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SOME 2-ADIC CONJECTURES CONCERNING POLYOMINO TILINGS OF AZTEC DIAMONDS

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Abstract

For various sets of tiles, we count the ways to tile an Aztec diamond of order n using tiles from that set. The resulting function $f(n)$ often has interesting behavior when one looks at n and $f(n)$ modulo powers of 2.

– Dedicated to Michael Larsen on the occasion of his 60th birthday

1. Introduction

The main result of [6, 7] was that the number of domino-tilings of an Aztec diamond of order n is $2^{n(n+1)/2}$, where a *domino* is a rectangle in $\mathbb{R}^2$ of the form $[i, i+1] \times [j, j+2]$ or $[i, i+2] \times [j, j+1]$ (with $i, j \in \mathbb{Z}$), the *Aztec diamond* of order n is the union of the squares $[i, i+1] \times [j, j+1]$ lying entirely within the region $\{(x, y) : \ |x| + |y| \leq n+1\}$, and a *domino-tiling* of a region R is a set of dominos whose interiors are disjoint and whose union is R. Figure 1 shows one of the $2^{(4)(5)/2}$ domino tilings of the Aztec diamond of order 4. The sequence of successive values of $2^{n(n+1)/2}$ appears as A006125; throughout this article such codes refer to entries in the Online Encyclopedia of Integer Sequences.

The main result of [6, 7] was that the number of domino-tilings Mihai Ciucu [2] proved combinatorially that the number of domino tilings of the $2n$-by-$2n$ square (A004003) can be written in the form $2^n f(n)^2$ where $f(n)$ is the number of domino tilings of the region exemplified for $n = 4$ in Figure 2 (A065072).

Henry Cohn [3] proved that the function sending n to $f(n)$ is uniformly continuous under the 2-adic metric and thus extends to a function defined on all of $\mathbb{Z}$ and indeed all of $\mathbb{Z}_2$; moreover, he showed that this extension satisfies

$$f(-1-n) = \begin{cases} f(n) & \text{when } n \equiv 0 \text{ or } 3 \pmod 4, \\ -f(n) & \text{when } n \equiv 1 \text{ or } 2 \pmod 4. \end{cases} \tag{1}$$

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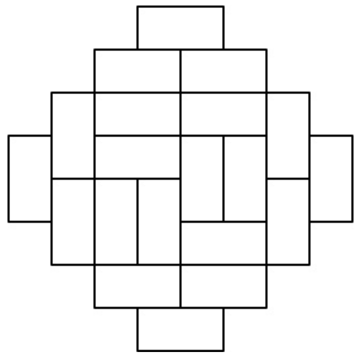

Figure 1: A domino tiling of the Aztec diamond of order 4.

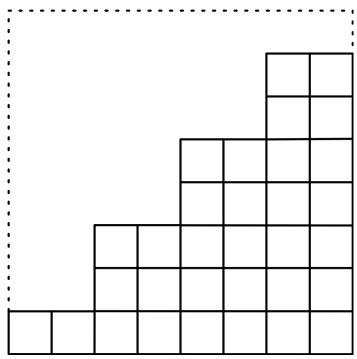

Figure 2: Ciucu's way of halving the 8-by-8 square.

Barkley and Liu [1] have recently proved results about 2-divisibility for the number of perfect matchings of a graph, including as a special case the number of domino tilings of a rectangle, but there is more refined work still to be done along the lines of Cohn's paper. For instance, the modulo 8 residue of the number of domino tilings of the $2n$-by-$(2n+2)$ rectangle appears to depend only on the modulo 4 residue of n; the same goes for the number of domino tilings of the $2n$-by-$4n$ rectangle.

In this article we extend the discussion to other sorts of tiles, specifically, tetrominos. A *tetromino* is a connected subset of the grid that is a union of four grid-squares, just as a domino is a union of two grid-squares. Up to symmetry, there are five kinds of tetrominos: straight tetrominos, skew tetrominos, L-tetrominos, square tetrominos, and T-tetrominos. They are shown in Figure 3, preceded by the domino. These six tiles can be placed on a square grid in 2, 2, 4, 8, 1, and 4 translationally-inequivalent ways, respectively (where rotations and reflections are permitted). These are the sorts of tiles considered in this article. *Trominos* – unions of three grid-squares – will be considered elsewhere.

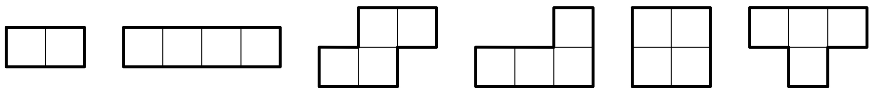

Figure 3: A domino, a straight tetromino, a skew tetromino, an L-tetromino, a square tetromino, and a T-tetromino.

2. Skew and Straight Tetrominos

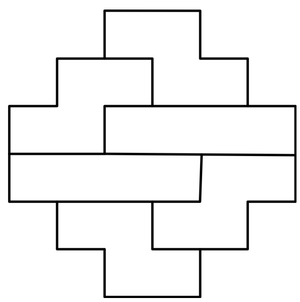

Figure 4: Tiling the Aztec diamond of order 3 with skew and straight tetrominos.

We will start with a warm-up puzzle that is roughly at the level of a math olympiad: prove that an Aztec diamond of order n can be tiled by skew and straight tetrominos (as shown in Figure 4 for $n = 3$) only if n is congruent to 0 or 3 modulo 4. The puzzle can be solved using a valuation argument (sometimes called a generalized coloring argument): one can construct a mapping from the grid-cells to an appropriate abelian group (a "weight function") and show that when n is congruent to 1 or 2 modulo 4, the sum of the weights of the tiles can not equal the sum of the weights of the region being tiled, where the weight of a tile or a region being tiled is the sum of the weights of the constituent cells. Readers who are already familiar with this technique might enjoy the challenge of attempting to solve the problem purely mentally.

3. Dominos and Square Tetrominos

In this section we use dominos and square tetrominos as tiles. Thus an Aztec diamond of order 1 (better known as the 2-by-2 square) can be tiled in 3 ways: with two horizontal dominos, two vertical dominos, or a single square tetromino. The Aztec diamond of order 2 can be tiled in $2^{(2)(3)/2} = 8$ ways using dominos, and can be tiled in an additional 11 ways if one or more square tetrominos are included, as shown in Figure 5. Thus there are a total of $8 + 11 = 19$ ways to tile an Aztec diamond of order 2 using dominos and square tetrominos.

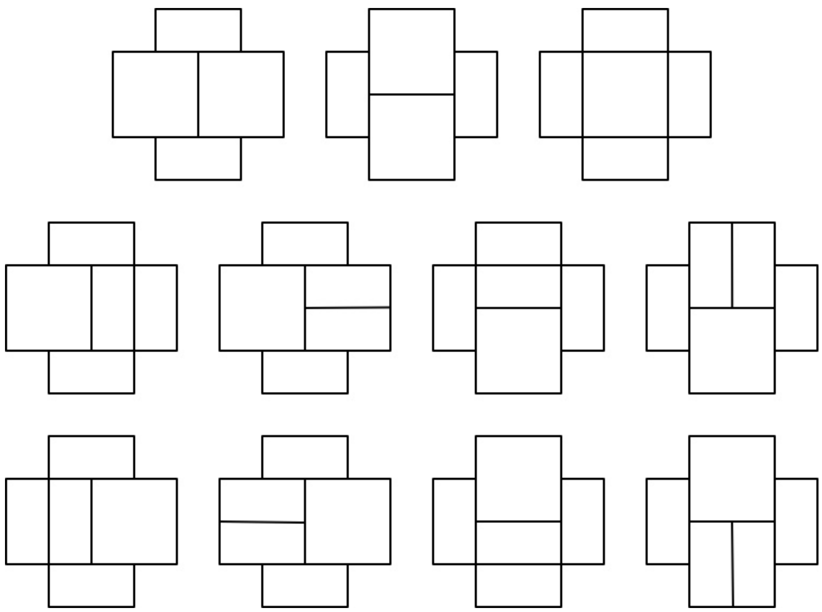

Figure 5: Tiling the Aztec diamond of order 2 with dominos and at least one square tetromino.

Define $M(n)$ (with $n \geq 0$) as the number of tilings of the Aztec diamond of order n using dominos and square tetrominos. This is A356512. Trivially we have $M(0) = 1$ (since the Aztec diamond of order 0 is empty) and we have already seen that $M(1) = 3$ and $M(2) = 19$. Figure 6 shows the terms of the sequence $M(n)$ for n ranging from 0 to 12, computed using a program written by David desJardins. The reader may wish to pause here to consider the problem of showing that $M(n)$ is always odd; a solution will be given in Section 5.

These numbers grow quadratic-exponentially as a function of n, and we have no conjectural formula for the nth term, nor a conjectural recurrence relation for the sequence, nor any efficient method of computing terms. Nonetheless, something

n	$M(n)$
0	1
1	3
2	19
3	293
4	10917
5	996599
6	222222039
7	121552500713
8	162860556763865
9	535527565429290907
10	4318205059450240425083
11	85475498697714319842817853
12	4151186175463797888945512144221

Figure 6: Enumeration of tilings of Aztec diamonds using dominos and square tetrominos.

systematic is going on. We have already mentioned that all the terms are odd. Taking this observation further, one notices that the numbers' residues modulo 4 are

$$1,\ 3,\ 3,\ 1,\ 1,\ 3,\ 3,\ 1,\ 1,\ 3,\ 3,\ 1,\ 1,$$

the residues modulo 8 are

$$1,\ 3,\ 3,\ 5,\ 5,\ 7,\ 7,\ 1,\ 1,\ 3,\ 3,\ 5,\ 5,$$

and the residues modulo 16 are

$$1,\ 3,\ 3,\ 5,\ 5,\ 7,\ 7,\ 9,\ 9,\ 11,\ 11,\ 13,\ 13.$$

Conjecture 1. For all $k \geq 1$, the modulo 2^k residue of $M(n)$ is periodic with period dividing 2^k. That is, 2^k divides $M(n+2^k) - M(n)$ for all k, n.

We tried to prove this conjecture by reducing it to an assertion about alternating-sign matrices but we were unsuccessful. Note that if the conjecture is true then $M(n) \equiv n+1+(1+(-1)^{n+1})/2 \pmod 8$. This congruence might also hold modulo 16 but it certainly cannot hold modulo 2^k for all k, since that would require that $M(n)$ actually equals $n+1+(1+(-1)^{n+1})/2$, which is clearly not the case for $n \geq 2$. And indeed $M(2) = 19 \not\equiv 3 \pmod{32}$.

A deeper consequence of Conjecture 1 is that the function sending n to $M(n)$ is 2-adically continuous. Moreover, the function appears to satisfy a kind of symmetry analogous to the functional Equation (1) mentioned at the end of Section 1.

Conjecture 2. For all $k \geq 1$, if $n + n' \equiv -3 \pmod{2^k}$ then $M(n) + M(n') \equiv 0 \pmod{2^k}$.

That is, if one extends $M : \mathbb{N} \to \mathbb{N}$ to the 2-adic function $\widehat{M} : \mathbb{Z}_2 \to \mathbb{Z}_2$, one has $\widehat{M}(-3 - n) = -\widehat{M}(n)$.

Although this article is focused on tilings of Aztec diamonds, we have also looked at tilings of other regions using dominos and square tetrominos, and the same phenomenon of 2-adic continuity arises fairly broadly there. For instance, for the $2n$-by-$2n$ square, the $2n$-by-$(2n + 2)$ rectangle, and the $2n$-by-$4n$ rectangle, the number of tilings with dominos and square tetrominos always seems to be congruent to $2n + 1$ modulo 8.

4. Skew Tetrominos and Square Tetrominos

In [13] we considered tilings of Aztec diamonds by skew tetrominos and square tetrominos. If we require that all skew tetrominos be horizontal, interesting numerical patterns appear. Of course we would get the same result if we required that all skew tetrominos be vertical. In this section we allow horizontal skew tetrominos and square tetrominos as seen in Figure 7, which depicts all six tilings of the Aztec diamond of order 3 using square tetrominos and horizontal skew tetrominos.

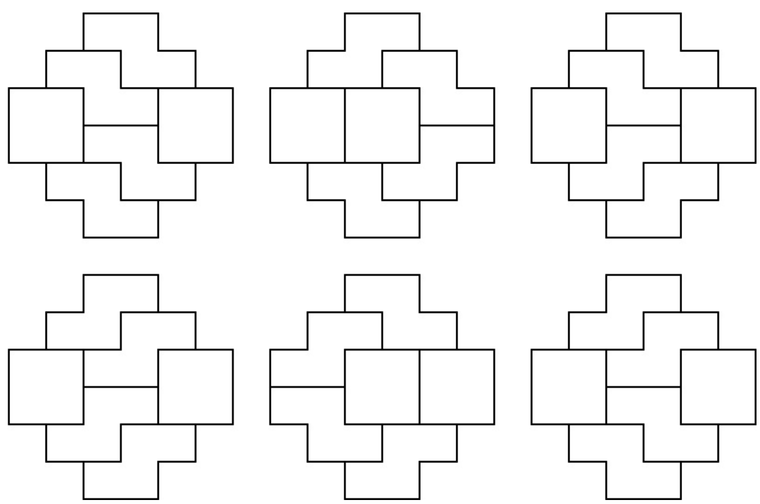

Figure 7: Tiling the Aztec diamond of order 3 with horizontal skew tetrominos and square tetrominos.

n	$L(n)$
0	1
1	1
2	2
3	6
4	40
5	364
6	7904
7	226152
8	15835008
9	1439900880
10	324189571584
11	94080051207136
12	68041472016287744
13	63145927127133361600
14	146637148542938673930240
15	435697213021432661980535936

Figure 8: Enumeration of tilings of Aztec diamonds using horizontal skew tetrominos and square tetrominos.

Define $L(n)$ (with $n \geq 0$) as the number of tilings of the Aztec diamond of order n using horizontal skew tetrominos and square tetrominos. This is A356513. Trivially we have $L(0) = 1$ and $L(1) = 1$. Figure 8 shows the terms of the sequence $L(n)$ for n ranging from 0 to 15, again computed using the program written by David desJardins.

The sequence grows quadratic-exponentially, and once again, we have no conjectural formula, but as before there are patterns that call out for explanation. Noticing that all but the first two terms are even, one might naturally think to look at the multiplicity of 2 in the prime factorization of $L(n)$, obtaining the sequence 0, 0, 1, 1, 3, 2, 5, 3, 7, 4, 9, 5, 11, 6, 13, 7, ..., which (once we throw out the initial 0) we recognize as an interspersal of the arithmetic progressions 0, 1, 2, 3, 4, 5, 6, 7, ... and 1, 3, 5, 7, 9, 11, 13, ...

Conjecture 3. For $n \geq 1$, the multiplicity of 2 in the prime factorization of $L(n)$ is $n - 1$ if n is even and $(n - 1)/2$ if n is odd.

Going further, let $L_0(m) = L(2m)/2^{2m-1}$ and $L_1(m) = L(2m-1)/2^{m-1}$, so that (if Conjecture 3 holds) $L_0(m)$ and $L_1(m)$ are odd integers for all m. These two new sequences are shown in Figure 9.

The modulo 4 residues of the L_0 sequence go 1, 1, 3, 3, 1, 1, 3 while those of the L_1 sequence go 1, 3, 3, 1, 1, 3, 3, 1. That is not much evidence to go on, so perhaps

m	$L_0(m)$	m	$L_1(m)$
1	1	1	1
2	5	2	3
3	247	3	91
4	123711	4	28269
5	633182757	5	89993805
6	33223375007953	6	2940001600223
7	17900042546745443595	7	986655111361458775
		8	3403884476729942671722937

Figure 9: Values of $L_0(m)$ and $L_1(m)$.

it would be prudent not to make a conjecture, but we choose to be hopeful.

Conjecture 4. For all $k \geq 0$, the modulo 2^k residue of $L_0(m)$ is periodic with period dividing 2^k. Likewise for $L_1(m)$.

One does not observe such patterns in the numbers of tilings when both horizontal and vertical skew tetrominos are allowed along with square tetrominos as in [13]. More specifically, if one counts tilings of Aztec diamonds in which one is permitted to use all four kinds of skew tetrominos as well as square tetrominos, the resulting sequence, taken modulo 4, goes 1, 0, 0, 0, 0, 0, 2, 2, 0, 0, 0, 0, $\ldots$; if there is a period here, and it is a power of 2, it must be at least 16.

The prime $p = 2$ appears to be special for the enumerative problems we described above; looking at the M and L sequences modulo 3 or modulo 5 yields no discernible patterns.

5. Assorted Congruential Problems

For each of the sixty-three nonempty subsets of the set of six tiles shown in Figure 3, we can ask in how many ways it is possible to tile the Aztec diamond of order n using only tiles from that set, allowing translations, rotations, and reflections of tiles. These are the enumerative problems considered in this section.

One could expand the set of tiling problems by distinguishing between different orientations of the tiles, as was done in the preceding section where we permitted horizontal skew tetrominos but forbade vertical skew tetrominos; since there are $2 + 2 + 4 + 8 + 1 + 4 = 21$ different tiles up to translation, we would obtain over two million different problems, and even if we mod out the $2^{21} - 1$ problems by a dihedral action of order 8, that is still too many problems to consider exhaustively. One that appears to be interesting is discussed at the end of this section.

In each of the sixty-three cases, we used the aforementioned program to count the tilings of the Aztec diamond of order n, with n going from 1 to 8, using the allowed tiles. Although no 2-adic continuity phenomena arose from these experiments, there were definite patterns in the parity, and in a few cases there were congruence patterns modulo higher powers of 2. Here we will adopt a six-bit code to represent the sixty-three tiling problems, in which the six successive bits (from left to right) equal 1 or 0 according to whether or not dominos, straight tetrominos, skew tetrominos, L-tetrominos, square tetrominos, and T-tetrominos are allowed. For instance, the case treated in Section 2, in which only straight tetrominos and skew tetrominos are allowed (see Figure 4), would be assigned the code 011000; the case treated in Section 3, in which only dominos and square tetrominos are allowed (see Figure 5), would be assigned the code 100010; and the case of unconstrained skew and square tetrominos (briefly discussed in Section 4) would be assigned the code 001010.

In one-third of the 63 cases, we observed that for all n between 1 and 8, the number of tilings of the Aztec diamond of order n is even. These were the cases associated with the six-bit codes 001001, 001100, 001101, 011001, 011100, 011101, 100001, 100100, 100101, 101000, 101001, 101100, 101101, 110000, 110001, 110100, 110101, 111000, 111001, 111100, and 111101. Presumably some (perhaps all) of these examples can be resolved by showing that there are no tilings that are invariant under the full dihedral group, since in that case all orbits would contain an even number of tilings.

Three of the 21 cases were especially interesting. In case 011100, all terms were divisible by 8; in case 100001, all terms after the first were divisible by 8; and in case 110001, all terms were congruent to 2 modulo 4. There were also four cases in which we observed that the number of tilings of the Aztec diamond of order n is even for all n between 2 and 8 (with the number of tilings being the odd number 1 in the case $n = 1$). These were the cases associated with the six-bit codes 001010, 001110, 011010, and 011110. In the cases 001101, 100001, 100011, and 111000 it appears that the exponent of 2 in the number of tilings may be going to infinity with n, though with such scant evidence it would be rash to place too much faith in this guess.

Additionally, there is one case in which the number of tilings of the Aztec diamond of order n is always odd, namely, tilings using only dominos and square tetrominos. Indeed, if we assign each tiling weight $(-1)^s$ where s is the number of square tetrominos, we claim that the sum of the weights is 1. The proof uses a partition of the tilings into equivalence classes, as depicted in Figure 10 for the 19 tilings of the Aztec diamond of order 2. There is a unique tiling T_0 that uses only horizontal dominos, shown at the top left of the Figure; the middle two rows of T_0 consist of 2-by-2 squares. Identify the vertices in the tiling as alternately even or odd so that the aforementioned squares in T_0 have even vertices at their corners.

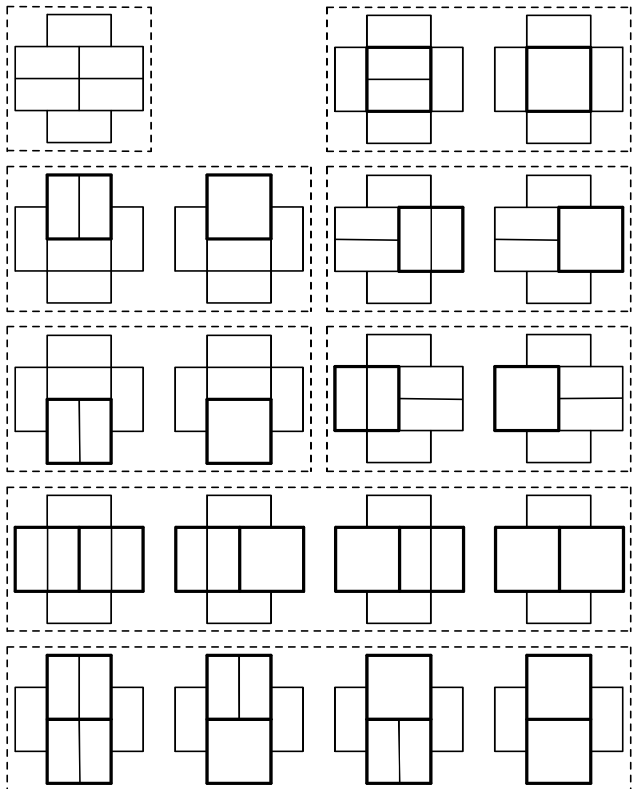

Figure 10: An equivalence relation on the 19 tilings of the Aztec diamond of order 3 by dominos and square tetrominos.

Call a square in the Aztec diamond even or odd according to whether its corners are even or odd. Given a tiling T of the Aztec diamond using dominos and square tetrominos, we say a 2-by-2 square S in the Aztec diamond is *hot* if one of three conditions holds: T contains S as a tile; S is even and T contains two vertical dominos that tile S; or S is odd and T contains two horizontal dominos that tile S. Note that for any T, the hot squares of T must be disjoint. The outlines of the hot squares are shown in bold in the Figure. We say two tilings T, T' are equivalent if they determine the same hot squares and if they agree outside of those squares. If the tiles in an equivalence class have $k > 0$ hot squares, then the class contains 2^k tilings, obtainable from one another by moves that switch an even hot square from being tiled by a square tetromino to being tiled by two vertical dominos and by moves that switch an odd hot square from being tiled by a square tetromino to being tiled by two horizontal dominos. It is clear that half of the tilings in the class have an odd number of square tetrominos and half have an even number of square tetrominos, so the sum of $(-1)^s$ over such a class is 0. It remains to consider tilings that are free of hot squares. Let T be such a tiling. T cannot contain any square tetrominos, since squares are automatically hot, so T must use only dominos. Moreover, T cannot have pairs of vertical dominos that tile an even square or pairs of horizontal dominos that tile an odd square. But Thurston's height-function theory [15] shows that such a tiling would be minimal in the lattice of tilings, and T_0 is the only minimal element.

Finally, leaving the small world of the $2^6 - 1$ problems and dipping our toe into the big world of the $2^{21} - 1$ problems, we consider tilings of the Aztec diamond of order n by dominos and horizontal straight tetrominos. This is A356523, and begins 1, 2, 11, 209, 12748, 2432209, 1473519065, ... It appears that the number of tilings is even when $n \equiv 1 \pmod 3$ and odd otherwise; this has been verified for $1 \le n \le 16$.

6. Reduction to Perfect Matchings

The L sequence from Section 4 has an interpretation in terms of perfect matchings. To see why, suppose we have a tiling of the Aztec diamond of order n using horizontal skew tetrominos and square tetrominos. Dividing each tetromino into two horizontal dominos gives us a tiling of the Aztec diamond by horizontal dominos, but it is easy to see that there is exactly one such tiling (call it T). Hence each tetromino is obtained by gluing together two dominos in T. That is, the tetromino tilings correspond to perfect matchings in the graph whose vertices correspond to the dominos in T with an edge joining two vertices if the corresponding dominos form a horizontal skew tetromino or square tetromino. It is not hard to see that this graph is similar to the n-by-n square except that the diagonal has been "doubled"; for instance, the right panel of Figure 11 shows the graph for $n = 4$.

A similar analysis can be applied to tilings of Aztec diamonds using horizontal skew tetrominos and horizontal straight tetrominos. In this case the Aztec diamond splits into two non-interacting halves (top half and bottom half), each of which can be tiled independently of the other, and the tilings of either half correspond to perfect matchings of a triangle graph as shown in Figure 12. Thus the number of such tetromino tilings of the Aztec diamond of order n is equal to the square of the nth term of sequence A071093. Studying the first 25 terms, we find that the sequence seems to have 2-adic properties of its own. The largest power of 2 dividing the nth term of the sequence A071093 appears to be $\lfloor n/2 \rfloor$, and the 2-free part appears to satisfy 2-adic continuity: for instance, its value modulo 16 seems to be determined by n modulo 16.

What if we superimpose the two graphs, obtaining the graph shown at the right

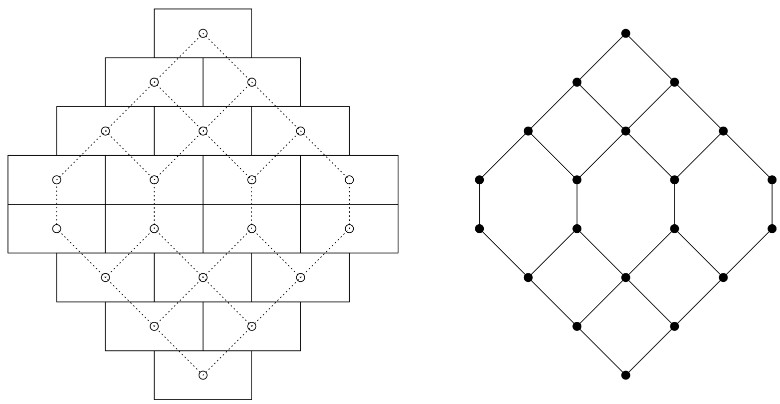

Figure 11: Deriving the square graph with doubled diagonal.

half of Figure 13? This is equivalent to tiling an Aztec diamond using horizontal skew tetrominos, horizontal straight tetrominos, and square tetrominos. Then, counting the tilings, we obtain the integer sequence 1, 2, 10, 116, 3212, 209152, 32133552, 11631456480, 9922509270288, 19946786274879008, 94492874103638971552, 1054865198752147761744448, ... This is A356514. It appears that the number of tilings is divisible by $2^{\lfloor n/2 \rfloor}$.

7. Some Thoughts

The articles of Lovász [11], Ciucu [2], Pachter [12], and Barkley and Liu [1] give ways to find the largest power of 2 that divides the number of perfect matchings of a graph. This should provide traction for Conjecture 3, since we saw in Section 6 that the L sequence has an interpretation in terms of perfect matchings of certain graphs. Graphs of this kind appear in the paper of Ciucu [2]; in particular, his Lemma 1.1 shows that the number of perfect matchings is divisible by $2^{\lfloor n/2 \rfloor}$. By bringing ideas from Pachter [12], one might be able to prove Conjecture 3, as well as some of the other 2-divisibility conjectures from this article.

The only work we know of that provides detailed 2-adic information about the 2-free part of numbers that count tilings is the work of Cohn [3]. Cohn's approach presupposes the existence of an exact formula (in Cohn's case, an explicit product of algebraic integers); perhaps something similar can be done for perfect matchings of the square graph with doubled diagonal, yielding a proof of Conjecture 4.

Conjectures 1 and 2 seem harder. The product formula exploited by Cohn was discovered by Temperley and Fischer [14] and independently by Kasteleyn [9] at about the same time; those researchers made use of the fact that, just as deter-

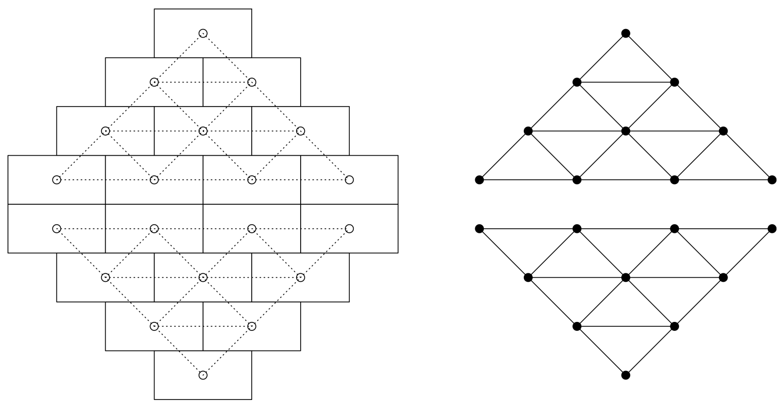

Figure 12: Deriving the double triangle graph.

minants and Pfaffians of matrices can be expressed as sums of terms associated with perfect matchings of the set of rows and columns, one can conversely express the number of perfect matchings of a planar graph in terms of the determinant or Pfaffian of an associated matrix. We know of no way to recast the M sequence as enumerating perfect matchings of graphs. However, it is easy to recast the M sequence as enumerating perfect matchings of certain hypergraphs. Can any of the existing notions of hyperdeterminants be brought to bear? Perhaps a reading of [8] would suggest possible approaches.

Kuperberg's elegant solution [10] to the alternating sign matrix conjecture exploits the power of the Yang-Baxter equation in statistical mechanics. It is possible that tools for analyzing the new problems described in this article will be found in the existing literature at the interface between algebra and statistical mechanics.

In any case, inasmuch as Conjectures 1 and 2 are reminiscent of Cohn's work, and inasmuch as Cohn's argument hinges on an exact product formula, one might hope that an exact formula of some kind can be found for the M sequence. Such an exact formula would have other uses. In [4] and [5], exact enumeration results are used to prove concentration theorems for random tilings. One might hope that the curious 2-adic phenomena discussed in this article hint at the existence of algebraic machinery that could be applied to the task of showing us what random tilings associated with Conjecture 1 look like in the limit as size goes to infinity. Preliminary experiments suggest that there is a "frozen region" near the boundary, but we have no idea how far into the interior it extends.

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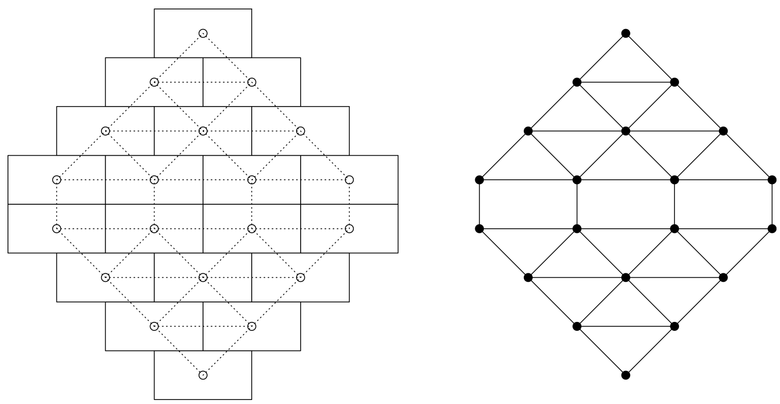

Figure 13: Superimposing Figures 11 and 12.

finding and fixing a mistake in one of the proofs, and the editor of Integers for his patience. Above all I thank Michael Larsen for his many years of friendship and mathematical camaraderie.

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