\overset{\text{Prop. 113}}{=} \{\mathfrak{I}\} \cup \{\mathfrak{I} + (p) \mid p \in \mathbb{Z} \text{ prime }\}.
\end{aligned}$$

So it suffices to show that neither $\mathfrak{I}$ nor $\mathfrak{I} + (p)$, with $p \in \mathbb{Z}$ prime, are associated primes of $\mathrm{Tor}_i^{RG}(RH_1, RH_2)$. By assumption, there exists an element $a \in \mathfrak{I}$ for which the multiplication on $\mathrm{Tor}_i^{RG}(RH_1, RH_2)$ is injective. In particular, for all elements $m \in \mathrm{Tor}_i^{RG}(RH_1, RH_2) \smallsetminus \{0\}$ the annihilator $\mathrm{ann}(m)$ does not contain a and thus the augmentation ideal $\mathfrak{I}$ is not a subset of $\mathrm{ann}(m)$. So neither $\mathfrak{I}$ nor $\mathfrak{I} + (p)$ is associated. $\square$

So our goal is to find elements $a_i \in \mathfrak{I}$ for $i > \mathrm{rank}(G) - (\mathrm{rank}(H_1) + \mathrm{rank}(H_2))$ so that the respective multiplication on $\mathrm{Tor}_i^{RG}(RH_1, RH_2)$ is injective. The next three lemmas yield an approach on how to find these elements:

Lemma 118. *Let G be a compact Lie group and H_1, H_2 two closed subgroups. Write $i\colon H_1 \hookrightarrow G$ for the inclusion of H_1 in G. In addition, let a be an element in RG for which the image under i^* is given by $i^*(a) = \prod_{j=1}^k b_j$ with $b_j \in RH_1$, $1 \leq j \leq k$. If for an $s \in \mathbb{N}_{>0}$ and all $1 \leq j \leq k$ the maps*

$$\mathrm{Tor}_s^{RG}(RH_1, RH_2) \xrightarrow{\cdot b_j} \mathrm{Tor}_s^{RG}(RH_1, RH_2)$$

induced by the multiplication with $b_j \in RH_1$ on RH_1 are injective, then the scalar multiplication

$$\mathrm{Tor}_s^{RG}(RH_1, RH_2) \xrightarrow{\cdot a} \mathrm{Tor}_s^{RG}(RH_1, RH_2)$$

with $a \in RG$ is injective as well.

Proof. By Proposition 83 the induced map on $\mathrm{Tor}_s^{RG}(RH_1, RH_2)$ by the multiplication with the scalar $a \in RG$ on H_1 is exactly the multiplication with a on $\mathrm{Tor}_s^{RG}(RH_1, RH_2)$ as an RG-module.

By the construction of the RG-module structure on RH_1, multiplication with $a \in RG$ is the same as multiplication with $i^*(a) \in RH_1$ in the ring RH_1. As $i^*(a)$ is the product $\prod_{j=1}^{k} b_j$, this map decomposes as a concatenation of the maps given by the multiplication with $b_j \in RH_1$, $1 \leq j \leq k$. Since $\mathrm{Tor}_s^{RG}(-, RH_2)$ is a functor, the multiplication with $i^*(a)$ on $\mathrm{Tor}_s^{RG}(RH_1, RH_2)$ decomposes in the same way. So multiplication with the scalar a on $\mathrm{Tor}_r^{RG}(H_1, H_2)$ is the same as the induced map by multiplication with $i^*(a)$ which is by assumption a concatenation of injective maps and thus injective. $\square$

By symmetry, the lemma is also true if we formulate it in terms of H_2. In general, the elements b_j of the previous lemma may not be in the image of the map $i^* \colon RG \to RH_1$. The lemma suggests that it might be useful to analyze the induced maps on Tor by arbitrary elements of RH_1 as well:

Lemma 119. *Let G be a compact Lie group and H_1, H_2 two closed subgroups of G and let RH_1 be an integral domain. Let $b \in RH_1$ be a nonzero element. If $\mathrm{Tor}_{k+1}^{RG}(RH_1/(b), RH_2)$ vanishes, then the induced multiplication*

$$\mathrm{Tor}_k^{RG}(RH_1, RH_2) \xrightarrow{\cdot b} \mathrm{Tor}_k^{RG}(RH_1, RH_2)$$

is injective. This is also true, if we reformulate the statement in terms of H_2.

Proof. Since RH_1 is an integral domain and $b \in RH_1$ a nonzero element, the multiplication with b on RH_1 is injective and we get a short exact sequence of RH_1-modules (that restricts to a short exact sequence of RG-modules) of the form:

$$0 \longrightarrow RH_1 \xrightarrow{\cdot b} RH_1 \longrightarrow RH_1/(b) \longrightarrow 0.$$

We take a look at the long exact Tor sequence resulting from the short exact sequence above:

$$\begin{tikzcd}
 & \dots \arrow[r] & \operatorname{Tor}^{RG}_{k+1}(RH_1/(b), RH_2) \arrow[dll] \\
\operatorname{Tor}^{RG}_{k}(RH_1, RH_2) \arrow[r, "\cdot b"] & \operatorname{Tor}^{RG}_{k}(RH_1, RH_2) \arrow[r] & \operatorname{Tor}^{RG}_{k}(RH_1/(b), RH_2) \arrow[dll] \\
\operatorname{Tor}^{RG}_{k-1}(RH_1, RH_2) \arrow[r] & \dots &
\end{tikzcd}$$

By assumption, $\operatorname{Tor}^{RG}_{k+1}(RH_1/(b), RH_2) = 0$ and thus by the exactness of the sequence, the map

$$\operatorname{Tor}^{RG}_{k}(RH_1, RH_2) \xrightarrow{\cdot b} \operatorname{Tor}^{RG}_{k}(RH_1, RH_2)$$

is injective. An analog proof works for H_2. □

Lemma 120. *Let G be a compact connected Lie group and H_1, H_2 two closed connected subgroups that satisfy the strict double coset condition. Assume that the conjecture holds if one of the occurring subgroups is of a lower rank than $\operatorname{rank}(H_1)$. If for $b \in RH_1$ the RG-module $RH_1/(b)$ is isomorphic to the representation ring of a closed connected subgroup H_1' of H_1 with $\operatorname{rank}(H_1') = \operatorname{rank}(H_1) - 1$, then the induced multiplication given by*

$$\operatorname{Tor}^{RG}_{i}(RH_1, RH_2) \xrightarrow{\cdot b} \operatorname{Tor}^{RG}_{i}(RH_1, RH_2)$$

is injective for all $i > \operatorname{rank}(G) - (\operatorname{rank}(H_1) + \operatorname{rank}(H_2))$.

Proof. Let the setting be as described in the lemma. It follows directly from the definition of the strict double coset condition, that for every subgroup H_1' of H_1, the groups H_1' and H_2 satisfy the strict double coset condition as well. We assumed that the conjecture holds if one of the occurring subgroups is of lower rank than H_1. So for a closed connected subgroup $H_1' \subseteq H_1$ with $\operatorname{rank}(H_1') = \operatorname{rank}(H_1) - 1$, we have $\operatorname{Tor}^{RG}_{i}(RH_1', RH_2) = 0$ for all $i > \operatorname{rank}(G) - (\operatorname{rank}(H_1) + \operatorname{rank}(H_2)) + 1$. Applying Lemma 119 yields the proof. □

To apply the previous lemma, it is important to understand the quotient $RH_1/(b)$. In the following example, we analyze $RT/(b)$ for a torus T and different choices of b:

Example 121. *Let T be a torus of rank k with representation ring $RT \cong \mathbb{Z}[x_1^{\pm 1}, \dots, x_k^{\pm 1}]$.*

1. *For $b = (x_j - 1)$, $1 \leq j \leq k$, the quotient $RT/(b)$ is isomorphic to RT_1 with $T_1 = \{(a_1, \ldots, a_k) \in (S^1)^k \mid a_j = 1\} \subseteq T$.*

2. *For $b = x_i - x_j$, $1 \leq i \neq j \leq k$, the RG-module $RT/(b)$ is isomorphic to the representation ring of the subtorus $T_2 = \{(a_1, \ldots, a_k) \in T_1 \mid a_i = a_j\} \subseteq T$.*

3. *For $b = (x_1^2 - 1)$, the ring $RT/(b)$ is not the representation ring of a compact connected Lie group: It contains x_1, an element satisfying $x_1^2 = 1$ that is neither 1 nor -1. Thus we have $(x_1 - 1)(x_1 + 1) = x_1^2 - 1 = 0$. This cannot occur in integral domains, in particular not in representation rings of compact connected Lie groups (see Corollary 45).*

We quickly sum up the current results: By Proposition 117 we can show that $\mathrm{Tor}_i^{RG}(RH_1, RH_2)$ vanishes by finding an element $a \in \mathfrak{I} \subseteq RG$ for which the induced multiplication on $\mathrm{Tor}_i^{RG}(RH_1, RH_2)$ is injective. In Lemma 118 we have seen that it suffices to consider the image of a in RH_1 under $i^*\colon RG \to RH_1$. If $i^*(a)$ decomposes as a product of elements in RH_1, showing that every factor induces an injective multiplication on $\mathrm{Tor}_i^{RG}(RH_1, RH_2)$ is sufficient. The same is true if we replace RH_1 by RH_2. Lemma 119 and Lemma 120 yield a way to show that elements in RH_1 (or RH_2) induce an injective multiplication on $\mathrm{Tor}_i^{RG}(RH_1, RH_2)$ for $i > \mathrm{rank}(G) - (\mathrm{rank}(H_1) + \mathrm{rank}(H_2))$. For $b \in RH_1$ this is the case if $RH_1/(b)$ is isomorphic to the representation ring of a closed connected subgroup H_1' of H_1 with $\mathrm{rank}(H_1') = \mathrm{rank}(H_1) - 1$ and if additionally the conjecture holds if one of the occurring subgroups is of rank lower than $\mathrm{rank}(H_1)$.

Ideas on how to choose the element a:

By Proposition 104 it suffices to only consider tori as subgroups. So let G by a compact connected Lie group and let T_1, T_2 be two subtori satisfying the strict double coset condition. We assume that Conjecture 100 is true if one of the occurring subgroups is of rank lower than $\mathrm{rank}(T_1)$. We can choose a maximal torus T in G that contains T_1. As we have seen in Proposition 43, the representation rings of T and T_1 can be expressed in the following form:

$$RT = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$
$$RT_1 = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_k^{\pm 1}]$$

with $k = \mathrm{rank}(T_1)$, $n = \mathrm{rank}(G) \geq k$ and with x_j corresponding to the projection to the j-th factor. Then the map $RT \to RT_1$ is given by setting $x_j = 1$ for all $j > k$. By Theorem 44, we know that RG consists of all elements in RT which are invariant under the action of the Weyl group W of G. So the map $RG \to RT_1$ is given by restricting $RT \to RT_1$ to RG.

Consider the element $x_1 \in RT$. It maps to the respective element $x_1 \in RT_1$. As we have seen in Example 121, multiplication with (x_1-1) is of the desired form and thus by Lemma 120, the multiplication $\mathrm{Tor}_i^{RG}(RT_1, RT_2) \xrightarrow{\cdot(x_1-1)} \mathrm{Tor}_i^{RG}(RT_1, RT_2)$ is injective for $i > \mathrm{rank}(G) - (\mathrm{rank}(T_1) + \mathrm{rank}(T_2))$. However, in general $(x_1 - 1)$ is not invariant under the action of the Weyl group and thus not in $\mathfrak{I} \subseteq RG$.

Let $W_{x_1} \subseteq W$ be the subset of the Weyl group W given by:

$$W_{x_1} = \{w \in W \mid wx_1 - 1 \notin \ker(RT \to RT_1)\}.$$

Then consider the following element:

$$a = \sum_{\substack{W' \subseteq W \\ |W'| = |W_{x_1}|}} \prod_{w' \in W'} (w'x_1 - 1) \in RT.$$

Lemma 122. *The element a is in $\mathfrak{I} \subseteq RG$.*

Proof. To begin with, we show that a is in RG. By Theorem 44, it suffices to check that a is invariant under the action of the Weyl group W of G: Let $w \in W$ be an arbitrary element of the Weyl group. The action of the Weyl group commutes with products and sums, so:

$$wa = \sum_{\substack{W' \subseteq W \\ |W'| = |W_{x_1}|}} \prod_{w' \in W'} ((ww')x_1 - 1) = \sum_{\substack{W' \subseteq W \\ |W'| = |W_{x_1}|}} \prod_{w' \in wW'} ((w')x_1 - 1)$$

with $wW' = \{ww' \mid w' \in W'\}$. Since W is a group, $ww' = ww''$ for $w, w', w'' \in W$ if and only if $w' = w''$. So for $w \in W$ and $W' \subseteq W$ we have $|W'| = |wW'|$. In addition, $wW' = wW''$ for $W', W'' \subseteq W$ of the same cardinality if and only if $W' = W''$. Thus w permutes the subsets $W' \subseteq W$ of cardinality $|W_{x_1}|$. It follows that $wa = a$ and thus $a \in RG$.

The element a is in the augmentation ideal $\mathfrak{I}$ of RG as well, since every $(w'x_1 - 1)$ maps to zero under the rank map as the action of the Weyl group does not change the rank of a representation. □

By construction of W_{x_1}, every subset $W' \neq W_{x_1}$ of cardinality $|W_{x_1}|$ contains an element w' so that $(w'x_1 - 1) \in \ker(RT \to RT_1)$. So under $i^* \colon RT \to RT_1$, the element a maps to:

$$b = \prod_{w \in W_{x_1}} (i^*(wx_1) - 1) \in RT_1.$$

If we could prove that for each $w \in W$ only the following two cases can occur, we would have proven the conjecture:

- $wx_1 - 1 \in \ker(RT \to RT_1)$.
- $RT_1/(wx_1 - 1)$ is isomorphic to the representation ring of a subtorus $T_1' \subseteq T_1$ of rank $k-1$ as RG-module.

As the action of W does not change the rank of a representation, using Proposition 40, we get that the element wx_1 has to be of the form $wx_1 = \prod_{r=1}^{n} x_r^{s_r}$ with $s_r \in \mathbb{Z}$.

Lemma 123. *If there exists an index $1 \leq t \leq k$ so that $s_t = \pm 1$, then $RT_1/(i^*(wx_1) - 1)$ is isomorphic to the representation ring of a subtorus of T_1 of rank $k-1$.*

Proof. The image of $wx_1 = \prod_{r=1}^{n} x_r^{s_r}$ under i^* is $i^*(wx_1) = \prod_{r=1}^{k} x_r^{s_r}$. In $RT_1/(wx_1 - 1)$, we have $\prod_{r=1}^{n} x_r^{s_r} = 1$. If there exists an $1 \leq t \leq k$ with $s_t = 1$, we can multiply the previous equation by x_t^{-1} and get $x_t^{-1} = \prod_{\substack{r=1 \\ r \neq t}}^{k} x_r^{s_r}$. As all x_j correspond to the respective projections, this is the representation ring of the subtorus T_1' of T_1 given by

$$T_1' = \left\{ (a_1, \dots, a_k) \in T_1 = (S^1)^k \;\middle|\; a_t = \prod_{\substack{r=1 \\ r \neq t}}^{k} a_r^{-s_r} \right\} \cong (S^1)^{k-1}.$$

A similar argument works for $s_t = -1$. □

However, if for example $wx_1 = x_2^2$, then non of the two cases above would occur, as seen in Example 121.

When does the idea work?

In the case that $G = T$ is a torus, the Weyl group W is trivial. Thus we have

$$a = \sum_{\substack{W' \subseteq W \\ |W'| = |W_{x_1}|}} \prod_{w' \in W'} (w'x_1 - 1) \quad = \quad (x_1 - 1)$$

and $b = (i^*(x_1) - 1) = (x_1 - 1) \in RT_1$ which induces an injective multiplication on $\mathrm{Tor}_i^{RT}(RT_1, RT_2)$ for $i > \mathrm{rank}(G) - (\mathrm{rank}(T_1) + \mathrm{rank}(T_2))$.

So if we have two subtori T_1 and T_2 of T with $\mathrm{rank}(T_1) = 2$, we can use the previous procedure to find an element $a \in \mathfrak{I} \subseteq RT$ that induces an injective multiplication on $\mathrm{Tor}_i^{RT}(RT_1, RT_2)$ for $i > \mathrm{rank}(T) - (2 + \mathrm{rank}(T_2))$. By Proposition 117 it follows that $\mathrm{Tor}_i^{RT}(RT_1, RT_2) = 0$ for $i > \mathrm{rank}(T) - (2 + \mathrm{rank}(T_2))$. Which proves the case that one of the occurring subgroups is of rank 2 or lower. We can go on inductively and gain:

Proposition 124. *Conjecture 100 holds if $G = T$ is a torus.*

We are also interested in how well this works for groups G that are not tori. So we again take a look at Example 99:

Example 125. *$G = \mathrm{SU}(6)$ with the (standard) maximal torus T given by the diagonal matrices in $\mathrm{SU}(6)$, i.e. diagonal matrices with entries in S^1 whose product is 1. The inclusions of the two subtori T_1 and T_2 were given by:*

$$i_1^*(T_1) = \left\{ \begin{pmatrix} a & & & & & \\ & b & & & & \\ & & c & & & \\ & & & a & & \\ & & & & b & \\ & & & & & c \end{pmatrix} \;\middle|\; a, b, c \in S^1 \text{ with } c = a^{-1}b^{-1} \right\}$$

$$i_2^*(T_2) = \left\{ \begin{pmatrix} a & & & & & \\ & b & & & & \\ & & c & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \;\middle|\; a, b, c \in S^1 \text{ with } c = a^{-1}b^{-1} \right\}.$$

The representation ring of T is given by $RT = \mathbb{Z}[x_1^{\pm 1}, \dots, x_6^{\pm 1}]/(x_1 \cdots x_6 - 1)$ where x_j is the projection to the j-th diagonal entry. One could also rewrite this as a Laurent ring with 5 variables, but to retrace the action of the Weyl group W, this form is the better choice. The Weyl group W of $\mathrm{SU}(6)$ is S_6 and acts on the given maximal torus by permuting the entries of the diagonal. So the corresponding action on RT permutes the indices of the variables.

The inclusions $T_1, T_2 \subseteq T$ yield the following homomorphisms on the respective representation rings:

$$RT = \mathbb{Z}[x_1^{\pm 1}, \dots, x_6^{\pm 1}]/(x_1 \cdots x_6 - 1) \longrightarrow \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]/(x_1 x_2 x_3 - 1) = RT_1$$
$$x_i \longmapsto x_{i \bmod 3}$$

$$RT = \mathbb{Z}[x_1^{\pm 1}, \dots, x_6^{\pm 1}]/(x_1 \cdots x_6 - 1) \longrightarrow \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]/(x_1 x_2 x_3 - 1) = RT_2$$
$$x_i \longmapsto \begin{cases} x_i & \text{if } i = 1, 2, 3 \\ 1 & \text{else.} \end{cases}$$

We take a look at the element $x_1 \in RT$. The orbit of x_1 under the action of the Weyl group is $W \cdot x_1 = \{x_1, \dots, x_6\}$.

Now, we can directly compute the elements a and b which we defined previously. We will do this for both subtori T_1 and T_2 and denote the respective elements with a_j and b_j, $j = 1, 2$.

We start with T_1: There is no $w \in W$ for which the element $wx_1 - 1$ is in the kernel of $RT \to RT_1$. So W_{x_1} is the whole Weyl group and

$$a_1 = \prod_{w \in W} (wx_1 - 1) = \prod_{j=1}^{6} (x_j - 1)^{120}.$$

For the occurring coefficients, note that S_6 consists of $6! = 720$ elements and if we fix sending 1 to $i \in \{1, \ldots, 6\}$, then there remain $5! = 120$ possibilities to split up the remaining elements. So for b_1 we have:

$$b_1 = \prod_{w \in W} (wx_1 - 1) = \prod_{j=1}^{6} (i_1^*(x_j) - 1)^{120} = \prod_{j=1}^{3} (x_j - 1)^{240}.$$

Every occurring factor is of the form $(x_j - 1)$ and fulfills that $RT_1/(x_j - 1)$ is the representation ring of the subtorus of T_1 of rank 1 given by restricting the respective entry to $1 \in S^1$. So the our idea works in this case.

Now, we will calculate a_2 and b_2 for T_2: Under $i_2^* \colon RT \to RT_2$, the elements $x_j - 1$ with $j = 4, 5, 6$ are sent to 0. It follows:

$$W_{x_1} = \{\pi \in S_6 \mid \pi(1) \neq \{4, 5, 6\}\} \quad \text{and} \quad |W_{x_1}| = 360.$$

The elements a_2 and b_2 are given by:

$$a_2 = \sum_{\substack{W' \subseteq W \\ |W'| = 360}} \prod_{w \in W'} (wx_1 - 1)$$

$$b_2 = \prod_{w \in W_{x_1}} (wx_1 - 1) = \prod_{j=1}^{3} (i_2^*(x_j) - 1)^{120} = \prod_{j=1}^{3} (x_j - 1)^{120}.$$

Again, the factors $(x_j - 1)$, $j = 1, 2, 3$, are of the desired form and the idea works for T_2 as well.

More generally, consider $G = \mathrm{U}(n), \mathrm{SU}(n), \mathrm{SO}(2n+1)$ or $\mathrm{Sp}(n)$ together with the standard maximal tori T which we studied in Subsection 1.5. If the subtori T_1 or T_2 result from T by either equalizing some diagonal entries (or blocks, as in the $\mathrm{SO}(n)$-cases) or setting some entries to 1, the idea works as well, because the Weyl group acts by permuting the respective entries (and thus permutes the elements x_j in RT).

However, it is possible that there also exist closed subgroups H_1 and H_2 that satisfy the strict double coset condition for which the respective maximal tori are not embedded like this. We can embed S^1 in the maximal torus of $\mathrm{U}(2)$ by:

$$S^1 \longrightarrow T_{\mathrm{U}(2)}, \quad z \longmapsto \begin{pmatrix} z & 0 \\ 0 & z^2. \end{pmatrix}$$

On the representations rings, the embedding yields the following map:

$$RT_{\mathrm{U}(n)} \cong \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}] \longrightarrow RS^1 \cong \mathbb{Z}[x_1^{\pm}], \qquad \begin{aligned} x_1 &\longmapsto x_1 \\ x_2 &\longmapsto x_1^2. \end{aligned}$$

with x_j, $j = 1, 2$ corresponding to the projections to the respective diagonal entry. The action of the Weyl group $W_{\mathrm{U}(2)} = S_2$ permutes the entries of the diagonal. So for the transposition $w \in W_{\mathrm{U}(2)}$ the element wx_1 is sent to x_1^2, an element that is not of the desired form. This is not yet a counter example for the procedure as we only have one group given. Yet, it suggests that it might be necessary to deal with further cases.

Conclusion

To start with, we quickly sum up the results of this thesis. We were able to prove our conjecture in the case that one of the occurring subgroups is of rank 1 or lower. The proof worked as follows: Restricting ourselves to considering only subtori, we were able to gain a free resolution of the representation ring of the subtorus of higher rank using a Koszul complex. We used this resolution to calculate $\mathrm{Tor}_i^{RG}(RT_1, RT_2)$. The crucial part was to show the vanishing for $i = \mathrm{rank}(G) - (\mathrm{rank}(T_1) - \mathrm{rank}(T_1)) + 1$. This was done by studying the primes in the respective supports and the associated primes. For this, some results of Segal's study of primes in compact Lie groups [Seg68] were of great importance as they allowed us to build a bridge between the strict double coset condition and the primes in the supports of the respective representation rings.

Afterwards, we presented an idea for a more general proof. The idea is based on the fact that modules over Noetherian rings are zero if and only if its set of associated primes is empty. For the vanishing of $\mathrm{Tor}_i^{RG}(RH_1, RH_2)$ it thus suffices to show that it has no associated prime. We showed that only $\mathfrak{I}$ and $\mathfrak{I}+(p)$, with $p \in \mathbb{Z}$ prime, can be associated primes of $\mathrm{Tor}_i^{RG}(RH_1, RH_2)$. So if there exists an element $a \in \mathfrak{I} \subseteq RG$ for which the multiplication on $\mathrm{Tor}_i^{RG}(RH_1, RH_2)$ is injective, it follows that neither of the previously mentioned ideals is associated. So the goal was to find such an element. We saw that for proving the injectivity of the multiplication on $\mathrm{Tor}_j^{RG}(RH_1, RH_2)$ it suffices to consider the image of a under $RG \to RH_1$. If the image is a product of elements in RH_1, each inducing an injective multiplication on $\mathrm{Tor}_j^{RG}(RH_1, RH_2)$, then $a \in RG$ does so as well. A way to show that certain elements b in RH_1 induce such a multiplication is to consider $RH_1/(b)$. If our conjecture is true if one of the subgroups is of rank lower than H_1 and if $RH_1/(b)$ is isomorphic to the representation ring of a subgroup of $H_1' \subseteq H_1$ of with $\mathrm{rank}(H_1') = \mathrm{rank}(H_1) - 1$, then the induced multiplication on $\mathrm{Tor}_j^{RG}(RH_1, RH_2)$ is injective for $j > \mathrm{rank}(G) - (\mathrm{rank}(H_1) + \mathrm{rank}(H_2))$. We then presented an idea for a possible choice of a and verified that it works if $G = T$ is a torus and for an example we previously regarded. However, in general it is highly dependent on the embedding of the subtori in the maximal torus and the respective Weyl group action and thus there might possibly be examples in which it does not work.

To further study this topic, it might be a good idea to examine the vanishing of $\mathrm{Tor}_i^{RG}(RH_1/(wx_1 - 1), RH_2)$ for arbitrary elements w of the Weyl group W, i.e. without assuming that $RH_1/(wx_1 - 1)$ is isomorphic to the representation ring of a closed connected subgroup. For this, it might be necessary to drop the condition, that all involved groups are connected and study representation rings of general compact Lie groups and their closed subgroups. Like this, we cannot reduce the problem to maximal tori, but representation rings that are not integral domains may appear as well.

Bibliography

[Ada69] John Frank Adams. *Lectures on Lie groups.* W. A. Benjamin, Inc., New York, 1969.

[AM07] Hossein Abbaspour and Martin Moskowitz. *Basic Lie Theory.* World Scientific, 2007.

[Ati68] Michael F. Atiyah. Bott periodicity and the index of elliptic operators. *The Quarterly Journal of Mathematics*, 19(1):113–140, 1968.

[Bos14] Siegfried Bosch. *Lineare Algebra.* Springer Spektrum, Berlin, Heidelberg, 2014.

[Bou06] Nicolas Bourbaki. *Algèbre commutative: Chapitres 5 à 7.* Éléments de Mathématique. Springer, 2006.

[BtD85] Theodor Bröcker and Tammo tom Dieck. *Representations of Compact Lie Groups*, volume 98 of *Graduate Texts in Mathematics.* Springer, Berlin, Heidelberg, 1985.

[Eis04] David Eisenbud. *Commutative Algebra: with a View Toward Algebraic Geometry*, volume 150 of *Graduate Texts in Mathematics.* Springer, New York, 2004.

[Esc84] Jost-Hinrich Eschenburg. *Freie isometrische Aktionen auf kompakten Lie-Gruppen mit positiv gekrümmten Orbiträumen*, volume 32 of *Schriftenreihe des Mathematischen Instituts der Universität Münster.* R. Remmert, 1984.

[HN12] Joachim Hilgert and Karl-Hermann Neeb. *Structure and Geometry of Lie Groups.* Springer Monographs in Mathematics. Springer, New York, 2012.

[Lee13] John M. Lee. *Introduction to Smooth Manifolds*, volume 218 of *Graduate Texts in Mathematics.* Springer, New York, 2013.

[LS15] Gerd Laures and Markus Szymik. *Grundkurs Topologie.* Springer Spektrum, Berlin, Heidelberg, 2nd edition, 2015.

[Mac95] Ian Grant Macdonald. *Symmetric Functions and Hall Polynomials.* Oxford Mathematical Monographs. Oxford University Press, Oxford, 2nd edition, 1995.

[Mat87] Hideyuki Matsumura. *Commutative Ring Theory*, volume 8 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1987.

[Nac65] Leopoldo Nachbin. *The Haar integral.* D. van Nostrand Company, Inc., Princeton, New Jersey, 1965.

[Pit72] Harsh V. Pittie. Homogeneous vector bundles on homogeneous spaces. *Topology*, 11:199–203, 1972.

[Seg68] Graeme Segal. The representation ring of a compact Lie group. *Publications Mathématiques de l'IHÉS*, 34:113–128, 1968.

[Sin93] Wilhelm Singhof. On the topology of double coset manifolds. *Mathematische Annalen*, 297(1):133–146, 1993.

[Ste75] Robert Steinberg. On a Theorem of Pittie. *Topology*, 14:173–177, 1975.

[Wei94] Charles A. Weibel. *An Introduction to Homological Algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1994.

[Wei13] Charles A. Weibel. *The K-book: An Introduction to Algebraic K-theory*, volume 145 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, Rhode Island, 2013.

[Zib] Marcus Zibrowius. The Theorem of Pittie and Steinberg. (Unpublished notes.).

[ZK09] Wolfgang Ziller and Martin Kerin. On Eschenburg's Habilitation on Biquotients. *arXiv:0909.0167*, 2009.

Erklärung

Hiermit versichere ich, die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt zu haben.

Düsseldorf, den 30. September 2020

(Alessandra Wiechers)