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ON LINEAR SUBSTITUTION OF VARIABLES AND ON SIMPLIFICATION OF EXPRESSIONS IN THE SPACE OF GENERALIZED FUNCTIONS

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Annotation. The rigorous mathematical theory of generalized functions contains the definition of a set of basic functions, the definition of continuous functional, rules for performing limit transitions, delta-like sequences, etc. Special literature is devoted to these questions, in which one can find formulations and proofs of the corresponding theorems (see, for example, the book by V.S.Vladimirov “Generalized functions in mathematical physics”, I.M.Gelfand and G.E.Shilov “Generalized functions and actions on them”, M.S.Agranovich “Generalized functions and Sobolev spaces”, M.A.Shubin “Lectures on the equations of mathematical physics” and many others). In this article, we will look at techniques for substitution a variable in an argument of generalized function, multiplying by a local-integrable function, and simplifying expressions. The theory of generalized functions has been chosen as a mathematical apparatus. The technique of generalized functions provides a convenient apparatus for solving a number of classical problems of an applied nature. The proposed article will be useful for students of physics and mathematics, engineering and physics specialties of higher educational institutions, as well as master students, teachers of relevant universities, studying and interested in the practical application of the theory of the generalized.

Key words: basic functions, locally integrable functions, generalized functions, regular generalized function, $\delta -$ function, linear substitution of variables

Introduction

The importance of generalized functions in mathematics itself and its applications needs no proof. The emergence of generalized functions is associated with the names of many mathematicians and physicists. The simplest generalized function - $\delta -$ the function was used by D.Maxwell (1873), O.Heaviside (1898), P.Dirac (1926). The foundations of the mathematical theory of generalized functions were laid by S.L.Sobolev (1939) and L.Schwartz (1950). The most important results in the theory of generalized functions were obtained by K.Friedrichs (1940). Further developments in the theory of generalized functions can be found in the works of I.M.Gelfand and G.E.Shilova (1959), N.N.Bogolyubov and O.S.Parasyuk (1955-1960), V.S.Vladimirov, L.Hormander, A.Zemanian, G.Bremerman (1968-1971) and others.

In modern mathematical analysis, the main mathematical tools are Sobolev spaces and the theory of generalized Schwarz functions.

Let us present some definitions and statements regarding generalized functions, which will be used in what follows [1, 3, 6].

1. Let's introduce the important space of $D.$ basic functions. We refer to the set $D = D(\mathrm{R})$ of all possible functions $\varphi(x)$, defined on the whole number line R and having the following properties:

1) each function $\varphi(x) \in C^{\infty}(R) -$ has derivatives of all orders on the whole number line;

2) for each function $\varphi(x)$ there is an interval outside of which it is equal to zero.

We denote the set of all points of x by X_{φ}, in which $\varphi(x) \neq 0$, and the closure of the set X_{φ} by $\overline{X_{\varphi}} -$, i.e. $\overline{X_{\varphi}} -$ is the union of the set X_{φ} and all of its limit points.

The set $\overline{X_{\varphi}}$ is called the carrier of the function $\varphi(x)$ and is denoted as

$$\operatorname{supp} \varphi(x) = \overline{X}_{\varphi} = \{\overline{x : \varphi(x) \neq 0}\}.$$

The set of basic functions, the carriers of which are contained in the given area G will be denoted by $D(G)$. In this way, $D(G) \subset D(\mathrm{R}^n) = D$.

2. Convergence in D is defined as follows. The sequence of basic functions $\varphi_1, \varphi_2, .., \{\varphi_n(x)\}$ from D converges to the function $\varphi(x)$ from D, if:

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1) there is an interval $(-a, a)$, such, that $\forall n \in N$:

$$supp\varphi_n(x) \in (-a, a);$$

2) $\forall k = 0,1,2,\dots$ sequence of derivatives $\left\{\varphi_n^{(k)}(x)\right\}$ evenly in G converges to $\varphi^{(k)}(x)$.

The designation: $\varphi_n(x) \to \varphi(x)$ at $n \to \infty$ in space D.

The function $\varphi: R^n \to R$ is called basic if $\varphi(x)$ is infinitely differentiable and $supp\varphi$ is a bounded subset of G.

Let us now make it clearer that the above construction in a certain sense extends $R_{1,loc}$ – the class of locally integrable functions on the number line. This class consists of functions f that have the property

$$\forall a, b \in \boldsymbol{R}(a < b) \Rightarrow f \in R_1[a, b].$$

The inclusion $R_1(\boldsymbol{R}) \subset R_{1,loc}$ takes place.

3. The generalized function $f \in D^{/}$ corresponds to each function $f \in R_{1,loc}$ defined by the equality

$$(f, \varphi) = \int f(x)\varphi(x)dx, \ \varphi \in D. \qquad (0.1)$$

The space $D^{/}$ is wider than $R_{1,loc}$, there are generalized functions that are not functionals of the form (0.1). Thus, the enclosure $R_{1,loc} \subset D^{/}$ is carried out.

4. Let's define $\delta -$ function. Any measure can be identified with a linear continuous functional on the space of continuous functions, and note that such a functional for the function $\delta(x)$ acts by the formula

$$(\delta(x), f(x)) = f(0), \qquad (0.2)$$

where $f(x)$ is any function continuous on the interval, for example, $f(x) \in C_{[-1,1]}$. Consequently, the function $\delta(x)$ is, by definition, a functional on $C_{[-1,1]}$, contrasting the function $f(x)$ with its value at point 0.

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$\delta -$ function is a mathematical expression of the density of a unit mass concentrated at the point $x = 0$. If such a mass is concentrated at the point $x = a$, we come to $\delta -$ the function $\delta(a) = \delta(x-a)$, that is defined by the equality

$$(\delta(x-a), \varphi(x)) \equiv (\delta(x), \varphi(x+a) = \varphi(x+a)|_{x=0} = \varphi(a).$$

The question arises: does locally integrable function $f \in L_{1,loc}(m.e. R_{1,loc})$ correspond to any linearly continuous functional g, i.e. is it true

$$g(\varphi) = (g, \varphi) = \int f(x)\,\varphi(x)dx?$$

The answer is negative. There is no function $f \in L_{1,loc}$ corresponding to the functional $(\delta, \varphi) = \varphi(0)$ in order to take place $\int f(x)\,\varphi(x)dx = \varphi(0)$. Consequently, $(\delta, \varphi) \notin L_{1,loc}$.

The following statements are true.

Theorem 4.1. $\delta -$ function can be represented as a limit in the space of $D^{/}$ sequence of regular generalized functions:

$$\forall\varphi(x) \in D: \lim_{a\to 0}(\delta_a(x), \varphi(x)) = \lim_{a\to 0}\int_{-\infty}^{+\infty} \delta_a\,(x)\varphi(x)dx = (\delta, \varphi) = \varphi(0),$$

$$\lim_{a\to +0} \boldsymbol{\delta}_a(\boldsymbol{x}) = \begin{cases} \mathbf{0}, \boldsymbol{x} \neq \mathbf{0}, \\ +\boldsymbol{\infty}, \boldsymbol{x} = \mathbf{0}. \end{cases}$$

Theorem 4.2. Any singular generalized function can be represented as the limit in the space of $D^{/}$ sequence of regular generalized functions.

In other words, the space of generalized functions is the completion of the space of classical locally integrable functions, i.e. is obtained by adding all limit elements to the space $R_{1,loc}$ in the sense of weak convergence (in the sense of generalized functions).

The space of generalized functions

Let's introduce the concept of a functional underlying the definition of generalized functions.

1. Any reflection $f : D(G) \to R$ of the space of basic functions to the set of real numbers is called a functional.

Definition 1. We will say that a functional is given on the space D if a rule is specified according to which a certain number $f(\varphi)$ is associated with each function $\varphi(x) \in D$.

The functional will also be denoted as (f, φ).

A functional f is called linear if for any real numbers a and b and any basic functions $\varphi_1(x), \varphi_2(x)$ from the space D there is carried out the equality

$$(f, a\varphi_1 + b\varphi_2) = a(f, \varphi_1) + b(f, \varphi_2).$$

2. We now introduce the concept of a continuous functional defined on the space D of basic functions.

Definition 2. The functional f defined on the space D is called continuous if for any sequence of basic functions $\{\varphi_n(x)\}$, converging to some function $\varphi(x) \in D(G)$, the numerical sequence (f, φ_n) converges to the number (f, φ).

We now formulate the central definition of this work.

Definition 3. A generalized function is a linear continuous functional on the space of basic functions.

The value of the functional f on an element $\varphi(x)$ of the space D, as it was said, will be denoted by (f, φ).

The collection of all generalized functions on G forms a linear space and is denoted by $D^{/}(G)$.

3. In the vector space $D^{/}$ of the generalized functions over the base space D , we give the definition of the concept of convergence.

The linear substitution of variables in generalized functions and simplification of expressions in the space of generalized functions

Here we give the definition of the operation of linear substitution of variable of generalized functions, which is known to us for ordinary functions: we will observe how an operation known to us for ordinary functions can be reformulated as an action of a regular generalized function on an arbitrary basic function, and then (already for all generalized functions, including singular ones), we will take the resulting identity as a definition.

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1. Multiplication by an infinitely differentiable function. Let $f \in R_{1,loc}(\mathrm{R}^1)$ and $a(x) -$ infinitely differentiable function on the whole line R^1: $a(x) \in C^\infty(\mathrm{R}^1)$. Then $a(x)f(x) -$ locally integrable function and $\forall \varphi(x) \in D$ function $a(x)\varphi(x) \in D$. Therefore, for a regular generalized function $a(x)f(x)$, we obtain the equality

$$(af,\varphi) = \int_{-\infty}^{+\infty} a(x)f(x)\varphi(x)dx = \int_{-\infty}^{+\infty} f(x)[a(x)\varphi(x)]dx = (f,a\varphi). \quad (1)$$

So, for any regular generalized function $f \in D^{/}(\mathrm{R}^1)$ and for any function $a(x) \in C^\infty(\mathrm{R}^1)$, the following equality is true:

$$(af,\varphi) = (f,a\varphi), \varphi(x) \in D.$$

For singular generalized functions, equality (1) is taken as the definition of the product of the generalized function by an infinitely differentiable function. Thus, we come to the following definition.

Definition. The product of the generalized function f by an infinitely differentiable function $a(x)$ is a generalized function af acting according to the rule

$$\forall \varphi(x) \in D: (af,\varphi) = (f,a\varphi).$$

Example 1. To find $a(x)\delta(x)$.

Decision.

$$(a(x)\delta(x),\varphi(x)) = (\delta(x),a(x)\varphi(x)) = a(0)\varphi(0) =$$
$$= a(0)(\delta,\varphi) = (a(0)\delta,\varphi),$$

e.i. multiplying $\delta -$ function by an infinitely differentiable function $a(x)$ is equivalent to multiplying $\delta -$ function by a number $a(0)$:

$$a(x)\delta(x) = a(0)\delta(x).$$

Example 2. To find $cosx\delta(x)$.

Decision.

$$(cosx\delta(x),\varphi(x)) = (\delta(x),\cos x\,\varphi(x)) = (\cos x\,\varphi(x))(0) =$$
$$= \cos 0\,\varphi(0) = 1\cdot\varphi(0) = \varphi(0) = (\delta,\varphi).$$

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Therefore, multiplying $\delta -$ function by a function $cos x$ is equivalent to multiplying $\delta -$ function by a number $cos\, 0$:

$$cos\, x\, \delta(x) = \delta(x).$$

2. Linear substitution of variables in generalized functions. Let $f \in R_{1,loc}(\mathrm{R}^1) -$ locally integrable in R^1 функция and $x = ay + b,\quad a, b -$ arbitrary numbers, ($a \neq 0$) – linear transformation of the space R^1 onto itself. Then for any function $\varphi(x) \in D$ we have

$$\left(f(ay+b), \varphi(y) \right) = \int_{-\infty}^{\infty} f(ay+b)\varphi(y)\, dy. \qquad (2)$$

Let’s do the substitution of the variable in the integral $x = ay + b$.

Then $dy = \frac{1}{a}dx,\ y = \frac{x-b}{a}$ and we come to the equalities

$$\int_{-\infty}^{\infty} f(ay+b)\varphi(y)\, dy = \frac{1}{|a|}\int_{-\infty}^{\infty} f(x)\varphi(a^{-1}(x-b))\, dx =$$

$$(3)$$

$$= \frac{1}{|a|}\int_{-\infty}^{\infty} f(y)\varphi(a^{-1}(y-b))\, dy = \frac{1}{|a|}\left(f, \varphi(a^{-1}(y-b)) \right).$$

From (2) and (3) it follows that for any regular generalized function $f(y)$ and any numbers a ($a \neq 0$) and b there is the equality

$$\left(f(ay+b), \varphi(y) \right) = \frac{1}{|a|}\left(f, \varphi(a^{-1}(y-b)) \right).$$

We will take this equality as the definition of a linear substitution of variables for singular generalized functions. Thus, we introduce the following definition.

Example 1. $\delta(-y)=\delta(y)$.

This follows from the calculation of the parity δ - the function and equality (7) at $a=-1$:

$$(\delta(-y),\varphi(y))=(\delta(y),\varphi(-y))=\varphi(0)=(\delta,\varphi),$$

$$\delta(-y)=\delta(y).$$

Example 2. Shifted $\delta-$ function:

$$(\delta(x-x_0),\varphi(x))=(\delta(y),\varphi(y+x_0))=\varphi(x_0).$$

Conclusion

The solution of many theoretical and practical problems is closely related to both the methodological approach and the mathematical apparatus used.

The theory of generalized functions is a powerful mathematical apparatus that makes it possible to solve a wide class of problems that, generally speaking, cannot be solved by the method of classical mathematical analysis. This apparatus makes it possible to rigorously substantiate the methods used and the results obtained and make it possible to construct a general theory.

The technique that formed the basis of this work allows us to formulate the correct understanding of concepts such as basic and generalized functions, substitution of variables in generalized functions. A large number of examples are given related to generalized functions that are built from a delta function using substitutions in the argument and multiplication by a locally integrable function. The given examples give an idea of the significance of the elements of the theory of generalized functions for their decidability. The theoretical material of the article is of a reference nature and helps teachers and students in preparing for practical classes, as well as in mastering the methods of the theory of generalized functions during their independent work on the course.

References:

1. Alexandrov V.A. Generalized functions. Practical manual. Novosibirsk State University, 2005. 46 p.

ISSN 2181-2632 www.involta.uz

2. Alenitsyn A.G., Grikurov V.E. Generalized functions in mathematical physics. SPbGU, 2001.64 p.
3. Butuzov V.F., Butuzova M.V. Fourier series and integral. Generalized functions. Practical manual. M., 2017. 60 p.
4. Pozharsky A.A. Toolkit. St. Petersburg State University, 2015.183 p.
5. Popova E.M., Chigireva O.Yu. Methodological features of the presentation of the topic “Generalized functions. Generalized derivatives. Delta-function of Dirac”// Scientific-methodical electronic journal” Concept”, 2018.
6. Sherstnev A.N. Lecture notes on mathematical analysis. Kazan Mathematical Society. 2009.
7. Daujanov A.Sh. Methodical statements of the elements of the theory of generalized functions//Scientific-methodical journal “ILM SARCHASHMALARI”, Urgench State University. 2020. No.10. pp. 11-25.