The Solution of Goldbach Conjecture

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Abstract: Goldbach conjecture is proved to be not valid by comparing the cardinality of a set containing all the even integers of which every even integer is the sum of two odd primes, to that of the subsets of even integers, such as a subset of all the even integers that can be factorized with a total of only three factors consisting of odd primes and at least one of 2.

Goldbach conjecture states that every even integer > 4 is the sum of two odd primes.

Let *P* represent the set of all odd primes, that is

$$P = \{p_1, p_2, \cdots, p_k, \cdots\} = \{3, 5, 7, 11, 13, \cdots\} \tag{1}$$

and *E* represent the set of all even integers:

$$E = \{2, 4, 6, 8, \cdots\} = \{x = 2n \mid n = 1, 2, 3, \cdots\} \tag{2}$$

From the set *P*, an even integer set *E*(1, 1) can be generated

$$E(1,1) = \{p_i + p_j \mid p_i \le p_j and\ p_i, p_j \in P\} \tag{3}$$

The counting function $P_x(1, 1)$ can be defined as the total number of pairs of two odd primes satisfying

$$p_i + p_j = x \quad for\ a\ given\ x > 4\ in\ E \tag{4}$$

Goldbach's conjecture is equivalent to one or the other of the following two equivalent questions

$$\begin{cases} [1] \begin{cases} P_x(1,1) \ge 1\ for\ all\ x > 4\ in\ E, Goldbach's\ conjeture\ is\ valid \\ P_x(1,1) = 0\ for\ some\ x > 4\ in\ E, Goldbach's\ conjeture\ is\ not\ valid \end{cases} \\ \\ [2] \begin{cases} |E(1,1)| \ge |E|, Goldbach's\ conjeture\ is\ valid \\ |E(1,1)| < |E|, Goldbach's\ conjeture\ is\ not\ valid \end{cases} \end{cases} \tag{5}$$

The attempts have been made to derive the analytical equation of the counting function $P_x(1, 1)$[1], but with little success. The difficulty is arisen from the prime distribution function π(x) that is not well understood. What is known about π(x) is that its asymptotic behavior at x→∞ is thought to satisfy the equation:

$$\lim_{x\to\infty}\frac{\pi(x)lnx}{x}=1 \tag{6}$$

It is not reasonable to use the asymptotic behavior of $\pi(x)$ at $x\to\infty$ as $\pi(x)$ itself in the whole region of x, such as did in the prime number theorem. After a method is established to derive an explicit expression of the counting function $\pi(x)$, $P_x(1, 1)$ and other counting functions of different prime patterns such as $\pi_2(x)$ could be obtained.

It was proved[2] that

$$|E(1,1)| \ll |\,E| \tag{7}$$

For the calculation of $|E(1, 1)|$ and $|E|$, let $|P|$ = N and then one has

$$|E(1,1)| < \frac{N(N+1)}{2} \tag{8}$$

$$\frac{1}{2}\zeta(1)=\left(\frac{1}{1-2^{-1}}-1\right)\prod_{i=1}^{N}\frac{1}{1-p_i^{-1}}\xrightarrow{\sim till\ the\ order\ of\ n}\left(\sum_{k=1}^{n}2^{-k}\right)\prod_{i=1}^{N}\left(\sum_{k=0}^{n}p_i^{-k}\right) \tag{9}$$

$$|E| = n\times(n+1)^N\ \ (\text{number of terms in the right side of equation (9)}) \tag{10}$$

When $n \to \infty$ and $|P| = N \to \infty$

$$\sum_{k=3}^{\infty}P_{2k}(1,1)=\lim_{N\to\infty}\frac{N(N+1)}{2}\approx\lim_{N\to\infty}N^2 \tag{11}$$

$$|E|\approx\lim_{N\to\infty}\lim_{n\to\infty}n^N \tag{12}$$

$$|E(1,1)|\approx\begin{cases}\lim\limits_{N\to\infty}N \quad if\sum\limits_{k=3}^{\infty}P_{2k}(1,1)\gg\ |E(1,1)|\\ \lim\limits_{N\to\infty}N^2 \quad if\sum\limits_{k=3}^{\infty}P_{2k}(1,1)\approx\ |E(1,1)|\end{cases} \tag{13}$$

Therefore, the equation (7) is proved by simply comparing of the equation (13) to the equation (12).

In the following discussions, the subset structure of even integers is analyzed and $|E(1, 1)|$ is compared with the cardinalities of the subsets of even integers.

The subset structure of odd integers

Let *O* represent the set containing all odd integers, that is,

$$O = \{1, 3, 5, 7, \cdots\} = \{2n - 1 \mid n = 1, 2, 3, \cdots\} \tag{14}$$

Because of the uniqueness of the prime factorization, it is natural to define the subset *O*(*n*) of set *O* based on the number of odd prime factors n

$$O(n) = \left\{O(n,k) \middle| \, O(n,k) = \prod_{i=1}^{n} p_{k_i}\,;\ O(n,k) < O(n,k+1), p_{k_i} \le p_{k_{i+1}} and\ p_{k_i} \in P\right\} \tag{15}$$

The *k*th element *O*(*n*, *k*) in the set *O*(*n*) is the product of n primes, in which the prime numbers are taken from the set *P* of all odd primes.

Obviously, there hold the following three equations

$$O = \bigcup_{n=0}^{\infty} O(n) \tag{16}$$

$$O(m) \cap O(n) = \begin{cases} \emptyset & if\ m \neq n \\ O(m) & if\ m = n \end{cases} \tag{17}$$

$$|O| = \sum_{n=0}^{\infty} |O(n)| \tag{18}$$

As some examples, $O(n, 1)=3^n$ and $O(n, 2)=3^{(n-1)}5$.

$O(0) = \{1\}$; $O(1) = P$ and $O(2) = \{p_i p_j \mid p_i \le p_j \in P\}$.

If | *P* | = N, then

$$|O(0)| = 1;\ |O(1)| = N;\ |O(2)| = \frac{N(N+1)}{2};\ |O(n)| = \frac{(N-1+n)!}{(N-1)!\,n!} (n = 3, 4, 5, \cdots) \tag{19}$$

$$|O(0)| < |O(1)| < |O(2)| < \cdots < |O(n)| < |O(n+1)| < \cdots \tag{20}$$

$$|O| = \lim_{M\to\infty} \sum_{n=0}^{M} |O(n)| = \lim_{M\to\infty} \frac{(N+M)!}{N!\,M!} \approx \lim_{M\to\infty} M^N \tag{21}$$

If | *P* | = N $\to \infty$, then

$$|O(0)| = 1$$

$$|O(1)| \approx \lim_{N\to\infty} N$$

$$|O(2)| \approx \lim_{N\to\infty} N^2$$

$$|O(n)| \approx \lim_{N\to\infty} N^n$$

$$|O(0)| \ll |O(1)| \ll |O(2)| \ll \cdots \ll |O(n)| \ll |O(n+1)| \ll \cdots \quad (22)$$

$$|O| = \lim_{M\to\infty} \sum_{n=0}^{M} |O(n)| \approx \lim_{N\to\infty} \lim_{M\to\infty} M^N \quad (23)$$

The subset structure of even integers

The subsets of the set *E* can be defined from the subsets *O*(n) of odd integers in two ways. The first method is simply adding 1 to every element in *O*(n), and the result subset is called *EO*(n).

$$EO(n) = \{EO(n,k) |\, E(n,k) = O(n,k) + 1 \text{ and } O(n,k) \in O(n)\} \quad (24)$$

As the examples,

EO(0) = {2};
EO(1) = $\{p_k+1 \mid p_k \in P\} = \{4, 6, 8, 12, 14, \cdots\}$
EO(2) = $\{p_i p_j+1 \mid p_i \le p_j \in P\}$.

Similarly, there hold the following three equations

$$E = \bigcup_{n=0}^{\infty} EO(n) \quad (25)$$

$$EO(m) \cap EO(n) = \begin{cases} \emptyset & if\ m \neq n \\ EO(m) & if\ m = n \end{cases} \quad (26)$$

$$|E| = \sum_{n=0}^{\infty} |EO(n)| \quad (27)$$

If | P | = N, then

$$|EO(0)| = 1;$$

$$|EO(1)| = N;$$

$$|EO(2)| = \frac{N(N+1)}{2}; \quad (28)$$

$$|EO(3)| = \frac{N(N+1)(N+2)}{6};$$

$$|EO(n)| = \frac{(N-1+n)!}{(N-1)!\,n!} (n = 4, 5, \cdots)$$

$$|EO(0)| < |EO(1)| < |EO(2)| < \cdots < |EO(n)| < |EO(n+1)| < \cdots$$

$$|E| = \lim_{M\to\infty} \sum_{n=0}^{M} |EO(n)| = \lim_{M\to\infty} \frac{(N+M)!}{N!\,M!} \approx \lim_{M\to\infty} M^N \quad (29)$$

If $|P| = N \to \infty$, then

$$|EO(0)| = 1;$$

$$|EO(1)| \approx \lim_{N\to\infty} N;$$

$$|EO(2)| \approx \lim_{N\to\infty} N^2; \quad (30)$$

$$|EO(3)| \approx \lim_{N\to\infty} N^3;$$

$$|EO(n)| \approx \lim_{N\to\infty} N^n (n = 4, 5, \cdots)$$

$$|EO(0)| \ll |EO(1)| \ll |EO(2)| \ll |EO(3)| \ll \cdots \ll |EO(n)| \ll |EO(n+1)| \ll \cdots \quad (31)$$

$$|E| = \lim_{M\to\infty} \sum_{n=0}^{M} |EO(n)| \approx \lim_{N\to\infty} \lim_{M\to\infty} M^N \quad (32)$$

The second method is as defined in equation (33). Every even number in E(n) is factorized by n factors consisting of 2 and odd primes.

$$E(n) = \bigcup_{m=0}^{n} 2^{n-m+1} O(m) \tag{33}$$

For example, $E(0) = \{2\}$; $E(1) = \{4, 2P\} = \{4, 6, 10, 14, 22, 26, \cdots\}$ and $E(2) = \{8, 4O(1), 2O(2)\}$.

$$E = \bigcup_{n=0}^{\infty} E(n) \tag{34}$$

$$E(m) \cap E(n) = \begin{cases} \emptyset & if\ m \neq n \\ E(m) & if\ m = n \end{cases} \tag{35}$$

$$|E| = \sum_{n=0}^{\infty} |E(n)| \tag{36}$$

If $|\,P\,| = N$, then

$$\begin{aligned} &|E(0)| = 1; \\ &|E(1)| = N + 1; \\ &|E(2)| = \frac{(N+1)(N+2)}{2}; \\ &|E(3)| = \frac{(N+1)(N+2)(N+3)}{6}; \\ &|E(n)| = \frac{(N+n)!}{(N)!\,n!} (n = 4, 5, \cdots) \end{aligned} \tag{37}$$

$$|E(0)| < |E(1)| < |E(2)| < \cdots < |E(n)| < |E(n+1)| < \cdots$$

$$|E| = \lim_{M\to\infty} \sum_{n=0}^{M} |E(n)| = \lim_{M\to\infty} \frac{(N+1+M)!}{(N+1)!\,M!} \approx \lim_{M\to\infty} M^{N+1} \tag{38}$$

If $|\,P\,| = N \to \infty$, then

$$|E(0)| = 1;$$

$$|E(1)| \approx \lim_{N\to\infty} N;$$

$$|E(2)| \approx \lim_{N\to\infty} N^2; \quad (39)$$

$$|E(3)| \approx \lim_{N\to\infty} N^3;$$

$$|E(n)| \approx \lim_{N\to\infty} N^n \ (n = 4, 5, \cdots)$$

$$|E(0)| \ll |E(1)| \ll |E(2)| \ll |E(3)| \ll \cdots \ll |E(n)| \ll |E(n+1)| \ll \cdots \quad (40)$$

$$|E| = \lim_{M\to\infty} \sum_{n=0}^{M} |E(n)| \approx \lim_{N\to\infty} \lim_{M\to\infty} M^N \quad (41)$$

Discussions

[1] If |*P*| = N is finite, the values of |*E*|] obtained respectively from equations (29) and (38) are different

$$\frac{\lim_{M\to\infty} M^N}{\lim_{M\to\infty} M^{N+1}} = \lim_{M\to\infty} \frac{M^N}{M^{N+1}} = \lim_{M\to\infty} \frac{1}{M} = 0$$

If |*P*| = N is infinite, the equations (29) and (38) are equivalent as shown in the equations (32) and (41) respectively. Therefore, there is infinite number of primes.

[2] From the equations (11), (13), (30), (32), (39) and (41), it can be concluded that

$$\begin{cases} |E(1+1)| \ll |EO(n)| \approx |E(n)|(n = 3, 4, 5, \cdots) \ll |E| \\ |E(1+1)| \ll or \approx \sum_{k=3}^{\infty} P_{2k}(1,1) \approx |EO(2)| \approx |E(2)| \end{cases} \quad (42)$$

[3] Let a represent a finite integer and *O*(a) a subset of odd integers, then one has

$$O(a) = \bigcup_{n=0}^{a} O(n) \quad (43)$$

$$|O(a)| = \lim_{N\to\infty} \sum_{n=0}^{a} |O(n)| = \lim_{N\to\infty} \frac{(N+a)!}{N!\,a!} \approx \lim_{N\to\infty} N^a \quad (44)$$

For any finite integers b ≥ a, subset E(a, b) of even integers can be defined by

$$E(a,b) = \{O(a,i) + O(b,j) \mid O(a,i) \le O(b,j), O(a,i) \in O(a)\ and\ O(b,j) \in O(b)\} \tag{45}$$

$$|E(a,b)| = |O(a)| \times \left(|O(b)| - \frac{|O(a)| - 1}{2}\right) \approx \lim_{N\to\infty} N^{a+b} \tag{46}$$

From the equations (12), (32), (41) and (46), it can be obtained that

$$\begin{cases} |E(a,b)| \ll |E|, if\ a\ and\ b\ are\ finite \\ |E(a,b)| \approx |E|, if\ a \to \infty\ or\ b \to \infty\ or\ both\ a\ and\ b \to \infty \end{cases} \tag{47}$$

[4] A subset $O(1, 1, 1)$ can be generated from P

$$O(1,1,1) = \{p_i + p_j + p_k \mid p_i \le p_j \le p_k and\ p_i, p_j, p_k \in P\} \tag{48}$$

$$|O(1,1,1)| = \lim_{N\to\infty} \frac{N(N+1)(N+2)}{6} \approx \lim_{N\to\infty} N^3 \tag{49}$$

By comparing between the equations (23) and (49), it is easy to conclude

$$|O(1,1,1)| \ll O(n)|\ (n = 4, 5, \cdots) \ll |O| \tag{50}$$

Generally, for any finite integers a≤b≤c,

$$O(a,b,c) = \{O(a,i) + O(b,j) + O(c,k) \mid O(a,i) \le O(b,j) \le O(c,k), O(a,i) \in O(a), O(b,j) \in O(b)\ and\ O(c,k) \in O(c)\} \tag{51}$$

$$|O(a,b,c)| \approx \lim_{N\to\infty} N^{a+b+c} \ll |O| \tag{52}$$

References:

[1] J. R. Chen, "*On the Representation of a Large Even Integer as the Sum of a Prime and the Product of at Most Two Primes*", Kexue Tongbao, 17, 385 (1966)

[2] K. Ding, "*Only a subset of all even integers in which every even integer can be expressed as the sum of two primes*", (non-peer-reviewed), DOI: 10.5281/zenodo.4552678, (2021)