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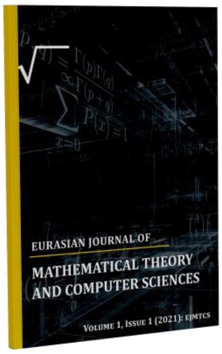

ON THE PROPERTIES OF THE SYSTEM OF HEAT CONDUCTION EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

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https://www.doi.org/10.5281/zenodo.7878765

ARTICLE INFO

Received: 19th April 2023
Accepted: 28th April 2023
Online: 29th April 2023

KEY WORDS

ABSTRACT

In this paper, we study the properties of the systems of heat conduction equations in a two-component environment and to obtain high estimates of global solutions and lower estimates of unbounded solutions using comparison theorems.

Consider the following nonlinear parabolic system of heat conduction equations a two-component environment coupled with nonlinear boundary conditions

$$\begin{cases} \rho_1(x)\dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(\left|\dfrac{\partial u}{\partial x}\right|^{p_1-2}\dfrac{\partial u^{m_1}}{\partial x}\right), x>0, t>0 \\ \rho_2(x)\dfrac{\partial \upsilon}{\partial t} = \dfrac{\partial}{\partial x}\left(\left|\dfrac{\partial \upsilon}{\partial x}\right|^{p_2-2}\dfrac{\partial \upsilon^{m_2}}{\partial x}\right), x>0, t>0 \end{cases} \quad (1.1)$$

$$\begin{cases} -\left|\dfrac{\partial u}{\partial x}\right|^{p_1-2}\dfrac{\partial u^{m_1}}{\partial x}\Bigg|_{x=0} = \upsilon^{q_1}(0,t), t>0 \\ -\left|\dfrac{\partial \upsilon}{\partial x}\right|^{p_2-2}\dfrac{\partial \upsilon^{m_2}}{\partial x}\Bigg|_{x=0} = u^{q_2}(0,t), t>0 \end{cases} \quad (1.2)$$

$$u(x,0) = u_0(x),\ \upsilon(x,0) = \upsilon_0(x), x>0 \quad (1.3)$$

where $m_i \ge 1$, $p_i > 1+1/m_i$, $q_i > 0$ $(i=1,2)$, $\rho_1(x) = |x|^n$, $\rho_2(x) = |x|^k$, $n > -p_1$, $k > -p_2$, $u_0(x)$ and $\upsilon_0(x)$ are nonnegative continuous functions with compact support in $\mathbb{R}_+$.

Mathematical problem (1)-(3) used for describe different nonlinear process of biological population, chemical reactions, diffusion and etc. For instance, the functions u and υ may be considered as the densities of two biological population during migration. Different particular cases of the problem (1.1)-(1.3) considered in many works (for example see [6],[7] and literature therein).

Now we recall some known results. In [3], Yongsheng Mi, Chunlai Mu and Botao Chen studied the following nonlinear filtration problem with slow diffusion:

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$u_t=(|u_x|^{p_1}(u^{m_1})_x)_x$, $\upsilon_t=(|\upsilon_x|^{p_2}(\upsilon^{m_2})_x)_x$, $x>0, t>0$,

$$\begin{cases} -|u_x|^{p_1}(u^{m_1})_x(0,t)=\upsilon^{q_1}(0,t) \\ -|\upsilon_x|^{p_2}(\upsilon^{m_2})_x(0,t)=u^{q_2}(0,t) \end{cases} \quad (1.4)$$

$u(x,0)=u_0(x)$, $\upsilon(x,0)=\upsilon_0(x)$

They proved that for the problem (1.4) every nonnegative solution is global in time, if $q_1q_2\le\frac{(2p_1+m_1+1)(2p_2+m_2+1)}{(p_1+2)(p_2+2)}$.

And for the case $q_1q_2>\frac{(2p_1+m_1+1)(2p_2+m_2+1)}{(p_1+2)(p_2+2)}$ are set:

a. If $\max\{l_1-k_1,l_2-k_2\}<0$, then every nonnegative nontrivial solution of the problem (4.1.4)-(4.1.6) blows up in finite time;

b. If $\min\{l_1-k_1,l_2-k_2\}>0$ and the initial data is small enough, then any solution of the problem (1.4) is global.

To state our results, we need to introduce the following numbers. Let

$$\alpha_1=\frac{(p_1+n)(p_2-1)q_1+(p_1-1)(p_2(k+2)+m_2(k+1)-2k-3)}{q_1q_2(p_1+n)(p_2+k)-(p_1(n+2)+m_1(n+1)-2n-3)(p_2(k+2)+m_2(k+1)-2k-3)}$$

$$\alpha_2=\frac{(p_1-1)(p_2+k)q_2+(p_2-1)(p_1(n+2)+m_1(n+1)-2n-3)}{q_1q_2(p_1+n)(p_2+k)-(p_1(n+2)+m_1(n+1)-2n-3)(p_2(k+2)+m_2(k+1)-2k-3)}$$

$$\beta=\frac{\alpha_2q_1-\alpha_1(p_1+m_1-2)}{p_1-1}=\frac{\alpha_1q_2-\alpha_2(p_2+m_2-2)}{p_2-1}$$

Theorem 1.

If $q_1q_2\le\frac{(p_1(n+2)+m_1(n+1)-2n-3)(p_2(k+2)+m_2(k+1)-2k-3)}{(p_1+n)(p_2+k)-(p_1(n+2)+m_1(n+1)}$, then every nonnegative solutions of the system (1.1)-(1.3) is global in time.

Proof.

We shall prove the theorem by constructing of a self-similar super-solution. We are looking for a self-similar solution in the form of

$$\begin{cases} \widehat{u}(x,t)=e^{L_1t}M_1\left(K+e^{-\xi_1}\right)^{\frac{1}{m_1}},\ \xi_1=(1+x)e^{-J_1t},\quad x\ge0,\ t\ge0, \\ \widehat{\upsilon}(x,t)=e^{L_2t}M_2\left(K+e^{-\xi_2}\right)^{\frac{1}{m_2}},\ \xi_2=(1+x)e^{-J_2t},\quad x\ge0,\ t\ge0, \end{cases} \quad (2.1)$$

where

$$K=\max\left\{\|u_0\|_\infty^{m_1}\Big/M_1^{\frac{1}{m_1}},\|\upsilon_0\|_\infty^{m_2}\Big/M_2^{\frac{1}{m_2}}\right\}-e^{-1},\ J_1=\frac{m_1(p_1-1)-1}{p_1+n}L_1,$$

$$J_2=\frac{m_2\left(p_2-1\right)-1}{p_2+k}L_2,\ M_1=\left(\frac{\left(K+e^{-1}\right)^{\frac{q_1}{m_2}}}{e^{-(p_1-1)}}\right)^{\frac{1}{m_1(p_1-1)-q_1}},$$

$$M_2=\left(\frac{\left(K+e^{-1}\right)^{\frac{q_2}{m_1}}}{e^{-(p_2-1)}}\right)^{\frac{1}{m_2(p_2-1)-q_2}},\ L_1>0,\ L_2>0.$$

We show that the constructed functions (2.1) will be the supper-solution of the problem (1.1)-(1.3). To do this, according to the principle of the solution comparison, they must satisfy the following system of inequalities

$$\begin{cases}\rho_1(x)\dfrac{\partial\hat{u}}{\partial t}\ge\dfrac{\partial}{\partial x}\left(\left|\dfrac{\partial\hat{u}}{\partial x}\right|^{p_1-2}\dfrac{\partial\hat{u}^{m_1}}{\partial x}\right),x>0,t>0\\ \rho_2(x)\dfrac{\partial\hat{\upsilon}}{\partial t}\ge\dfrac{\partial}{\partial x}\left(\left|\dfrac{\partial\hat{\upsilon}}{\partial x}\right|^{p_2-2}\dfrac{\partial\hat{\upsilon}^{m_2}}{\partial x}\right),x>0,t>0\end{cases}\quad(2.2)$$

$$\begin{cases}-\left|\dfrac{\partial\hat{u}}{\partial x}\right|^{p_1-2}\dfrac{\partial\hat{u}^{m_1}}{\partial x}\Bigg|_{x=0}\ge\hat{\upsilon}^{q_1}(0,t),t>0\\ -\left|\dfrac{\partial\hat{\upsilon}}{\partial x}\right|^{p_2-2}\dfrac{\partial\hat{\upsilon}^{m_2}}{\partial x}\Bigg|_{x=0}\ge\hat{u}^{q_2}(0,t),t>0\end{cases}\quad(2.3)$$

After the following calculations

$$\rho_1\left(x\right)\hat{u}_t=e^{(L_1+nJ_1)t}\xi_1^n\left(K+e^{-\xi_1}\right)^{\frac{1}{m_1}-1}\left(K+e^{-\xi_1}+\frac{M_1}{m_1}J_1\right)\ge e^{(L_1+nJ_1)t}\left(K+e^{-1}\right)^{\frac{1}{m_1}},$$

$$\frac{\partial}{\partial x}\left(\left|\frac{\partial\hat{u}}{\partial x}\right|^{p_1-2}\frac{\partial\hat{u}^{m_1}}{\partial x}\right)=M_1^{m_1(p_1-1)}(p_1-1)e^{(L_1(m_1-J_1)(p_1-1)-J_1)t}e^{-(p_1-1)\xi_1}$$

$$\rho_2\left(x\right)\hat{\upsilon}_t=e^{(L_2+kJ_2)t}\xi_2^k\left(K+e^{-\xi_2}\right)^{\frac{1}{m_2}-1}\left(K+e^{-\xi_2}+\frac{M_2}{m_2}J_2\right)\ge e^{(L_2+kJ_2)t}\left(K+e^{-1}\right)^{\frac{1}{m_2}},$$

$$\frac{\partial}{\partial x}\left(\left|\frac{\partial\hat{\upsilon}}{\partial x}\right|^{p_2-2}\frac{\partial\hat{\upsilon}^{m_2}}{\partial x}\right)=M_2^{m_2(p_2-1)}(p_2-1)e^{(L_2(m_2-J_2)(p_2-1)-J_2)t}e^{-(p_2-1)\xi_2}$$

It is easy to make sure that if

$$q_1q_2\le\frac{(p_1(n+2)+m_1(n+1)-2n-3)(p_2(k+2)+m_2(k+1)-2k-3)}{(p_1+n)(p_2+k)-(p_1(n+2)+m_1(n+1)}$$

and by definitions $M_i,J_i,L_i,(i=1,2),K$, the systems of inequalities (2.2) and (2.3) will always be valid.

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Theorem 2.

Let $q_1 q_2 > \frac{(p_1(n+2)+m_1(n+1)-2n-3)(p_2(k+2)+m_2(k+1)-2k-3)}{(p_1+n)(p_2+k)-(p_1(n+2)+m_1(n+1)}$, then every solution of the system (1.1)-(1.3) is unbounded with sufficiently large initial data.

Proof.

To prove the unsolvability in time of the solution, we construct unlimited self-similar solutions of the following form

$$\begin{cases} u\_(x,t) = (T-t)^{-\alpha_1} f(\xi) \\ \upsilon\_(x,t) = (T-t)^{-\alpha_2} g(\xi) \end{cases} \qquad (3.1)$$

$$\xi = x(T-t)^{\beta}$$

where $T > 0$, the functions $f(\xi)$ and $g(\xi)$ with compact support are defined below.

After the following simple calculations

$$\frac{\partial u\_}{\partial t} = (T-t)^{-\alpha_1 - 1}(\alpha_1 f + \beta\xi \frac{df}{d\xi})$$

$$\frac{\partial u\_}{\partial x} = (T-t)^{-\alpha_1 - \beta} \frac{df}{d\xi}$$

$$\left|\frac{\partial u\_}{\partial x}\right|^{p_1-2} \frac{\partial u\_^{m_1}}{\partial x} = (T-t)^{-\alpha_1(p_1+m_1-2)-\beta(p_1-1)} \left|\frac{df}{d\xi}\right|^{p_1-2} \frac{df^{m_1}}{d\xi}$$

$$\frac{\partial}{\partial x}\left(\left|\frac{\partial u\_}{\partial x}\right|^{p_1-2} \frac{\partial u\_^{m_1}}{\partial x}\right) = (T-t)^{-\alpha_1(p_1+m_1-2)-\beta p_1} \frac{d}{d\xi}\left(\left|\frac{df}{d\xi}\right|^{p_1-2} \frac{df^{m_1}}{d\xi}\right)$$

$$\frac{\partial \upsilon\_}{\partial t} = (T-t)^{-\alpha_2 - 1}(\alpha_1 f + \beta\xi \frac{dg}{d\xi})$$

$$\frac{\partial \upsilon\_}{\partial x} = (T-t)^{-\alpha_2 - \beta} \frac{dg}{d\xi}$$

$$\left|\frac{\partial \upsilon\_}{\partial x}\right|^{p_2-2} \frac{\partial \upsilon\_^{m_2}}{\partial x} = (T-t)^{-\alpha_2(p_2+m_2-2)-\beta(p_2-1)} \left|\frac{dg}{d\xi}\right|^{p_2-2} \frac{dg^{m_2}}{d\xi}$$

$$\frac{\partial}{\partial x}\left(\left|\frac{\partial \upsilon\_}{\partial x}\right|^{p_2-2} \frac{\partial \upsilon\_^{m_2}}{\partial x}\right) = (T-t)^{-\alpha_2(p_2+m_2-2)-\beta p_2} \frac{d}{d\xi}\left(\left|\frac{dg}{d\xi}\right|^{p_2-2} \frac{dg^{m_2}}{d\xi}\right)$$

and by definition $\alpha_i, \beta (i = 1,2)$ for the system, we get the following

$$\begin{cases} \frac{d}{d\xi}\left(\left|\frac{df}{d\xi}\right|^{p_1-2} \frac{df^{m_1}}{d\xi}\right) - \beta\xi^{n+1}\frac{df}{d\xi} - \alpha_1 \xi^n f = 0 \\ \frac{d}{d\xi}\left(\left|\frac{dg}{d\xi}\right|^{p_2-2} \frac{dg^{m_2}}{d\xi}\right) - \beta\xi^{k+1}\frac{dg}{d\xi} - \alpha_2 \xi^k g = 0 \end{cases} \qquad (3.2)$$

$$\begin{cases} -\left|\dfrac{df}{d\xi}\right|^{p_1-2}\dfrac{df^{m_1}}{d\xi}(1)=g^{q_1}(1) \\ -\left|\dfrac{dg}{d\xi}\right|^{p_2-2}\dfrac{dg^{m_2}}{d\xi}(1)=f^{q_2}(1) \end{cases} \quad (3.3)$$

We show that the $f(\xi)$ and $g(\xi)$ with compact support defined by the formula (3.1) will be the subsolution of the system. Consider the following functions with a compact support

$$\begin{cases} \bar f(\xi)=A_1\left(a-\xi^{\frac{p_1+n}{p_1-2}}\right)_+^{\frac{p_1-1}{p_1-m_1-3}} \\ \bar g(\xi)=A_2\left(a-\xi^{\frac{p_2+n}{p_2-2}}\right)_+^{\frac{p_2-1}{p_2-m_2-3}} \end{cases} \quad (3.4)$$

where $A_i, a (i=1,2)$ are constants to be determined.

Then, according to the condition of the solution comparison theorem, the functions $\bar f(\xi)$, $\bar g(\xi)$ must satisfy the following problem

$$\begin{cases} \dfrac{d}{d\xi}\left(\left|\dfrac{d\bar f}{d\xi}\right|^{p_1-2}\dfrac{d\bar f^{m_1}}{d\xi}\right)-\beta\xi^{n+1}\dfrac{d\bar f}{d\xi}-\alpha_1\xi^n\bar f\ge 0 \\ \dfrac{d}{d\xi}\left(\left|\dfrac{d\bar g}{d\xi}\right|^{p_2-2}\dfrac{d\bar g^{m_2}}{d\xi}\right)-\beta\xi^{k+1}\dfrac{d\bar g}{d\xi}-\alpha_2\xi^k\bar g\ge 0 \end{cases} \quad (3.5)$$

$$\begin{cases} -\left|\dfrac{d\bar f}{d\xi}\right|^{p_1-2}\dfrac{d\bar f^{m_1}}{d\xi}(1)\le \bar g^{q_1}(1) \\ -\left|\dfrac{d\bar g}{d\xi}\right|^{p_2-2}\dfrac{d\bar g^{m_2}}{d\xi}(1)\le \bar f^{q_1}(1) \end{cases} \quad (3.6)$$

On the other hand, the boundary conditions in (1.2) are satisfied if we have

$$\begin{cases} A_1^{m_1(p_1-1)}\left(\dfrac{m_1(p_1-1)(p_1+n)}{p_1(m_1(p_1-1)-1)}\right)^{p_1-1}(a_1+1)^{\frac{p_1-1}{m_1(p_1-1)-1}}\le A_2^{q_1}(a_2+1)^{\frac{q_1(p_2-1)}{m_2(p_2-1)-1}}, \\ A_2^{m_2(p_2-1)}\left(\dfrac{m_2(p_2-1)(p_2+k)}{p_2(m_2(p_2-1)-1)}\right)^{p_2-1}(a_2+1)^{\frac{p_2-1}{m_2(p_2-1)-1}}\le A_1^{q_2}(a_1+1)^{\frac{q_2(p_1-1)}{m_1(p_1-1)-1}}, \end{cases} \quad (3.7)$$

The following restrictions follow from (3.7)

$$\begin{cases} \left(\dfrac{m_1(p_1-1)(p_1+n)}{p_1(m_1(p_1-1)-1)}\right)^{p_1-1}(a_1+1)^{\frac{p_1-1}{m_1(p_1-1)-1}}(a_2+1)^{-\frac{q_1(p_2-1)}{m_2(p_2-1)-1}}\le A_1^{-m_1(p_1-1)}A_2^{q_1}, \\ \left(\dfrac{m_2(p_2-1)(p_2+k)}{p_2(m_2(p_2-1)-1)}\right)^{p_2-1}(a_1+1)^{-\frac{q_2(p_1-1)}{m_1(p_1-1)-1}}(a_2+1)^{\frac{p_2-1}{m_2(p_2-1)-1}}\le A_1^{q_2}A_2^{-m_2(p_2-1)}. \end{cases} \quad (3.8)$$

It is easy to check that for any

$$q_1 q_2 > \frac{(p_1(n+2)+m_1(n+1)-2n-3)(p_2(k+2)+m_2(k+1)-2k-3)}{(p_1+n)(p_2+k)-(p_1(n+2)+m_1(n+1)}$$

there are constants $A_i, a(i=1,2)$, satisfying inequality (3.8). According to the principle of the theorem of comparison of solutions for initial data, we have

$$\begin{cases} u_0(x) \ge T^{-\alpha_1}\bar{f}(\xi) \\ \upsilon_0(x) \ge T^{-\alpha_2}\bar{g}(\xi) \end{cases}$$

The proof is complete.

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