Gamma Function derived from Factorial Function based-Pi Function

Chinnaraji Annamalai
School of Management, Indian Institute of Technology, Kharagpur, India
Email: anna@iitkgp.ac.in
https://orcid.org/0000-0002-0992-2584

Abstract: Several professors of mathematics from the renowned universities in Australia, Canada, Europe, India, USA, etc. argue with me that the gamma function is not related to the factorial function. For them, this paper describes the derivation of gamma function from the factorial function.

MSC Classification codes: 05A10, 33B15

Keywords: computation, integral, real number

Euler's Factorial Function and Gamma Function

Euler's factorial function, generally known as Pi ($\mathbf{\Pi}$) function, is the basis for gamma function. The pi function is given below:

$$\mathbf{\Pi}(x) = x! = x(x-1)(x-2)\cdots 1. \qquad (1)$$

Swiss mathematician Leonhard Euler was defined the pi function by the integral as follows:

$$\mathbf{\Pi}(x) = \int_0^1 (-\ln t)^n \, dt, \forall\, n \in \boldsymbol{W}. \qquad (2)$$

Here, $\boldsymbol{W}$ denotes a system of whole numbers.

By substituting $t = e^{-x}$ in (2), we get

$$\mathbf{\Pi}(x) = \int_0^\infty x^n \, e^{-x} dx, \qquad \forall\, n \in W. \qquad (3)$$

By integrating (2), we obtain

$$= [-x^n e^{-x}]_0^\infty - \int_0^\infty -nx^{n-1} \, e^{-x} dx.$$

$$\mathbf{\Pi}(x) = n \int_0^\infty x^{n-1} \, e^{-x} dx. \qquad (4)$$

$$\therefore \mathbf{\Pi}(x) = n\mathbf{\Pi}(x-1). \qquad (5)$$

The gamma function $\Gamma(x)$ is obtained as follows.

$$\mathbf{\Pi}(x) = n\mathbf{\Pi}(x-1) \Longrightarrow \Gamma(x+1) = n\Gamma(x). \qquad (6)$$

$$n\Gamma(x) = n \int_0^\infty x^{n-1} \, e^{-x} dx \Longrightarrow \Gamma(x) = \int_0^\infty x^{n-1} \, e^{-x} dx. \qquad (7)$$

The gamma function, therefore, is derived from the Euler's factorial function that uses the actual factorial function.