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A NEW APPROACH TO NUMERICAL SIMULATION OF BOUNDARY VALUE PROBLEMS OF THE THEORY OF ELASTICITY IN STRESSES AND STRAINS

Abduvali Khaldjigitov
Department of Mechanics and Mathematical Modeling[1]

Umidjon Djumayozov
Department of software engineering
Samarkand branch Tashkent University of Information Technologies
47A Shokhrukh str., Mirzo, Samarkand, Uzbekistan, 140100

Otajon Tilovov✉
Department of Mechanics and Mathematical Modeling[1]
otajontilovov95@gmail.com

[1]*National University of Uzbekistan*
4 Universitetskaya str., Tashkent, Uzbekistan, 100174

✉**Corresponding author**

Abstract

The main parameters characterizing the process of deformation of solids are displacements, strain and stress tensors. From the point of view of the strength and reliability of the structure and its elements, researchers and engineers are mainly interested in the distribution of stresses in the objects under study. Unfortunately, all boundary value problems are formulated and solved in solid mechanics mainly with respect to displacements, or an additional stress functions. And the required stresses are calculated from known displacements or stress functions. In this case, the accuracy of stress calculation is strongly affected by the error of numerical differentiation, as well as the approximation order of the boundary conditions. The formulation of boundary value problems directly with respect to stresses or strains allows to increase the accuracy of stress calculation by bypassing the process of numerical differentiation.

Therefore, the present work is devoted to the formulation and numerical solution of boundary value problems directly with respect to stresses and strains. Using the well-known Beltrami-Miechell equation, and considering the equilibrium equation as ah additional boundary condition, a boundary value problem (BVP) is formulated directly with respect to stresses. In a similar way, using the strain compatibility condition, the Beltrami-Mitchell type equations for strains are written.

The finite difference equations for two-dimensional BVP are constructed and written in convenient a form for the use of iterative method. A number of problems on the equilibrium of a rectangular plate under the action of various loads applied on opposite sides are numerically solved. The reliability of the results is ensured by comparing the numerical results of the 2D elasticity problems in stresses and strains, and with the exact solution, as well as with the known solutions of the plate tension problem under parabolic and uniformly distributed loads.

Keywords: differential equations, stress, strain, compatibility conditions, equilibrium equations, finite-difference schemes, iterative method.

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1. Introduction

Determining the safety margins and reliability of structures is directly related to the study of stress distribution in solids. But, boundary value problems (BVP) are usually formulated with respect to displacements. At the same time, stresses and strains from known displacements are calculated with a some approximation errors of numerical differentiation. Therefore, the formulation of boundary value problems directly with respect to stresses and strains is an actual problem of the mathematical modeling.

It is known that boundary value problems (BVP) relative to stresses are based on the strain compatibility condition, which is usually using Hooke's law and the equilibrium equations reduced to the Beltrami-Michel equations [1]. It is known that the classical boundary value problem in stresses consists of six Beltrami-Michell equations, three equilibrium equations and three boundary conditions [2]. In the BVP the number of equations is nine and the unknowns are the six, and there are only three boundary conditions, and three more conditions are missing [3]. At the same time, it is not clear which of the six Beltrami-Mitchell equations are independent. In [4, 5] it is show that that six compatibility equations in terms of stresses can be split into two dependent groups consisting of three equations each of them. The first group of equations can be found from the second group of equations, and vice versa. However, for the Beltrami-Michel equations, such a division into two independent groups has not been proven. It was noted in [6] that they are independent.

Similar reasoning is valid for the «Beltrami-Michell type» strain equations obtained from the strain compatibility conditions using Hooke's law and the equilibrium equation expressed with respect to deformations [7].

In [8, 9], a new formulation of the boundary value problem in stresses is proposed, where the number of equations is equal to six. Considering the equilibrium equations on the boundary of a given domain is shown correctness of the boundary problems in stresses. In [10, 11] the theorem if existence and uniqueness of the solutions of new BVP in stresses and the equivalence of the new and classical formulations are proved.

The fundamental principles of strain compatibility conditions and their current state are discussed in more detail in the review article [12, 13]. It is well known, that the two-dimensional elasticity problems in stresses, using the Airy's stress function, are reduced to the solution of a biharmonic equation [14–16]. The boundary value problems in stresses, for isotropic parallelepiped by classical and new variation methods were solved numerically, respectively in [17, 18]. Using the three dimensional Maxwell's and Morera's stress functions were considered some elasticity problems in [19, 20].

It should be noted, that at present despite the progress made in the area of stress problems there is no correctly formulated and numerically solved boundary value problems directly with respect to stresses in the literature.

This work is devoted to the formulation of boundary value problems of the theory of elasticity with respect to stresses and strains and, their numerical solution, in contrast to existing methods, directly with respect to stresses and strains.

2. Materials and methods

It is known that the boundary value problem of the theory of elasticity in stresses consists of the equilibrium equation, the Beltrami-Michell equation with the corresponding boundary conditions [1, 8], i.e.:

$$\sigma_{ij,j} + X_i = 0, \tag{1}$$

$$\nabla^2\sigma_{ij} + \frac{1}{1+v}S_{i,j} = -(X_{i,j} + X_{j,i}) - \frac{v}{1-v}\delta_{ij}X_{k,k},\ \ S = \sigma_{kk}, \tag{2}$$

$$\sigma_{ij}n_j\big|_{\Sigma_1} = S_i. \tag{3}$$

In the absence of body forces, the boundary value problem (1)–(3) has the form [11, 21]:

$$\sigma_{ij,j} = 0, \tag{4}$$

$$\nabla^2\sigma_{ij} + \frac{1}{1+v}S_{i,j} = 0,\ \ S = \sigma_{kk}, \tag{5}$$

$$\sigma_{ij}n_j\big|_{\Sigma_2} = S_i. \tag{6}$$

The system of equations (4)–(6) consists of six Beltrami-Michel equations and three equilibrium equations with the corresponding three boundary conditions. According to studies [8, 10], the equilibrium equation can be considered as a boundary condition on the surface of a given domain, i.e.:

$$\sigma_{ij,j}\big|_{\Sigma} = 0. \tag{7}$$

Then the system of equations (4)–(7) is a closed boundary value problem of the theory of elasticity in stresses. Note that the system (2)–(7) includes nine differential equations with six boundary conditions, i.e. the system has been overdetermined. As noted in [4], these equations are not independent. According to [5], of the six compatibility equations, only three are independent.

Similar to the Beltrami-Michell equations, the compatibility conditions can be written in the form of differential equations with respect to strains. That is why let's consider the compatibility condition [1]:

$$\nabla^2\varepsilon_{ij} + \theta_{,ij} - \varepsilon_{ik,kj} - \varepsilon_{jk,ki} = 0, \tag{8}$$

which, taking into account the relation:

$$\varepsilon_{ij,j} = -\frac{\lambda}{2\mu}\theta_{,i} - \frac{1}{2\mu}X_{i,j}, \tag{9}$$

obtained from the equilibrium equation (1) using following Hooke's law:

$$\sigma_{ij} = \lambda\theta\delta_{ij} + 2\mu\varepsilon_{ij}, \tag{10}$$

can be written as a system of six differential equations with respect to the strain tensor [1]:

$$\mu\nabla^2\varepsilon_{ij} + (\lambda+\mu)\theta_{,ij} + \frac{1}{2}\left(X_{i,j} + X_{j,i}\right) = 0. \tag{11}$$

Adding to (11) three equilibrium equations expressed in terms of deformations:

$$\lambda\theta_{,i} + 2\mu\varepsilon_{ij,j} + X_i = 0, \tag{12}$$

and, boundary conditions:

$$(\lambda\theta\delta_{ij} + 2\mu\varepsilon_{ij})n_j\,|_{\Sigma_2} = S_i, \tag{13}$$

let's obtain the boundary value problem of the theory of elasticity in strains. Similarly, to the boundary value problem in stresses, it is necessary to add the equilibrium equations to equations (11)–(13) as additional boundary conditions, i.e.:

$$(\lambda\theta_{,i} + 2\mu\varepsilon_{ij,j} + X_i)\big|_{\Sigma} = 0. \tag{14}$$

Thus, equations (11)–(14) represent the boundary value problem of the theory of elasticity in strains. In the case of a plane stress state equations (4), (5) take the following form:

$$\frac{\partial\sigma_{11}}{\partial x} + \frac{\partial\sigma_{12}}{\partial y} = 0, \tag{15}$$

$$\frac{\partial\sigma_{21}}{\partial x} + \frac{\partial\sigma_{22}}{\partial y} = 0, \tag{16}$$

$$\frac{2+\nu}{1+\upsilon}\frac{\partial^2\sigma_{11}}{\partial x^2} + \frac{\partial^2\sigma_{11}}{\partial y^2} + \frac{1}{1+\upsilon}\frac{\partial^2\sigma_{22}}{\partial x^2} = 0, \tag{17}$$

$$\frac{2+\nu}{1+\upsilon}\frac{\partial^2\sigma_{22}}{\partial y^2} + \frac{\partial^2\sigma_{22}}{\partial x^2} + \frac{1}{1+\upsilon}\frac{\partial^2\sigma_{11}}{\partial y^2} = 0, \tag{18}$$

$$\frac{\partial^2\sigma_{12}}{\partial x^2} + \frac{\partial^2\sigma_{12}}{\partial y^2} + \frac{1}{1+\upsilon}\left(\frac{\partial^2\sigma_{11}}{\partial x\partial y} + \frac{\partial^2\sigma_{22}}{\partial x\partial y}\right) = 0. \tag{19}$$

The system of equations (15)–(19) depends on two normal and one tangential components of the stress tensor. It is known [1, 14] that in the two-dimensional case, the compatibility condition has the form:

$$\frac{\partial^2 \varepsilon_{11}}{\partial y^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial x \partial y}, \tag{20}$$

and, using Hooke's law and the equilibrium equation, can be written as a harmonic equation:

$$\nabla^2(\sigma_{11} + \sigma_{22}) = 0. \tag{21}$$

It can be shown that equations (17), (18) are equivalent to the harmonic equation (21). Differentiating the equilibrium equations with respect to x and y, respectively, one can find that:

$$\frac{\partial^2 \sigma_{11}}{\partial x^2} = -\frac{\partial^2 \sigma_{12}}{\partial y \partial x}, \; \frac{\partial^2 \sigma_{11}}{\partial y^2} = -\frac{\partial^2 \sigma_{12}}{\partial y \partial x}. \tag{22}$$

Substituting (22) into equations (17), (18), i.e.:

$$\frac{2+\nu}{1+v}\left(-\frac{\partial^2 \sigma_{12}}{\partial x \partial y}\right) + \frac{\partial^2 \sigma_{11}}{\partial y^2} + \frac{1}{1+v}\frac{\partial^2 \sigma_{22}}{\partial x^2} = 0, \tag{23}$$

$$\frac{2+\nu}{1+v}\left(-\frac{\partial^2 \sigma_{xy}}{\partial x \partial y}\right) + \frac{\partial^2 \sigma_{y}}{\partial x^2} + \frac{1}{1+v}\frac{\partial^2 \sigma_{x}}{\partial y^2} = 0, \tag{24}$$

and subtracting the second ratio from the first, let's find that:

$$\frac{\partial^2 \sigma_{11}}{\partial y^2} = \frac{\partial^2 \sigma_{22}}{\partial x^2}. \tag{25}$$

Using the last relation, equations (23), (24) can be reduced to the well-known harmonic equation (21).

Thus, equations (15), (16) and (21), i.e.:

$$\frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y} = 0, \; \frac{\partial \sigma_{21}}{\partial x} + \frac{\partial \sigma_{22}}{\partial y} = 0, \; \nabla^2(\sigma_{11} + \sigma_{22}) = 0, \tag{26}$$

is a well-known plane problem of the theory of elasticity, which can be solved with the introduction the Airy's stress function [1, 14], i.e.:

$$\sigma_{11} = \frac{\partial^2 \varphi}{\partial y^2}, \; \sigma_{22} = \frac{\partial^2 \varphi}{\partial x^2}, \; \sigma_{12} = -\frac{\partial^2 \varphi}{\partial x \partial y}, \tag{27}$$

based on the biharmonic equation [1, 14]:

$$\nabla^2 \nabla^2 \varphi = 0. \tag{28}$$

Similarly, to (26), the plane problem of elasticity theory can be formulated on the basis of the equilibrium equations (15), (16) and (19), i.e.:

$$\frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y} = 0, \; \frac{\partial \sigma_{22}}{\partial y} + \frac{\partial \sigma_{21}}{\partial x} = 0, \; \frac{\partial^2 \sigma_{12}}{\partial x^2} + \frac{\partial^2 \sigma_{12}}{\partial y^2} + \frac{1}{1+v}\left(\frac{\partial^2 \sigma_{11}}{\partial x \partial y} + \frac{\partial^2 \sigma_{22}}{\partial x \partial y}\right) = 0, \tag{29}$$

with boundary conditions:

$$\begin{aligned}(\sigma_{11}n_1+\sigma_{12}n_2)\big|_{\Gamma} &= S_1,\\ (\sigma_{21}n_1+\sigma_{22}n_2)\big|_{\Gamma} &= S_2,\end{aligned} \tag{30}$$

and, with additional boundary conditions found using the equilibrium equation, i.e.:

$$\left.\left(\frac{\partial\sigma_{11}}{\partial x}+\frac{\partial\sigma_{12}}{\partial y}\right)\right|_{\Gamma}=0,$$

$$\left.\left(\frac{\partial\sigma_{22}}{\partial y}+\frac{\partial\sigma_{21}}{\partial x}\right)\right|_{\Gamma}=0. \tag{31}$$

Thus, equations (29)–(31), in contrast to the well-known plane problem consisting of equations (26) and (30), (31), represent a new version of the plane boundary value problem of the theory of elasticity in stresses. Looking ahead, it is possible to say that the numerical results of these two considered plane problems, as shown in Section 6, very close.

Let's consider the boundary and additional boundary conditions in detail for a rectangular area. Let the rectangle be under the action of tensile forces on both sides along the OX axis, the other sides are free from loads (**Fig. 1**), i.e.:

$$\text{for } x=\pm a:\ \sigma_{11}\big|_{x=\pm a}=S,\ \ \sigma_{12}\big|_{x=\pm a}=0, \tag{32}$$

$$\text{for } y=\pm b:\ \sigma_{22}\big|_{y=\pm b}=0,\ \ \sigma_{21}\big|_{y=\pm b}=0. \tag{33}$$

In this case, additional boundary conditions (31) with respect to the rectangular domain have the form:

$$\text{for } x=\pm a:\ \left.\left[\frac{\partial\sigma_{22}}{\partial y}\right]\right|_{x=\pm a}=-\left.\left[\frac{\partial\sigma_{21}}{\partial x}\right]\right|_{x=\pm a}, \tag{34}$$

$$\text{for } y=\pm b:\ \left.\left[\frac{\partial\sigma_{11}}{\partial x}\right]\right|_{y=\pm b}=-\left.\left[\frac{\partial\sigma_{12}}{\partial y}\right]\right|_{y=\pm b}. \tag{35}$$

Thus, according to relations (32)–(35), there are three boundary conditions on each side of the rectangle, namely, two boundary conditions and one additional condition.

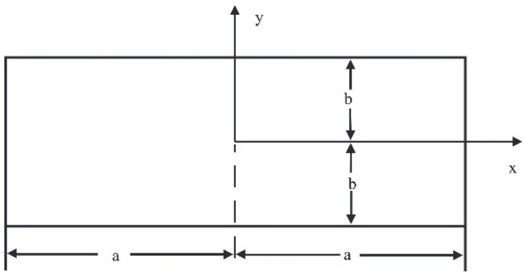

Fig. 1. Rectangular area of size ($2a$, $2b$)

Let's consider the boundary value problem (11), (12), in plane strain case without body forces, i.e.:

$$\mu\left(\frac{\partial^2\varepsilon_{11}}{\partial x^2}+\frac{\partial^2\varepsilon_{11}}{\partial y^2}\right)+(\lambda+\mu)\frac{\partial^2(\varepsilon_{11}+\varepsilon_{22})}{\partial x^2}=0, \tag{36}$$

$$\mu\left(\frac{\partial^2\varepsilon_{22}}{\partial x^2}+\frac{\partial^2\varepsilon_{22}}{\partial y^2}\right)+(\lambda+\mu)\frac{\partial^2(\varepsilon_{11}+\varepsilon_{22})}{\partial y^2}=0, \tag{37}$$

$$\mu\left(\frac{\partial^2\varepsilon_{12}}{\partial x^2}+\frac{\partial^2\varepsilon_{12}}{\partial y^2}\right)+(\lambda+\mu)\left(\frac{\partial^2\varepsilon_{11}}{\partial x\partial y}+\frac{\partial^2\varepsilon_{22}}{\partial x\partial y}\right)=0, \tag{38}$$

$$(\lambda+2\mu)\frac{\partial\varepsilon_{11}}{\partial x}+\lambda\frac{\partial\varepsilon_{22}}{\partial x}+2\mu\frac{\partial\varepsilon_{12}}{\partial y}=0, \tag{39}$$

$$(\lambda+2\mu)\frac{\partial\varepsilon_{22}}{\partial y}+\lambda\frac{\partial\varepsilon_{11}}{\partial y}+2\mu\frac{\partial^2\varepsilon_{12}}{\partial x}=0, \tag{40}$$

which consists of three strain equations (36)–(38) and two equilibrium equations (39), (40) expressed using Hooke's law with respect to strain tensor components. It can be shown that equations (36), (37) are equivalent to the compatibility condition (20).

For what, equations (36), (37) are written in the following form:

$$(\lambda+2\mu)\frac{\partial^2\varepsilon_{11}}{\partial x^2}+\mu\frac{\partial^2\varepsilon_{11}}{\partial y^2}+(\lambda+\mu)\frac{\partial^2\varepsilon_{22}}{\partial x^2}=0. \tag{41}$$

$$(\lambda+2\mu)\frac{\partial^2\varepsilon_{22}}{\partial y^2}+\mu\frac{\partial^2\varepsilon_{22}}{\partial x^2}+(\lambda+\mu)\frac{\partial^2\varepsilon_{11}}{\partial y^2}=0. \tag{42}$$

Further, differentiating the equilibrium equations (39), (40) with respect to x and y, let's find, respectively, i.e.:

$$(\lambda+2\mu)\frac{\partial^2\varepsilon_{11}}{\partial x^2}+\lambda\frac{\partial^2\varepsilon_{22}}{\partial x^2}=-2\mu\frac{\partial^2\varepsilon_{12}}{\partial x\partial y}, \tag{43}$$

$$(\lambda+2\mu)\frac{\partial^2\varepsilon_{22}}{\partial y^2}+\lambda\frac{\partial^2\varepsilon_{11}}{\partial y^2}=-2\mu\frac{\partial^2\varepsilon_{12}}{\partial x\partial y}. \tag{44}$$

Taking into account relations (43), (44), equations (41), (42) can be reduced to the form:

$$-2\mu\frac{\partial^2\varepsilon_{12}}{\partial x\partial y}+\mu\frac{\partial^2\varepsilon_{11}}{\partial y^2}+\mu\frac{\partial^2\varepsilon_{22}}{\partial x^2}=0, \tag{45}$$

$$-2\mu\frac{\partial^2\varepsilon_{12}}{\partial x\partial y}+\mu\frac{\partial^2\varepsilon_{22}}{\partial x^2}+\mu\frac{\partial^2\varepsilon_{11}}{\partial y^2}=0. \tag{46}$$

Adding these two equations, wit is possible to find the well-known compatibility condition (20):

$$\frac{\partial^2\varepsilon_{11}}{\partial y^2}+\frac{\partial^2\varepsilon_{22}}{\partial x^2}=2\frac{\partial^2\varepsilon_{12}}{\partial x\partial y}. \tag{47}$$

This proves the equivalence of equations (36), (37) and (47). On the other hand, adding equations (43), (44) i.e.:

$$2\frac{\partial^2\varepsilon_{12}}{\partial x\partial y}=-\left(1+\frac{\lambda}{2\mu}\right)\left(\frac{\partial^2\varepsilon_{11}}{\partial x^2}+\frac{\partial^2\varepsilon_{22}}{\partial y^2}\right)-\frac{\lambda}{2\mu}\left(\frac{\partial^2\varepsilon_{11}}{\partial y^2}+\frac{\partial^2\varepsilon_{22}}{\partial x^2}\right), \tag{48}$$

and, substituting the resulting expression into (47), the compatibility condition (47) can be written as the following harmonic equation:

$$\nabla^2(\varepsilon_{11}+\varepsilon_{22})=0. \tag{49}$$

Thus, equations (49) and (39), (40) with the corresponding boundary conditions, similarly to the plane elasticity problem in stresses (26), constitute a plane problem of the theory of elasticity in strains.

Now, considering the equation (38), together with the equilibrium equations (39), (40), it is possible to formulate another version of the plane problem of elasticity theory in strains, i.e.:

$$(\lambda+2\mu)\frac{\partial \varepsilon_{11}}{\partial x}+\lambda\frac{\partial \varepsilon_{22}}{\partial x}+2\mu\frac{\partial \varepsilon_{12}}{\partial y}=0,$$

$$(\lambda+2\mu)\frac{\partial \varepsilon_{22}}{\partial y}+\lambda\frac{\partial \varepsilon_{11}}{\partial y}+2\mu\frac{\partial \varepsilon_{21}}{\partial x}=0,$$

$$\mu\left(\frac{\partial^2 \varepsilon_{12}}{\partial x^2}+\frac{\partial^2 \varepsilon_{12}}{\partial y^2}\right)+(\lambda+\mu)\left(\frac{\partial^2 \varepsilon_{11}}{\partial x\partial y}+\frac{\partial^2 \varepsilon_{22}}{\partial x\partial y}\right)=0, \tag{50}$$

with appropriate boundary conditions:

$$((\lambda(\varepsilon_{11}+\varepsilon_{22})+2\mu\varepsilon_{11})n_1+(2\mu\varepsilon_{12})n_2)\big|_\Gamma=S_1,$$
$$((2\mu\varepsilon_{12})n_1+(\lambda(\varepsilon_{11}+\varepsilon_{22})+2\mu\varepsilon_{22})n_2)\big|_\Gamma=S_2. \tag{51}$$

To complete the boundary value problem (50), (51), it is necessary to add to it the equilibrium equations (39), (40) as boundary conditions, i.e.:

$$\left[(\lambda+2\mu)\frac{\partial \varepsilon_{11}}{\partial x}+\lambda\frac{\partial \varepsilon_{22}}{\partial x}+2\mu\frac{\partial \varepsilon_{12}}{\partial y}\right]\Bigg|_\Gamma=0, \tag{52}$$

$$\left[(\lambda+2\mu)\frac{\partial \varepsilon_{22}}{\partial y}+\lambda\frac{\partial \varepsilon_{11}}{\partial y}+2\mu\frac{\partial \varepsilon_{21}}{\partial x}\right]\Bigg|_\Gamma=0, \tag{53}$$

where Γ is the boundary of the given area. Thus, equations (50)–(53) represent a second version of plane boundary value problem of the theory of elasticity in strains.

Let's consider the boundary (51) and additional conditions (52), (53) for a rectangle under the action of tensile loads applied from two opposite sides along the *OX* axis, the remaining sides are free from loads (**Fig. 1**, (32), (33)).

Boundary conditions (32), (33), using Hooke's law [11]:

$$\varepsilon_{11}=\frac{1}{E_1}\sigma_{11}-\frac{\nu_1}{E}\sigma_{22},$$

$$\varepsilon_{22}=\frac{1}{E_1}\sigma_{22}-\frac{\nu_1}{E}\sigma_{11},$$

$$\varepsilon_{12}=\frac{1}{2\mu}\sigma_{12}, \tag{54}$$

where

$$E_1=\begin{cases}\dfrac{E}{1-\nu^2} & \text{plane strain state;}\\ E & \text{plane stress state,}\end{cases}\qquad \nu_1=\begin{cases}\dfrac{\nu}{1-\nu} & \text{p.s.s;}\\ \nu & \text{p.s.s,}\end{cases}$$

can be written with respect to the strains i.e.:

$$\varepsilon_{22}\big|_{y=0}=0,\ \ \varepsilon_{21}\big|_{y=0}=0,$$
$$\varepsilon_{22}\big|_{y=l_2}=0,\ \ \varepsilon_{21}\big|_{y=l_2}=0, \tag{55}$$

$$\varepsilon_{11}\big|_{x=0}=\frac{1}{E_1}S,\ \ \varepsilon_{12}\big|_{x=0}=0,$$

$$\varepsilon_{11}\big|_{x=l_1}=\frac{1}{E_1}S,\ \ \varepsilon_{12}\big|_{x=l_1}=0. \tag{56}$$

From condition (53), it is possible to find the following additional boundary conditions:

$$\left[\frac{\partial \varepsilon_{11}}{\partial y}\right]\Bigg|_{y=0,\, l_2} = -\left[\frac{2\mu}{\lambda}\frac{\partial \varepsilon_{21}}{\partial x}\right]\Bigg|_{y=0,\, l_2}. \tag{57}$$

Let's note that the first term of the boundary condition (53) at $y=0$, l_2 does not depend on the argument y. Similarly, for $x=0$, l_1 from equation (52) one can find the following condition:

$$\left[\frac{\partial \varepsilon_{22}}{\partial x}\right]\Bigg|_{x=0,\, l_1} = -\left[\frac{2\mu}{\lambda}\frac{\partial \varepsilon_{12}}{\partial y}\right]\Bigg|_{x=0,\, l_1}. \tag{58}$$

Thus, according to relations (55)–(58) on each side of the rectangular plate there are three boundary conditions for strain tensor components.

3. Results and discussion

This section is devoted to the construction of finite-difference equations for two-dimensional boundary value problems of elasticity theory formulated with respect to stresses (29)–(31) and strains (50)–(53). Boundary problems are considered with respect to a rectangular plate loaded on opposite sides perpendicular to the OX axis. The other sides are free from loads (**Fig. 1**).

Let the boundary value problem in stresses (29), (31) be considered in a rectangular domain with side lengths $l_1=2a$, $l_2=2b$ (**Fig. 1**). Dividing the sides of the rectangle by n_k let's obtain the mesh step $h_k=l_k/N_k$, $k=1,2$. Then nodal point coordinates have the form $x_i=ih_1$, $i=0..n_1$, $y_j=jh_2$, $j=0..n_2$ [22, 23]. Replacing the derivatives in equations (29) with the corresponding finite difference relations, it is possible to find that:

$$\frac{\sigma^{11}_{i+1,j}-\sigma^{11}_{i,j}}{h_1}+\frac{\sigma^{12}_{i,j+1}-\sigma^{12}_{i,j-1}}{2h_2}=0, \tag{59}$$

$$\frac{\sigma^{22}_{i,j+1}-\sigma^{22}_{i,j}}{h_2}+\frac{\sigma^{12}_{i+1,j}-\sigma^{12}_{i-1,j}}{2h_1}=0, \tag{60}$$

$$\frac{\sigma^{12}_{i+1,j}-2\sigma^{12}_{ij}+\sigma^{12}_{i-1,j}}{h_1^2}+\frac{\sigma^{12}_{i,j+1}-2\sigma^{12}_{ij}+\sigma^{12}_{i,j-1}}{h_2^2}+$$
$$+\frac{1}{1+v}\left(\frac{\sigma^{11}_{i+1,j+1}-\sigma^{11}_{i-1,j+1}-\sigma^{11}_{i+1,j-1}+\sigma^{11}_{i-1,j-1}}{4h_1h_2}+\frac{\sigma^{22}_{i+1,j+1}-\sigma^{22}_{i-1,j+1}-\sigma^{22}_{i+1,j-1}+\sigma^{22}_{i-1,j-1}}{4h_1h_2}\right)=0. \tag{61}$$

Resolving the finite difference equations (59)–(61) with respect to σ^{11}_{ij}, σ^{22}_{ij} and σ^{12}_{ij} respectively, and accepting the following redesignations $\sigma^{11}_{ij}=\underset{11}{\sigma}{}^{(k)}_{ij}$, $\sigma^{22}_{ij}=\underset{22}{\sigma}{}^{(k)}_{ij}$ and $\sigma^{12}_{ij}=\underset{12}{\sigma}{}^{(k)}_{ij}$ one can find,

$$\underset{11}{\sigma}{}^{(k+1)}_{ij}=\underset{11}{\sigma}{}^{(k)}_{ij}+\frac{h_1}{2h_2}\left(\underset{12}{\sigma}{}^{(k)}_{i,j+1}-\underset{12}{\sigma}{}^{(k)}_{i,j-1}\right), \tag{62}$$

$$\underset{22}{\sigma}{}^{(k+1)}_{ij}=\underset{22}{\sigma}{}^{(k)}_{i,j+1}+\frac{h_2}{2h_1}\left(\underset{12}{\sigma}{}^{(k)}_{i+1,j}-\underset{12}{\sigma}{}^{(k)}_{i-1,j}\right), \tag{63}$$

$$\underset{12}{\sigma}{}^{(k+1)}_{ij}=\left(\frac{\underset{12}{\sigma}{}^{(k)}_{i+1,j}+\underset{12}{\sigma}{}^{(k)}_{i-1,j}}{h_1^2}+\frac{\underset{12}{\sigma}{}^{(k)}_{i,j+1}+\underset{12}{\sigma}{}^{(k)}_{i,j-1}}{h_2^2}+\frac{1}{1+v}\left(\frac{\underset{11}{\sigma}{}^{(k)}_{i+1,j+1}-\underset{11}{\sigma}{}^{(k)}_{i-1,j+1}-\underset{11}{\sigma}{}^{(k)}_{i+1,j-1}+\underset{11}{\sigma}{}^{(k)}_{i-1,j-1}}{4h_1h_2}+\right.\right.$$
$$\left.\left.+\frac{\underset{22}{\sigma}{}^{(k)}_{i+1,j+1}-\underset{22}{\sigma}{}^{(k)}_{i-1,j+1}-\underset{22}{\sigma}{}^{(k)}_{i+1,j-1}+\underset{22}{\sigma}{}^{(k)}_{i-1,j-1}}{4h_1h_2}\right)\right)\Bigg/\left(\frac{2}{h_1^2}+\frac{2}{h_2^2}\right), \tag{64}$$

where indexes change at internal points i.e. $1\le i\le n_1-1$, $1\le j\le n_2-1$, k-iteration number.

In order to construct symmetric difference equations, in equation (59) let's replace the first term with the left finite relation, i.e.:

$$\frac{\sigma_{i,j}^{11}-\sigma_{i-1,j}^{11}}{h_1}+\frac{\sigma_{i,j+1}^{12}-\sigma_{i,j-1}^{12}}{2h_2}=0. \tag{65}$$

Subtracting (66) from (59) it is possible to find that:

$$\sigma_{i+1,j}^{11}-2\sigma_{ij}^{11}+\sigma_{i-1,j}^{11}=0. \tag{66}$$

Similarly, from equation (60) let's find:

$$\sigma_{i,j+1}^{22}-2\sigma_{ij}^{22}+\sigma_{i,j-1}^{22}=0. \tag{67}$$

Then resolving the equations (66), (67) with respect to σ_{ij}^{11} and σ_{ij}^{22} one can find the following relations, which can be considered instead of relations (62), (63), respectively:

$$\underset{11}{\sigma}{}_{ij}^{(k+1)}=\frac{\underset{11}{\sigma}{}_{i+1,j}^{(k)}+\underset{11}{\sigma}{}_{i-1,j}^{(k)}}{2}, \tag{68}$$

$$\underset{22}{\sigma}{}_{ij}^{(k+1)}=\frac{\underset{22}{\sigma}{}_{i,j+1}^{(k)}+\underset{22}{\sigma}{}_{i,j-1}^{(k)}}{2}. \tag{69}$$

The boundary conditions (32), (33) for nodal points have the form:

$$\text{for } x=\pm a \quad \begin{aligned}&\underset{11}{\sigma}{}^{(0)}{}_{0j}=S,\quad \underset{12}{\sigma}{}^{(0)}{}_{0j}=0,\\ &\underset{11}{\sigma}{}^{(0)}{}_{N_1 j}=S,\quad \underset{12}{\sigma}{}^{(0)}{}_{N_1 j}=0,\end{aligned} \tag{70}$$

$$\text{for } y=\pm b \quad \begin{aligned}&\underset{22}{\sigma}{}^{(0)}{}_{i0}=0,\quad \underset{12}{\sigma}{}^{(0)}{}_{i0}=0,\\ &\underset{22}{\sigma}{}^{(0)}{}_{iN_2}=0,\quad \underset{12}{\sigma}{}^{(0)}{}_{iN_2}=0.\end{aligned} \tag{71}$$

According to relations (34), (35), the additional boundary conditions have the form:

$$\text{for } x=\pm a \quad \begin{aligned}&\underset{22}{\sigma}{}^{(0)}{}_{0j}=\underset{22}{\sigma}{}^{(0)}{}_{0,j+1}+h_2\frac{\underset{12}{\sigma}{}^{(0)}{}_{1j}-\underset{12}{\sigma}{}^{(0)}{}_{0j}}{h_1},\\ &\underset{22}{\sigma}{}^{(0)}{}_{n_1 j}=\underset{22}{\sigma}{}^{(0)}{}_{n_1,j+1}+h_2\frac{\underset{12}{\sigma}{}^{(0)}{}_{n_1,j}-\underset{12}{\sigma}{}^{(0)}{}_{n_1-1,j}}{h_1},\end{aligned} \tag{72}$$

$$\text{for } y=\pm b \quad \begin{aligned}&\underset{11}{\sigma}{}^{(0)}{}_{i0}=\underset{11}{\sigma}{}^{(0)}{}_{i+1,0}+h_1\frac{\underset{12}{\sigma}{}^{(0)}{}_{i1}-\underset{12}{\sigma}{}^{(0)}{}_{i0}}{h_2},\\ &\underset{11}{\sigma}{}^{(0)}{}_{in_2}=\underset{11}{\sigma}{}^{(0)}{}_{i+1,n_2}+h_1\frac{\underset{12}{\sigma}{}^{(0)}{}_{in_2}-\underset{12}{\sigma}{}^{(0)}{}_{i,n_2-1}}{h_2}.\end{aligned} \tag{73}$$

Using the iterative relations (62), (63), (70)–(73) can be found searched values of the stress tensor components. At zero approximation i.e., at $k=0$, the values of these quantities at internal points are considered to be zero. The convergence of the iterative method for relations (62), (64) is ensured by the fulfillment of the diagonal dominance condition [24].

Similarly, for plane boundary problem in strains (50)–(53) considered in a rectangular domain, the following finite-difference equations can be constructed, which can be solved by the iterative method [22, 25]:

$$\underset{11}{\varepsilon}{}_{i,j}^{(k+1)}=\frac{\underset{11}{\varepsilon}{}_{i+1,j}^{(k)}+\underset{11}{\varepsilon}{}_{i-1,j}^{(k)}}{2},\ \underset{22}{\varepsilon}{}_{i,j}^{(k+1)}=\frac{\underset{22}{\varepsilon}{}_{i,j+1}^{(k)}+\underset{22}{\varepsilon}{}_{i-1,j}^{(k)}}{2}, \tag{74}$$

$$\varepsilon_{12}{}^{(k+1)}_{i,j} = \left(\left(1+\frac{\lambda}{\mu}\right) \frac{\varepsilon_{11}{}^{(k)}_{i+1,j+1} - \varepsilon_{11}{}^{(k)}_{i+1,j-1} - \varepsilon_{11}{}^{(k)}_{i-1,j+1} + \varepsilon_{11}{}^{(k)}_{i+1,j+1}}{4h_1h_2} + \frac{\varepsilon_{12}{}^{(k)}_{i+1,j} + \varepsilon_{12}{}^{(k)}_{i-1,j}}{h_1^2} + \right.$$

$$\left. + \frac{\varepsilon_{12}{}^{(k)}_{i,j+1} + \varepsilon_{12}{}^{(k)}_{i,j-1}}{h_2^2} + \left(1+\frac{\lambda}{\mu}\right) \frac{\varepsilon_{22}{}^{(k)}_{i+1,j+1} - \varepsilon_{22}{}^{(k)}_{i+1,j-1} - \varepsilon_{22}{}^{(k)}_{i-1,j+1} + \varepsilon_{22}{}^{(k)}_{i+1,j+1}}{4h_1h_2} \right) \Bigg/ \left(\frac{2}{h_1^2} + \frac{2}{h_2^2}\right), \tag{75}$$

with nodal boundary conditions:

$$\begin{gathered} \varepsilon_{22}{}^{(0)}_{i0} = 0, \quad \varepsilon_{12}{}^{(0)}_{i0} = 0, \\ \varepsilon_{22}{}^{(0)}_{iN_2} = 0, \; \varepsilon_{12}{}^{(0)}_{iN_2} = 0, \end{gathered} \tag{76}$$

$$\begin{gathered} \varepsilon_{11}{}^{(0)}_{0j} = \frac{1}{E_1}S, \quad \varepsilon_{21}{}^{(0)}_{0j} = 0, \\ \varepsilon_{11}{}^{(0)}_{N_1 j} = \frac{1}{E_1}S, \; \varepsilon_{21}{}^{(0)}_{N_1 j} = 0. \end{gathered} \tag{77}$$

In this case, additional boundary conditions for $\varepsilon_{11}^{(k)}\Big|_{y=0,l_2}$ and $\varepsilon_{22}^{(k)}\Big|_{x=0,l_1}$ are found from equations (57), (58):

– for $y=0$ and $y=l_2$:

$$\begin{gathered} \varepsilon_{11}{}^{(0)}_{i0} = \varepsilon_{11}{}^{(0)}_{i1} + \frac{\mu h_2}{\lambda} \frac{\varepsilon_{12}{}^{(0)}_{i+1,0} - \varepsilon_{12}{}^{(0)}_{i-1,0}}{h_1}, \\ \varepsilon_{11}{}^{(0)}_{iN_2} = \varepsilon_{11}{}^{(0)}_{i,N_2-1} - \frac{\mu h_2}{\lambda} \frac{\varepsilon_{12}{}^{(0)}_{i+1,N_2} - \varepsilon_{12}{}^{(0)}_{i-1,N_2}}{h_1}, \end{gathered} \tag{78}$$

– for $x=0$ and $x=l_1$:

$$\begin{gathered} \varepsilon_{22}{}^{(0)}_{0,j} = \varepsilon_{22}{}^{(0)}_{1,j} + \frac{\mu h_1}{\lambda} \frac{\varepsilon^{12}_{0,j+1} - \varepsilon^{12}_{0,j-1}}{h_2}, \\ \varepsilon_{22}{}^{(0)}_{N_1,j} = \varepsilon_{22}{}^{(0)}_{N_1-1,j} - \frac{\mu h_1}{\lambda} \frac{\varepsilon_{12}{}^{(0)}_{N_1,j+1} - \varepsilon_{12}{}^{(0)}_{N_1,j-1}}{h_2}. \end{gathered} \tag{79}$$

Let's note that the finite-difference schemes of boundary value problems are symmetric. The convergence of iterative methods is provided within the framework of the generalized theorem on the convergence of iterative methods [24, 26].

Let's now consider the numerical solution of the plane problem of the theory of elasticity in stresses (29)–(31) and strains (50)–(53) and compare their results. Discrete analogs of boundary value problems are constructed by the finite-difference method and solved by the iterative method. A number of two-dimensional problems of tension of a rectangular plate with different boundary conditions are solved.

Let a rectangular plate of size $l_1=2a$, $l_2=2b$ is in an equilibrium state under the action of a parabolic load applied on opposite sides perpendicular to the OX axis **(Fig. 2)**. The rest of the sides are free of loads. This problem using the Airy's stress function and strain energy was solved in the work of [11, 27]. In our case, this problem was formulated in two ways, namely with differential equations for stresses (29)–(31) and strains (50)–(53) with appropriate boundary conditions. In this case, the boundary conditions have the form [14, 15]:

$$\begin{gathered} \text{for } x=\pm a: \sigma_{11} = S_0\left(1-\frac{y^2}{a^2}\right), \; \sigma_{12}=0, \\ \text{for } y=\pm b: \sigma_{22}=0, \; \sigma_{12}=0. \end{gathered} \tag{80}$$

Finite-difference equations corresponding to this boundary value problem (BVP) formulated relative to stress and strains are defined by equations (62)–(64), (70)–(73) and (74)–(79), respectively. The finite-difference equations are solved by the iterative method. The initial data had the following values:

$$\lambda = 0.78,\ \mu = 0.5,\ a = 1,\ b = 1,\ h_1 = h_2 = 0.2.$$

Table 1 shows the values of stresses σ_{11} in the section $x = 0$, obtained by solving a plane problem of the theory of elasticity in stresses and strains, respectively, and compared with the result [14, 15]. In **Table 1**, taking into account the symmetry conditions, the numerical results are given for one fourth of the rectangle. It took $k = 76$ iterations to obtain numerical results. The closeness of the obtained results shows the validity of the formulated boundary value problems in stresses and strains. **Fig. 3** shows the distribution of σ_{11} over the cross section $x = 0$ according to the results of the boundary value problem formulated with respect to stresses (III) and strains(I), as well as according to [14] (II). The yellow solid curve in the **Fig. 3** represents part of the parabolic load applied on the perpendicular sides of the rectangle.

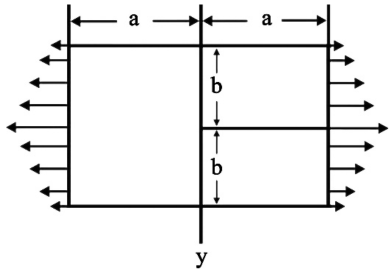

Fig. 2. Rectangular plate under parabolic load

Table 1
Stress values for $x = 0$

BVP	$y = 0$	$y = 0.2$	$y = 0.4$	$y = 0.6$	$y = 0.8$	$y = 1$
In strains ($k = 70$)	0.2161	0.4351	0.6153	0.8076	0.9229	0.9614
Timoshenko [14]	0.3404	0.5166	0.6536	0.7515	0.8102	0.8298
In stresses ($k = 76$)	0.2297	0.4493	0.6217	0.8160	0.9326	0.9714

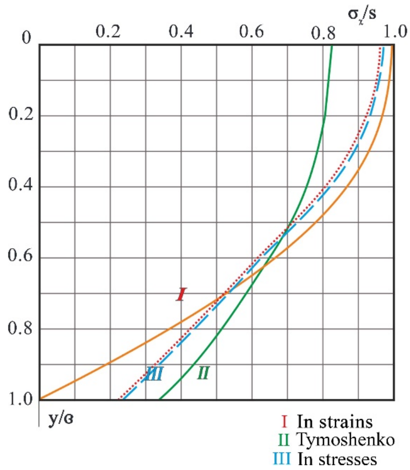

Fig. 3. Stress distribution over the cross-section $x = 0$ of a rectangular plate according to BVP in stresses (III) and strains (I), as well as according to [14] (II)

Let's now consider a rectangle of size $l_1 = a$, $l_2 = b$ in equilibrium state under the action of a uniformly distributed loads applied on opposite sides perpendicular to the OX-axis. In this case, the boundary conditions have the form:

$$x = \pm a: \quad \sigma_{11} = S_0 = 1, \quad \sigma_{12} = 0,$$
$$y = \pm b: \quad \sigma_{22} = 0, \qquad \sigma_{12} = 0. \tag{81}$$

Table 2 shows the numerical results for σ_{11} obtained by solving the equations (74)–(79) under boundary conditions (81), as well as the results of the boundary value problem in stresses (62)–(64), (70)–(73). The initial data for both boundary value problems had the following values:

$$\lambda = 0.78,\ \mu = 0.5,\ a = 0.5,\ b = 0.5,\ h_1 = h_2 = 0.1.$$

Table 2
Stress values

BVP	$y=0$	$y=0.1$	$y=0.2$	$y=0.3$	$y=0.4$	$y=0.5$
In strains ($k=85$)	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
In stresses ($k=73$)	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

A rectangle subjected to a uniformly distributed load applied on opposite sides behaves like a one-dimensional rod and the stress values are equal to the given load. It took 85 and 73 iterations to solve the boundary value problems in strains and stresses, respectively.

Let a boundary value problem in strains (51)–(56) be considered in a rectangle $l_1 \cdot l_2$. Consider the functions:

$$\varepsilon_{11} = y(y - l_2),\ \varepsilon_{22} = x(x - l_1),\ \varepsilon_{12} = xy(x - l_1)(y - l_2), \tag{82}$$

satisfying equations (51), with the following right-hand sides (body forces):

$$X_1 = -(x^2 - xl_1)(2y - l_2),\ X_2 = -(2x - l_1)(y^2 - yl_2),\ X_{12} = \frac{1}{2\mu}\left(\frac{\partial X_1}{\partial y} + \frac{\partial X_2}{\partial x}\right). \tag{83}$$

Then, according to (82), the boundary and additional boundary conditions have the form:

$$x = 0,\ l_1: \quad \varepsilon_{11} = y(y - l_2), \quad \varepsilon_{12} = 0, \quad \varepsilon_{22} = 0,$$
$$y = 0,\ l_2: \quad \varepsilon_{22} = x(x - l_1), \quad \varepsilon_{12} = 0, \quad \varepsilon_{11} = 0. \tag{84}$$

The first row of **Table 3** shows the values of the strain tensor component ε_{11} found by equations (74), (75) taking into account the boundary conditions (84). The second line is calculated according to the exact solution (82). The strain values given in the third line are calculated from the results of the boundary value problem in stresses (64), (68), (69), according to Hooke's law (54). The initial data has the following meanings:

$$\lambda = 0.78,\ \mu = 0.5,\ l_1 = 1,\ l_2 = 1,\ h_1 = h_2 = 0.1.$$

Table 3
Strain values

BVP	$x=0$	$x=0.1$	$x=0.2$	$x=0.3$	$x=0.4$	$x=0.5$
In strains ($k=74$)	–0.2491	–0.2493	–0.2493	–0.2495	–0.2496	–0.2500
Exact solution (63)	–0.2500	–0.2500	–0.2500	–0.2500	–0.2500	–0.2500
In stresses ($k=68$)	–0.2490	–0.2490	–0.2491	–0.2493	–0.2494	–0.2496

Let's note that the paper mainly presents numerical examples of solving linear boundary value problems and for rectangular domains. They can be continued taking into account temperature and plastic deformations.

4. Conclusions

In contrast to the well-known formulation of the boundary problem in stresses based on the Airy's function, a new version of the boundary value problem is formulated directly with respect to stresses. In the two-dimensional case, the boundary value problem consists of one (third) Beltrami-Mitchell equation and two equilibrium equations. In this case, it is important to view the equilibrium equation as additional boundary conditions.

Based on Beltrami-Mitchel type equations written for strains, a new version of boundary value problem in strains is formulated. The boundary value problem, in plane strain case, consists of one strain compatibility equation, and two equilibrium equations expressed with respect to strains. The equilibrium equation is also considered as an additional condition on the border.

For two-dimensional boundary value problems formulated with respect to stresses and strains by the finite-difference method, symmetric grid equations are constructed and written in a convenient form that ensures the convergence of the iterative method.

A number of problems on the equilibrium of a rectangular plate under the action of various loads applied on opposite sides have been solved numerically. The reliability of the results is ensured by comparing the numerical results of boundary value problems in stresses and strains, with the exact solution, as well as with the known solutions of the problem of plate tension with parabolic and uniformly distributed loads.

Numerical results of boundary value problems in stresses and strains are more accurate compared to boundary value problems formulated using Airy functions, due to the absence or insignificance of the numerical differentiation errors.

Conflict of interest

The authors declare that they have no conflict of interest in relation to this research, whether financial, personal, authorship or otherwise, that could affect the research and its results presented in this paper.

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Data availability

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