On the relationship between Singularity Exponents and Finite Size Lyapunov Exponents in remote sensed images of the ocean

Lluïsa Puig Moner lluisapuigmoner@gmail.com

Estrella Olmedo Antonio Turiel

GHRSST Talks
20th April 2023, online

Institut de Ciències del Mar

Universitat Autònoma de Barcelona

Detection of transport barriers and fronts in the ocean (turbulent flow)

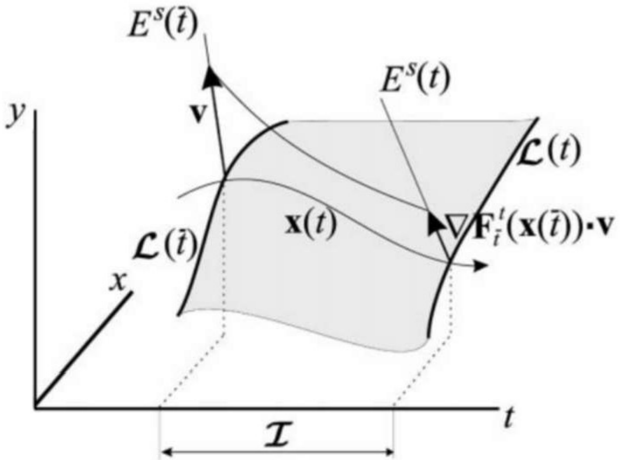
Finite Lyapunov Exponents	Singularity Analysis
Lagrangian analysis	**Eulerian analysis**
Exponential separation ←	Irregularity / diferenciability
Lagrangian Coherent Structures	Multifractal structure
Need to know the velocity field	Independent of the velocity field
	From remote sensed images of ADVECTED flow.
Units of time^{-1}	Dimensionless

Lagrangian structures detection via FTLE

$$\boldsymbol{F}_{\boldsymbol{t_0}}^{\boldsymbol{t}} : \boldsymbol{x_0} \rightarrow \boldsymbol{x}(t_0 + t, t_0, \boldsymbol{x_0})$$

Lagrangian coherent structure

$$\mathcal{E}_{t_0}^{t}(\boldsymbol{x_0}) = \max_{|\boldsymbol{e}|=1} |(\boldsymbol{\nabla}_{\boldsymbol{x_0}} \boldsymbol{F}_{\boldsymbol{t_0}}^{\boldsymbol{t}}) \boldsymbol{e}_{\boldsymbol{t_0}}| \equiv \|(\boldsymbol{\nabla}_{\boldsymbol{x}_0} \boldsymbol{F}_{\boldsymbol{t_0}}^{\boldsymbol{t}})\| = \sqrt{\lambda_{max}\left((\boldsymbol{\nabla}_{\boldsymbol{x_0}} \boldsymbol{F}_{\boldsymbol{t_0}}^{\boldsymbol{t}})^T (\boldsymbol{\nabla}_{\boldsymbol{x_0}} \boldsymbol{F}_{\boldsymbol{t_0}}^{\boldsymbol{t}})\right)}$$



$$\Lambda(t, t_0, \boldsymbol{x_0}) = \frac{1}{2(t - t_0)} log_e\left(\lambda_{max}\left((\boldsymbol{\nabla}_{\boldsymbol{x_0}} \boldsymbol{F}_{\boldsymbol{t_0}}^{\boldsymbol{t}})^T (\boldsymbol{\nabla}_{\boldsymbol{x_0}} \boldsymbol{F}_{\boldsymbol{t_0}}^{\boldsymbol{t}})\right)\right)$$

$$\mathcal{E}_{t_0}^{t}(\boldsymbol{x_0}) = e^{\Lambda(t, t_0, \boldsymbol{x_0})(t - t_0)}$$

$$\dot{\boldsymbol{x}}(t) = \Lambda \boldsymbol{x}(t)$$

$$\boldsymbol{x}(t) = \boldsymbol{x}(t_0) e^{\Lambda(t - t_0)} = \boldsymbol{x}(t_0) \mathcal{E}_{t_0}^{t}(\boldsymbol{x_0})$$

FSLE

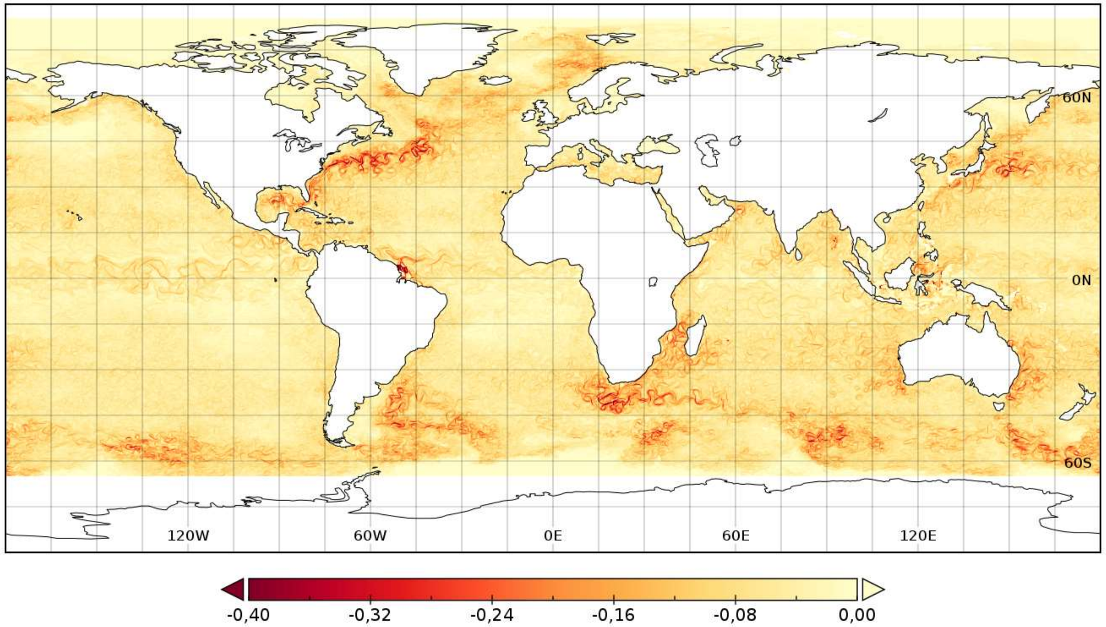
Finite-size Lyapunov Exponents (FSLE)

$$\boldsymbol{\lambda}(\boldsymbol{x_0}, t_0, \boldsymbol{\delta_0}, \delta_f) = \frac{1}{\tau} log\left(\frac{\delta_f}{|\boldsymbol{\delta_0}|}\right)\frac{\boldsymbol{\delta_0}}{|\boldsymbol{\delta_0}|}$$

- Backward in time

↓

- Negative exponents
- Transport barriers



SOURCE: *AVISO+ [4]*
<<Backward-in-time, Finite Size Lyapunov Exponents and orientations of associated eigenvector>>
Units of day−1
Results for January 25th, 2022
0.25º

Relation between FSLE and FTLE

Finite Time Lyapunov exponents (FTLE)

$$\Lambda(t, t_0, \boldsymbol{x_0}) =$$

$$= \frac{1}{2(t-t_0)} log_e\left(\lambda_{max}\left((\boldsymbol{\nabla}_{\boldsymbol{x_0}} \boldsymbol{F}^{\boldsymbol{t}}_{\boldsymbol{t_0}})^T (\boldsymbol{\nabla}_{\boldsymbol{x_0}} \boldsymbol{F}^{\boldsymbol{t}}_{\boldsymbol{t_0}})\right)\right)$$

Fixing t to find r

Finite Size Lyapunov Exponents (FSLE)

$$\Pi(r, t_0, \boldsymbol{x_0}) = \frac{log_e(r)}{2(t-t_0)}$$

$$r = \lambda_{max}\left((\boldsymbol{\nabla}_{\boldsymbol{x_0}} \boldsymbol{F}^{\boldsymbol{t}}_{\boldsymbol{t_0}})^T (\boldsymbol{\nabla}_{\boldsymbol{x_0}} \boldsymbol{F}^{\boldsymbol{t}}_{\boldsymbol{t_0}})\right)$$

Fixing r to find t

$$\boldsymbol{\lambda}(\boldsymbol{x_0}, t_0, \boldsymbol{\delta_0}, \delta_f) = \frac{1}{\tau} log\left(\frac{\delta_f}{|\boldsymbol{\delta_0}|}\right) \frac{\boldsymbol{\delta_0}}{|\boldsymbol{\delta_0}|}$$

Singularity exponents

The ocean, a turbulent environment with multifractal structure, Leonardo da Vinci

$$s(\boldsymbol{x}+\boldsymbol{r}) - s(\boldsymbol{x}) = \alpha(\boldsymbol{x}) r^{h(\boldsymbol{x})} + \mathcal{O}(r^{h(\boldsymbol{x})}), \qquad r \to 0$$

Fractal components

$$F_h = \{\boldsymbol{x} \ \textit{such that} \ h(\boldsymbol{x}) = h\}$$

Computation of the singularity exponents

"Equation of the singularity exponents"

$$s(\boldsymbol{x}+\boldsymbol{r}) - s(\boldsymbol{x}) = \alpha(\boldsymbol{x}) r^{H(\boldsymbol{x})} + \mathcal{O}(r^{H(\boldsymbol{x})}), \qquad r \to 0$$

Wavelet transform

$$\mathcal{T}_\psi s(\boldsymbol{x}, r) = \int ds(\boldsymbol{y}) \frac{1}{r^d} \psi\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{r}\right)$$

"Singularity exponents equation for the wavelet transformation" (invariant)

$$\mathcal{T}_\psi s(\boldsymbol{x}, r) = \alpha_\psi(\boldsymbol{x}) r^{H(\boldsymbol{x})} + \mathcal{O}(r^{H(\boldsymbol{x})}), \qquad r \to 0$$

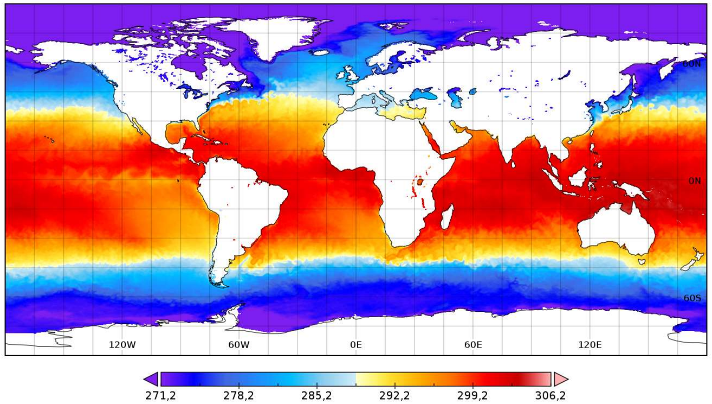
"Singularity exponents equation for the gradient"

$$\mathcal{T}_\psi \boldsymbol{\nabla} s(\boldsymbol{x}, r) = \alpha_\psi(\boldsymbol{x}) r^{h(\boldsymbol{x})} + \mathcal{O}(r^{h(\boldsymbol{x})}), \qquad r \to 0$$

$$\boxed{h(\boldsymbol{x}) = H(\boldsymbol{x}) - 1}$$

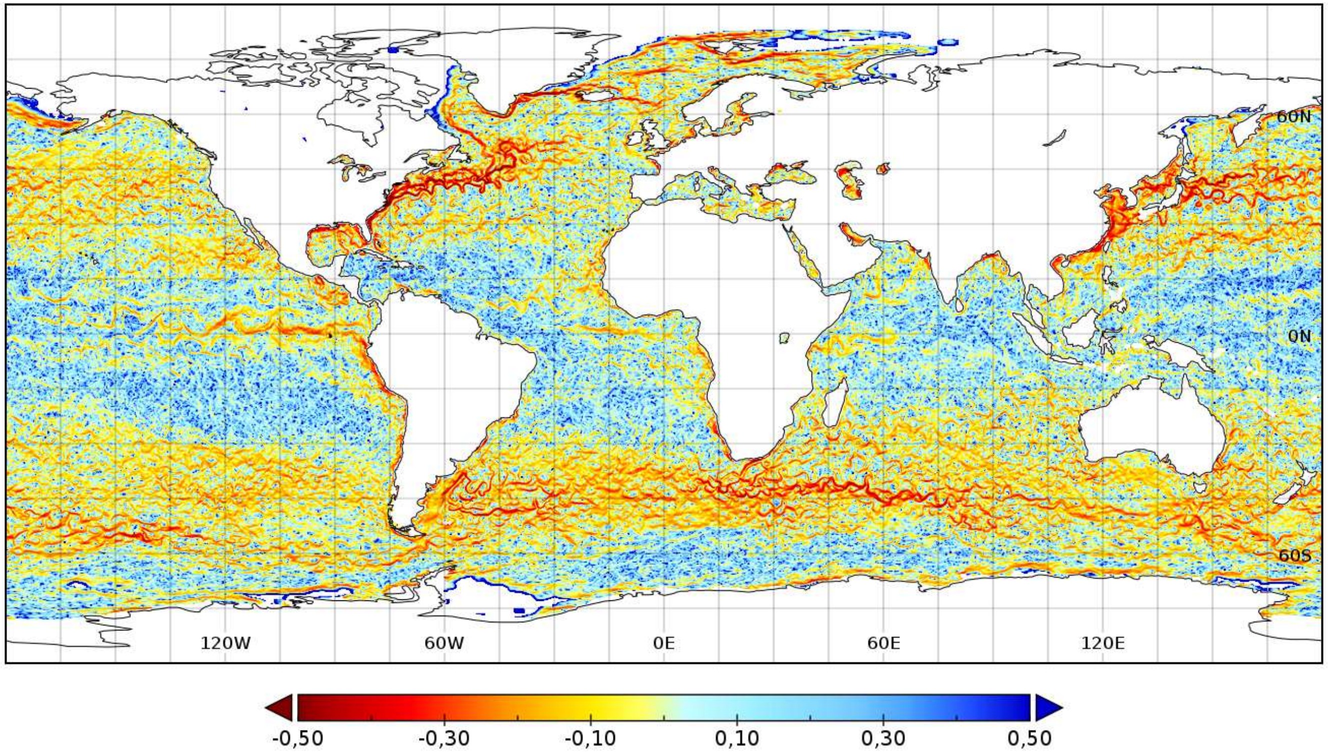
Singularity exponents from SST and ADT

SST



SOURCE: *SST global images from [5] Global Ocean OSTIA Sea Surface Temperature and Sea Ice Analysis Results for January 25th, 2022. 0.25º spatial resolution*

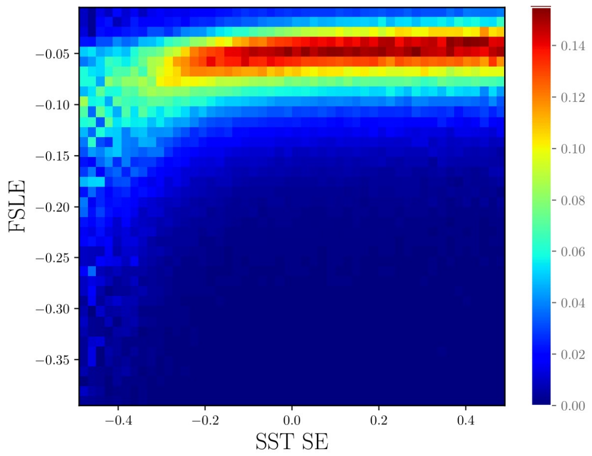
SST SE

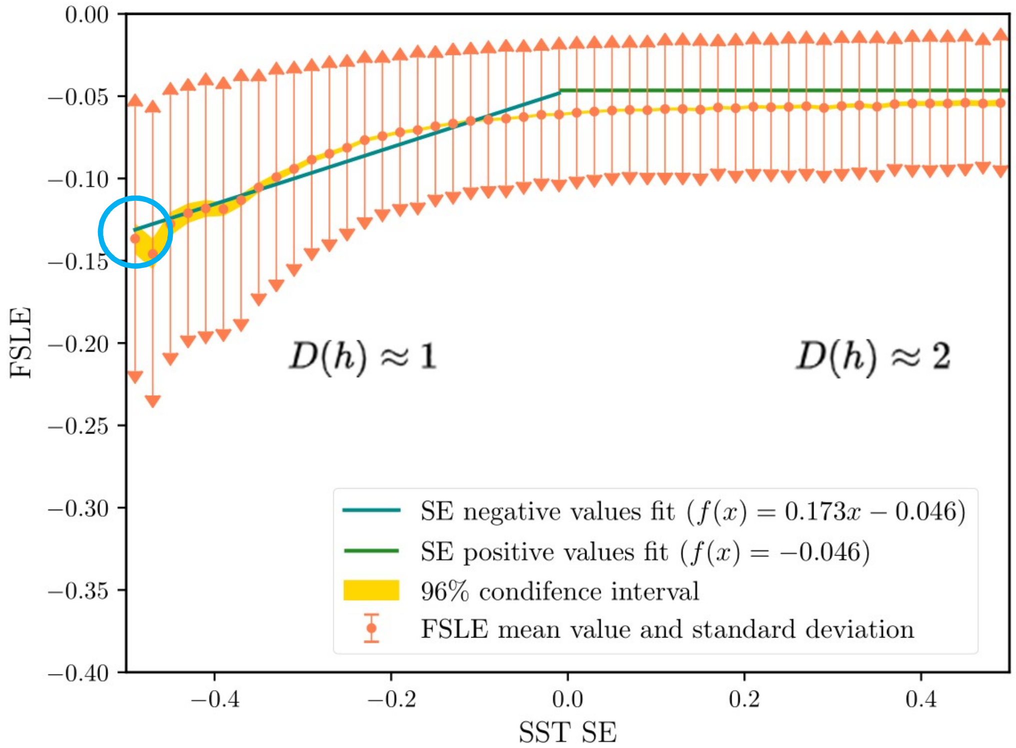


The curvilinear shape of the negative fractal components ->
Fractal dimension: $D(h) \approx 1$

Positive fractal components with unclear shape ->
Maxim fractal dimension: $D(h) \approx 2$

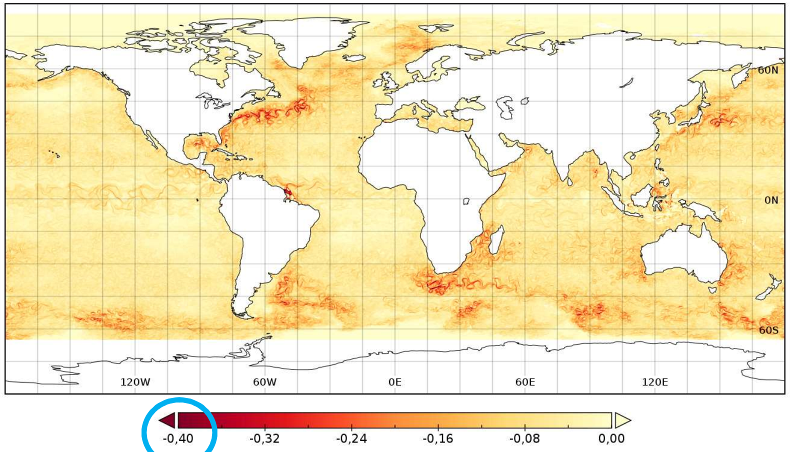
Functional relation FSLE i SST SE (Global)



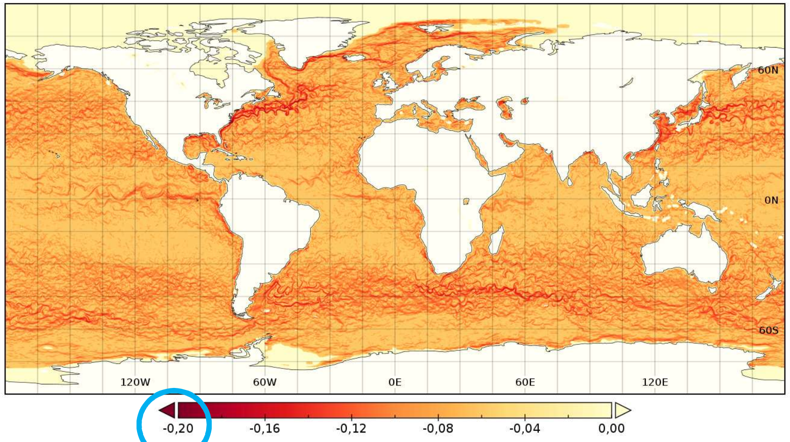


Reconstruction of FSLE via SST SE (Global)

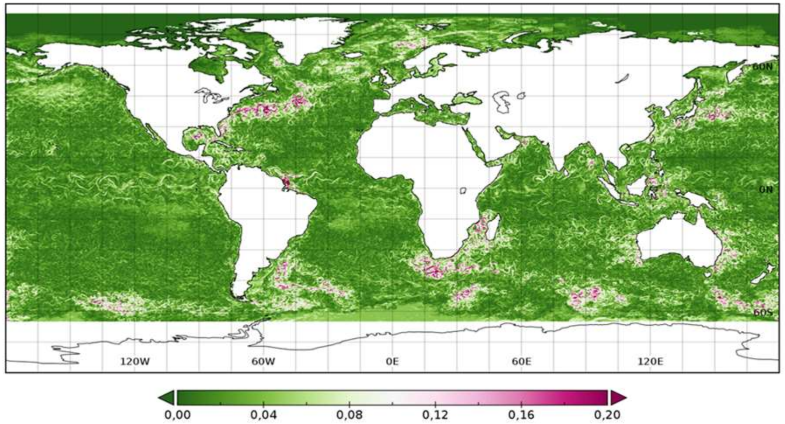
FSLE



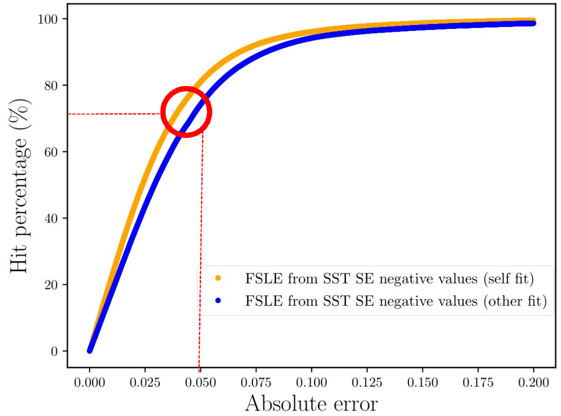
Reconstructed FSLE via SST SE



Absolut error of the reconstructed map



70% of the pixels with an accuracy of 0.05



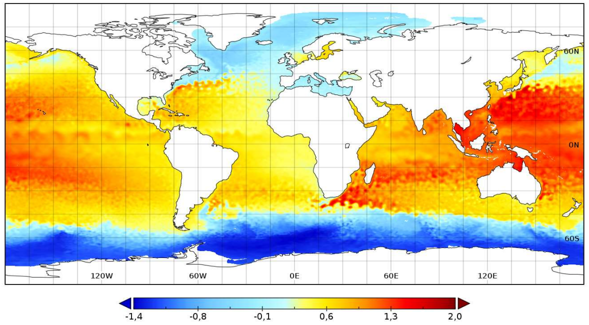
Institut de Ciències del Mar

Universitat Autònoma de Barcelona

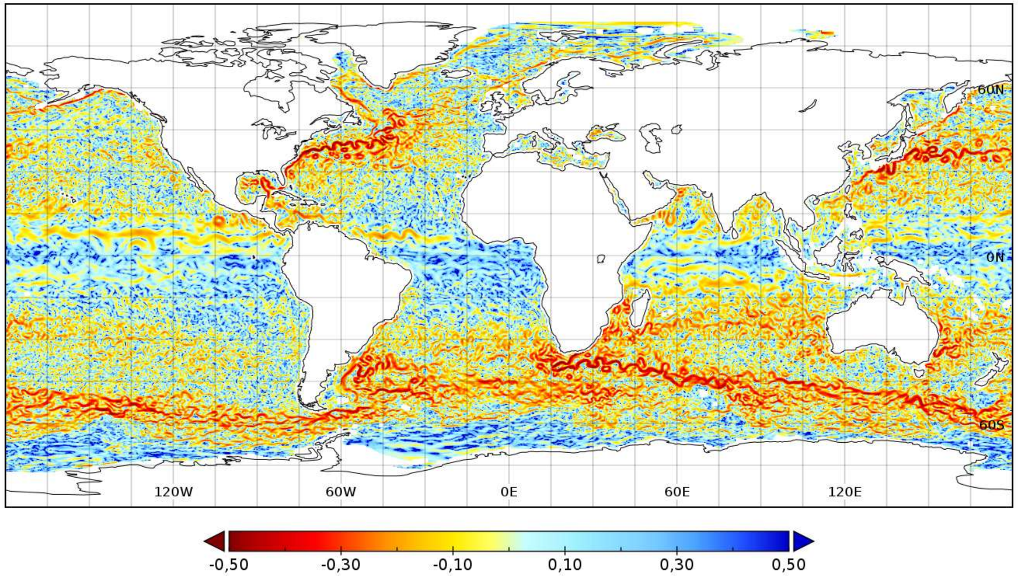
Singularity exponents from ADT

SOURCE: *ADT global images from [5]. NRT merged all satellites Global Ocean Gridded SSALTO/DUACS Sea Surface Height L4 product and derived variables.*
Results for January 25th, 2022.
0.25º scale resolution

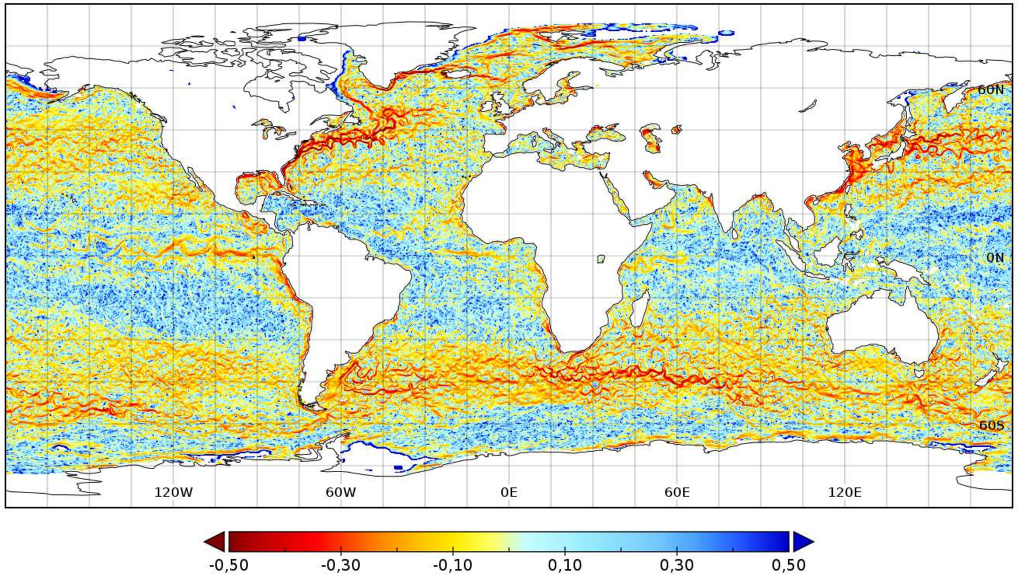
ADT



ADT SE

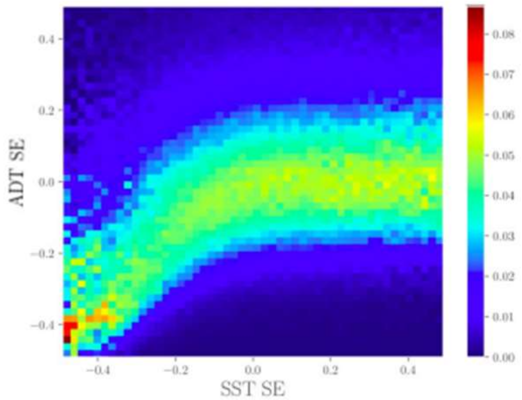


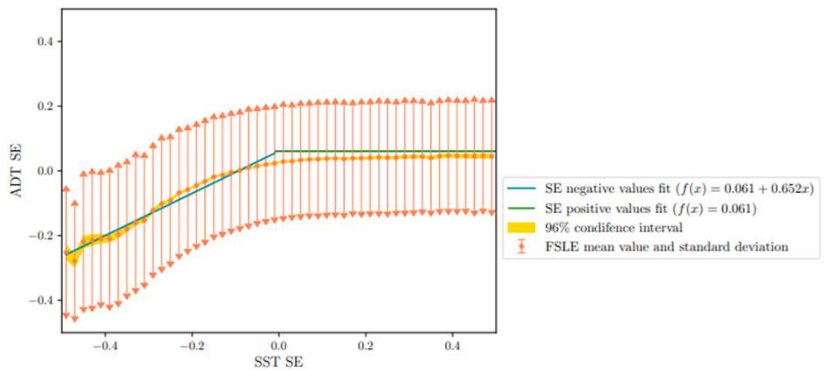
SST SE



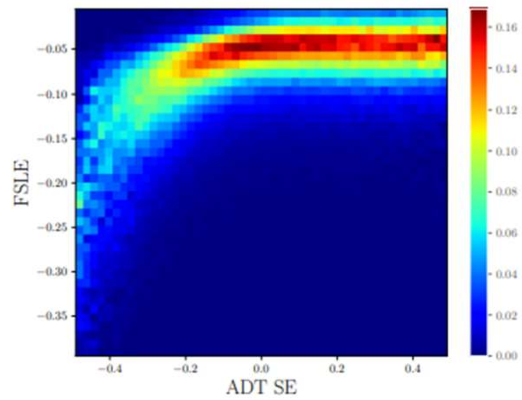
Altimetry SE (Global analysis) 11

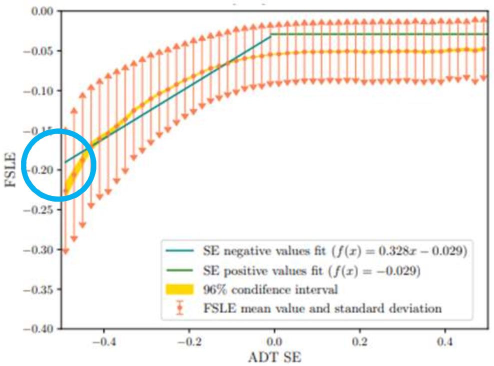
ADT SE vs SST SE





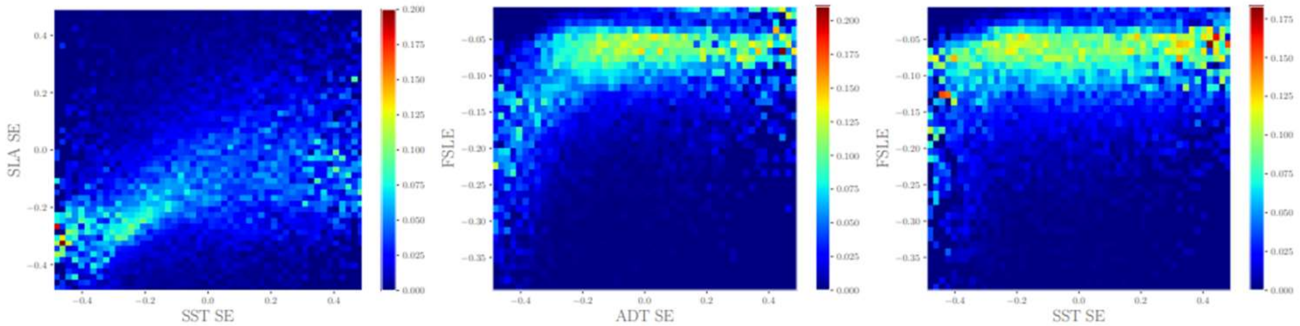
FSLE vs ADT SE



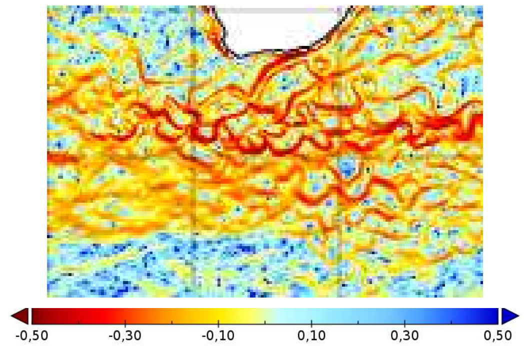


South Africa – Angulhas current [0E, 45E E] lon, [60S, 30S] lat

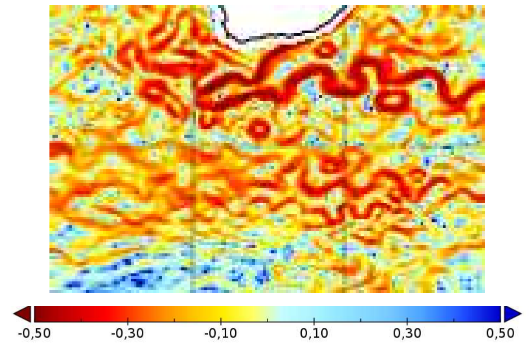
(a) SST SE and ADT SE (b) ADT SE and FSLE (c) SST SE and FSLE



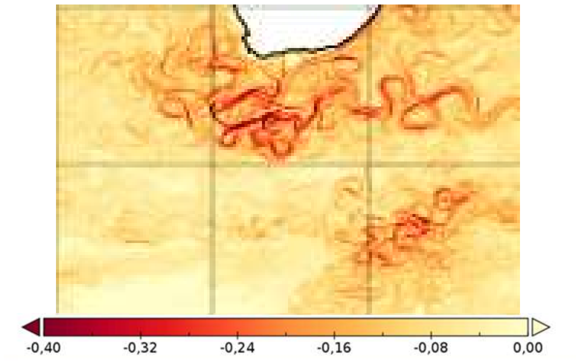
SST SE



ADT SE



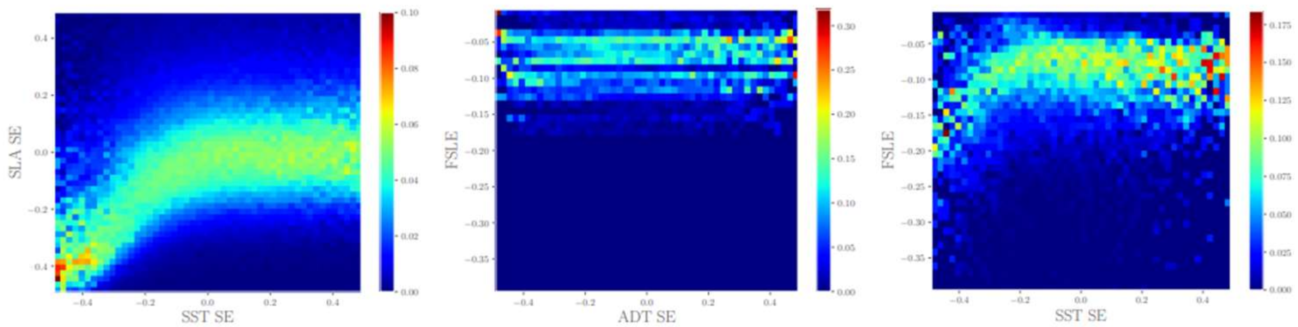
FSLE



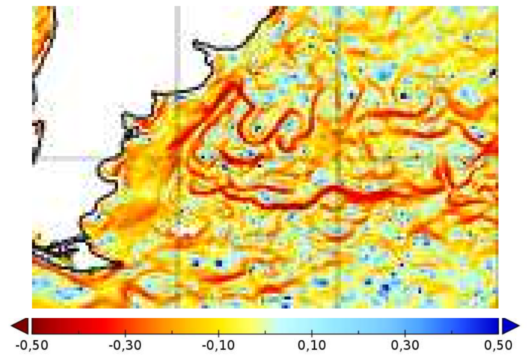
Regional analysis

South America – Brazil-Malvinas Confluence (BMC) [75W, 30W] lon, [60S, 30S] lat

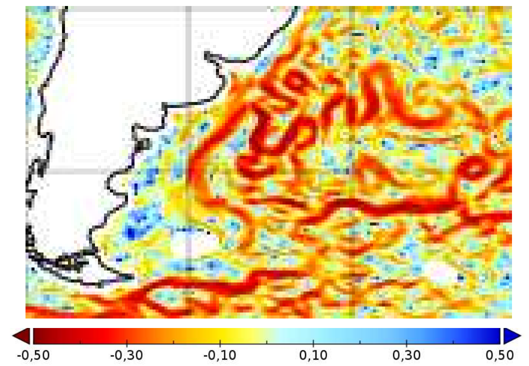
(a) SST SE and ADT SE (b) ADT SE and FSLE (c) SST SE and FSLE



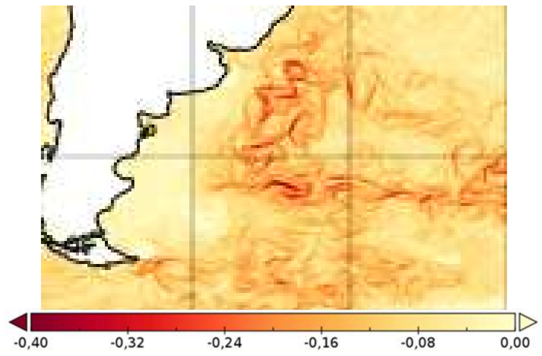
SST SE



ADT SE



FSLE



Further work

- Use of extended temporal series of satellite images
- Combination of different advected variables (SST, ADT, SSS) to produce singularity exponents
- Establishment of a finer functional relation (higher order polynomial, DL algorithm, …)
- Geometrical study on the coincidence of the tracks or streams
- Consideration of repealing LCS also

- Temporal stability of the relationship (using correlation coefficient)
- More robust relationship

- **Is the equivalence also holding in the areas with fractal dimension 2?**
- **Should we consider both attractive and repulsive LCS?**

References

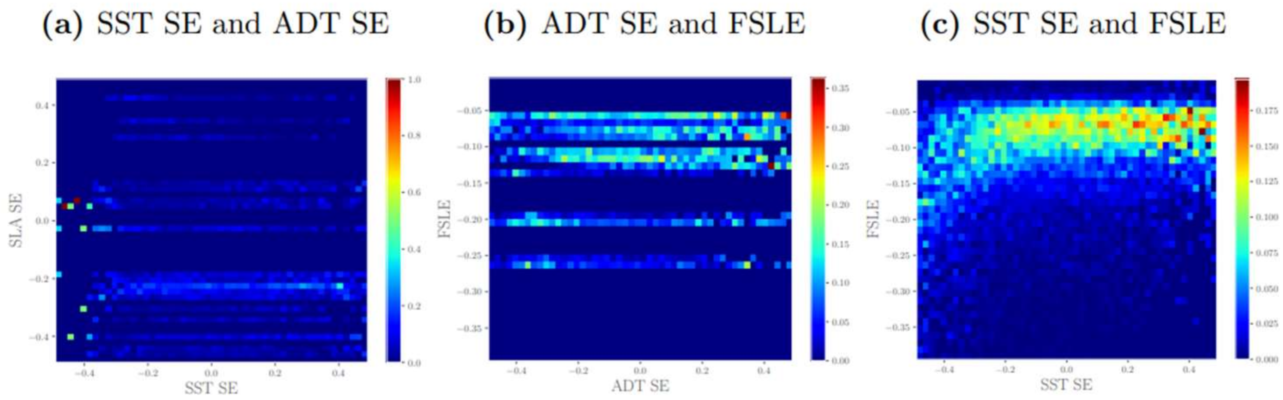
[1]: D'Ovidio F., C. Lopez, E. Hernandez-Garcia, V. Fernandez, 2004, "Mixing structures in the Mediterranean sea from Finite-Size Lyapunov Exponents", Geophys. Res. Lett., 31, L17203.
[2]: Pujol M.-I., Y. Faugere, O. Titaud, F.Briol, F. d'Ovidio, R. Morrow,E. Bronner, N. Picot, "20 years of reprocessed Lyapunov Exponents from altimetry available on AVISO+", Poster EGU2015-4542, EGU12 17 April 2015, Vienna Australia
[3]: Oriol Pont, Antonio Turiel & Hussein Yahia (2013) Singularity analysis of digital signals through the evaluation of their unpredictable point manifold, International Journal of Computer Mathematics, 90:8, 1693-1707, DOI: 10.1080/00207160.2012.748895.
[4]: FSLE description: Aviso+. (n.d.). AVISO+. Retrieved March 22, 2022, from https://www.aviso.altimetry.fr/en/data/products/value-added-products/fsle-finite-size-lyapunov-exponents/fsle-description.html
[5]: Data | Copernicus Marine. (n.d.). Copernicus.Eu. Retrieved March 22, 2022, from https://resources.marine.copernicus.eu/product-detail/SST_GLO_SST_L4_NRT_OBSERVATIONS_010_001/INFORMATION
[6]:J. Isern-Fontanet, X. Capet, A. Turiel, E. Olmedo, C. González-Haro, 2022, "On the Seasonal Cycle of the Statistical Properties of Sea Surface Temperature", Geophys. Res. , 49, e2022GL0980

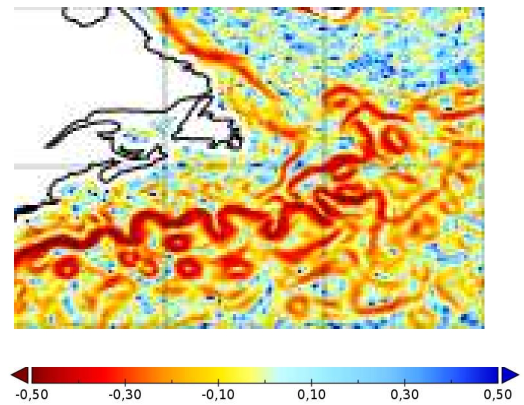
A summary of the presentation can be found in poster format in [Link to the poster]

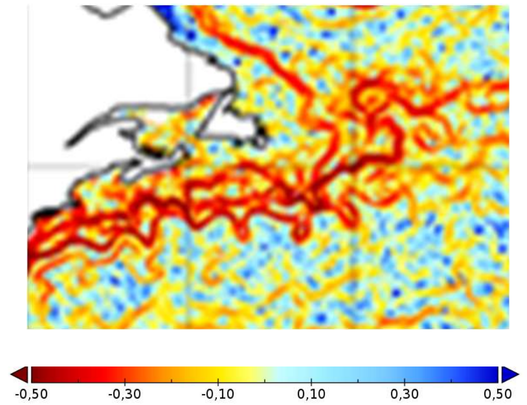
Institut de Ciències del Mar
Universitat Autònoma de Barcelona

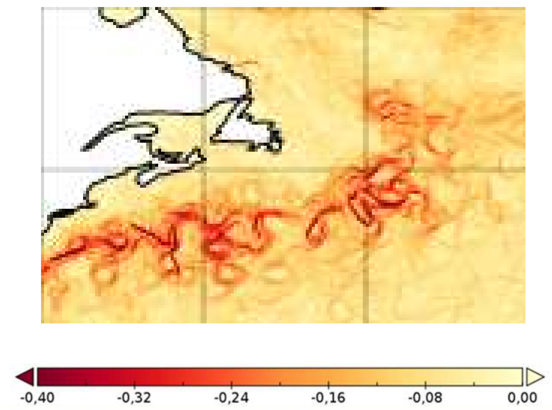
Regional analysis

Gulf Stream [75W, 30W] lon, [30N, 60N] lat









Singularity exponents

The ocean, a turbulent environment with multifractal structure, Leonardo da Vinci

$$s(\boldsymbol{x}+\boldsymbol{r}) - s(\boldsymbol{x}) = \alpha(\boldsymbol{x}) r^{h(\boldsymbol{x})} + \mathcal{O}(r^{h(\boldsymbol{x})}), \qquad r \to 0$$

Fractal components

$$F_h = \{\boldsymbol{x} \ \textit{such that} \ h(\boldsymbol{x}) = h\}$$

Most Singular Component (MSC)

$$F_\infty = \{\boldsymbol{x} \ \textit{such that} \ h(\boldsymbol{x}) \in \ ]h_\infty - \Delta, h_\infty + \Delta[\ \ \} \qquad h_\infty \equiv$$ Smallest exponent

Data sources

- SST global images from [5] Global Ocean OSTIA Sea Surface Temperature and Sea Ice Analysis
  Results for January 25th, 2022.
- ADT global images from [5]. NRT merged all satellites Global Ocean Gridded SSALTO/DUACS
  Sea Surface Height L4 product and derived variables.
  Results for January 25th, 2022.
- FSLE used in this project are <<Backward-in-time, Finite Size Lyapunov Exponents and orientations of associated eigenvector>> provided by AVISO+ (Archiving, Validation and Interpretation of Satellite Oceanographic data) [4]
  Units of day−1
  Results for January 25th, 2022
  0.25º (720X1440)