

Inconsistency of all formal systems that include \mathbb{N} from a not-finitist point of view

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Abstract

Considering the set of natural numbers \mathbb{N} , then in the context of Peano axioms, we find a fundamental contradiction from a not-finitist point of view.

1 Introduction

A formal system, constituted of a grammar, inference rules, axioms, and a reference set, can produce formalized propositions and deductions (theorems) through with a finite number of steps, that is a finitist approach [5] [10].

A system is consistent whether a proposition and its negation are not deduced. Godel's incompleteness theorems [4], developed on the basis of the system of Principia Mathematica including the axiom of infinity, represent a fortress of logic and consistency against inconsistency. But at the same time they represent a prelude of inconsistency. They give us necessary conditions of consistency, not sufficient ones (undecidable propositions and not demonstrable internal consistency are these necessary conditions).

2 The set of natural numbers

The existence of \mathbb{N} is granted by the axiom of infinity [11] [8] [9]. This existence imply that one of each element of the set, in an actual sense, not time dependent, so taken all together. Then the problem about the existence of natural numbers not yet effectively obtained (very large numbers) is avoided; in fact we could consider only effectively obtained numbers as natural numbers, but because the successor function $S(x)$ exists, also others not-obtained numbers have to exist. All numbers of the set are defined by Peano axioms [7] [6] [3], together their proprieties thanks to the axiom of induction. In particular we focus attention on successor function $S(x)$.

Nevertheless the axiom of infinity, usually natural numbers are thought one by one over time, specially implicitly. Probably this is a reason why the following contradiction had not yet been found. But a list of infinite elements

(nevertheless its existence?) is not contemplated by a finitist approach to obtain a deduction.

3 A fundamental contradiction

Considering the inductive axiom we simply get: $(x \in \mathbb{N})(\forall x(x < S(x)))$ with $S(x) \in \mathbb{N}$, that is:

$$0 < S(0)$$

$$0 < 1 < S(1)$$

$$0 < 1 < 2 < S(2)$$

.

.

.

$$0 < 1 < 2 < \dots < x < S(x) \quad \forall x$$

So we now imagine that all natural numbers x , to the left of " $S(x)$ " in the previous scheme, are used (it has to be true considering $\forall x$) and we have: $(x, y \in \mathbb{N})(\exists y \forall x(x < S(y)))$, that is, there is a greater number ($S(y)$) than all natural numbers, in contradiction with $S(y) \in \mathbb{N}$ (included in Peano axioms). $\forall y(\forall x((x \leq y) \rightarrow (x < S(y))) \rightarrow \exists y \forall x(x < S(y)))$ represents the previous scheme together the implication in symbols.

More explicitly, since the two sets are the same:
 $\{0, \{S(x) | x \in \mathbb{N}\}\} = \mathbb{N}$, but because

$$\sum_{x=0}^y x < \sum_{x=0}^y S(x) \quad \forall y \quad (\text{with} \quad \sum_{x=0}^y S(x) = 0 + \sum_{x=0}^y S(x)),$$

although there are infinite sums, there is the sum of all natural numbers (otherwise we will not have all x and then all y , in agreement with $\forall y$) and we find that the sum of all elements of a set is less than the other; in a set there are elements that are not in the other. But the set is the same, \mathbb{N} . Then there are fewer natural numbers than natural numbers, that is a contradiction.

We have another demonstration. We know that $\nexists y \forall x(x \leq y)$ is true. From this, considering all x and all y and the principle of the excluded third, we obtain $\exists x \forall y(y < x)$. It is very simple to understand this considering an oriented line with x, y marked points; transposing $\nexists y \forall x(x \leq y)$ on the line we see $\exists x \forall y(y < x)$.

$$\text{-----}x_1y_1\text{-----}x_2y_2\text{-----}x_my_m\text{-----}x_n\text{-----} >$$

So $\exists x \forall y(y < x)$, but because $x, y \in \mathbb{N}$, a natural number x cannot be greater than all natural numbers. So we have a contradiction.

These proofs seems to look like to Burali Forti antinomy [2] [1]. But we have only considered the \mathbb{N} set with its elements, while that one is centered on the "set" (actually named a class) of all ordinal numbers.

4 Conclusions

These proofs of inconsistency are not finitist because they need infinite totalities. But can we really summarize the finitist point of view as follows? "A list of infinity elements cannot be considered for deducing, although it has to exist in agreement with the axiom of infinity". Is this not an acceptable point of view and then are these not finitist proofs of inconsistency valid in a general way?

Anyway a view to avoid this inconsistency could be to consider finite sets or not extending $S(x)$ to all x (so it is possible that infinity doesn't exist), then revisiting Peano axioms.

References

- [1] Richard TW Arthur. Leibniz in cantor's paradise: A dialogue on the actual infinite. *Leibniz and the Structure of Sciences: Modern Perspectives on the History of Logic, Mathematics, Epistemology*, pages 71–109, 2019.
- [2] Cesare Burali-Forti. Una questione sui numeri transfiniti. *Rendiconti del Circolo Matematico di Palermo (1884-1940)*, 11(1):154–164, 1897.
- [3] Francesco Ciraulo. Elementi di logica matematica.
- [4] Kurt Gödel. On formally undecidable propositions of principia mathematica and related systems i 1 (1931). In *Gödel's Theorem in Focus*, pages 17–47. Routledge, 2012.
- [5] Jacques Herbrand. Sur la théorie de la démonstration. *Cambridge*, 12, 1971.
- [6] Yiannis Moschovakis. The natural numbers. *Notes on Set Theory*, pages 51–70, 2006.
- [7] Giuseppe Peano. *Arithmetices principia: Nova methodo exposita*. Fratres Bocca, 1889.
- [8] Jerzy Pogonowski. "mathematics is the logic of the infinite": Zermelo's project of infinitary logic. *Studies in Logic, Grammar and Rhetoric*, 66(3):673–708, 2021.
- [9] Bertrand Russell. *Introduction to mathematical philosophy*. Taylor & Francis, 2022.
- [10] Richard Zach. Numbers and functions in hilbert's finitism. 1998.
- [11] Ernst Zermelo. Investigations in the foundations of set theory i. *From Frege to Gödel*, pages 199–215, 1908.