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QUADRINOMIAL-LIKE VERSIONS FOR WOLSTENHOLME, MORLEY AND GLAISHER CONGRUENCES

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Abstract

The quadrinomial coefficient is defined as the coefficient of x^k in the polynomial expansion of $\left(1+x+x^2+x^3\right)^n$, where n and k are nonnegative integers. In the present paper, we derive some congruences involving the quadrinomial coefficients. For instance, we establish two congruences that are analogous to those of Morley and Wolstenholme.

1. Introduction

Let $s \geq 1$, $n \geq 0$ and k be integers. The bi^{s}nomial coefficient denoted by $\binom{n}{k}_s$ is defined as

$$\binom{n}{k}_s := \begin{cases} \left[t^k\right]\left(1+t+\cdots+t^s\right)^n, & \text{for } 0 \leq k \leq sn, \\ 0, & \text{for } k<0 \text{ or } k>sn, \end{cases} \tag{1}$$

where $\left[t^k\right] f(t)$ denotes the coefficient of t^k in the formal power series $f(t)$; see Belbachir et al. [3] and Comtet [6, p. 77]. The study of bi^{s}nomial coefficients dates back to de Moivre [7] and Euler [9]. Combinatorially, the bi^{s}nomial coefficient $\binom{n}{k}_s$ counts the number of different ways of distributing k objects among n cells where each cell contains at most s objects [10]. For $s=1$, one obtains the binomial coefficient $\binom{n}{k}_1 = \binom{n}{k}$. Some known properties of the bi^{s}nomial coefficients are as follows:

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- Symmetry relation
$$\binom{n}{k}_s = \binom{n}{sn-k}_s;$$
- De Moivre alternating summation
$$\binom{n}{k}_s = \sum_{i=0}^{s} \binom{n-1}{k-i}_s;$$
- The explicit form of the bi^{s}nomial coefficient in term of the binomial coefficients
$$\binom{n}{k}_s = \sum_{j_1+j_2+\cdots+j_s=k} \binom{n}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-1}}{j_s}.$$

Throughout the paper we consider p as an odd prime number.

In 1819, Babbage [2] established that
$$\frac{1}{2}\binom{2p}{p} = \binom{2p-1}{p-1} \equiv 1 \pmod{p^2},$$
and in 1862 Wolstenholme [24] showed the same result modulo p^3, for $p \geq 5$.
In 1895, Morley [18] showed that, for $p \geq 5$, it holds that
$$\binom{p-1}{(p-1)/2} \equiv (-1)^{(p-1)/2}\, 4^{p-1} \pmod{p^3}. \tag{2}$$
In 1900, Glaisher [11] proved that
$$\binom{np-1}{p-1} \equiv 1 - p^3 \frac{n(n-1)}{3} B_{p-3} \pmod{p^4}, \tag{3}$$
where $n \geq 1$ is an integer, and B_n is the n-th Bernoulli number given by the following generating function:
$$\frac{t}{\exp(t)-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!},\ 0 < |t| < 2\pi.$$
In 1949, Ljunggren [5] extended the Wolstenholme congruence to the following:
$$\binom{np}{mp} \equiv \binom{n}{m} \pmod{p^3}, \tag{4}$$
where n and m are positive integers.

When we set $s = 2$ in (1), we obtain the trinomial coefficients (A027907, OEIS [21]). The study of congruence properties of trinomial coefficients has recently been expanding. Apagodu and Liu [1, Theorem 1] demonstrated that
$$\binom{2p}{p}_2 \equiv 2 + \frac{2p^2}{3}\left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3},$$

where

$$B_m(t) = \sum_{k=0}^{m} \binom{m}{k} B_{m-k} t^k$$

is the Bernoulli polynomial, and $\left(\frac{n}{q}\right)$ is the Legendre symbol with q an odd prime number and n an integer, given by

$$\left(\frac{n}{q}\right) := \begin{cases} 0, & \text{if } q \text{ divides } n, \\ 1, & \text{if } n \text{ is a quadratic residue modulo } q, \\ -1, & \text{if } n \text{ is a quadratic nonresidue modulo } q. \end{cases}$$

Mao [16, Theorem 1.5] proved the following supercongruence:

$$\binom{np^r}{mp^r}_2 \equiv \binom{np^{r-1}}{mp^{r-1}}_2 \pmod{p^{r+1}},$$

where n, m, r are nonnegative integers with $r \geq 1$. Elkhiri and Mihoubi [8] showed that

$$\binom{np-1}{p-1}_2 \equiv \begin{cases} 1 + npq_p(3) \pmod{p^2}, & \text{if } p \equiv 1 \pmod 3, \\ -1 - npq_p(3) \pmod{p^2}, & \text{if } p \equiv 2 \pmod 3, \end{cases}$$

and

$$\binom{np-1}{(p-1)/2}_2 \equiv \begin{cases} 1 + np\left(2q_p(2) + \frac{1}{2}q_p(3)\right) \pmod{p^2}, & \text{if } p \equiv 1 \pmod 6, \\ -\frac{1}{2}pnq_p(3) \pmod{p^2}, & \text{if } p \equiv 5 \pmod 6, \end{cases}$$

where $q_p(x) := \dfrac{x^{p-1}-1}{p}$ is called the Fermat quotient and x is coprime with p. For more congruences involving the trinomial coefficients we refer the reader to Ömür et al. [19] and Sun [23]. Other congruences involving the bi^{s}nomial coefficients can be found in the work of Belbachir and Igueroufa [4].

For $s = 3$ in (1), we get the quadrinomial coefficients (A008287, OEIS [21]). For a nonnegative integer n, the central quadrinomial coefficient κ_n is the coefficient of x^{3n} in the expansion of $\left(1 + x + x^2 + x^3\right)^{2n}$ (A005721, OEIS [21]).

Motivated by the previous results, we study some congruence properties of quadrinomial coefficients. Our focus lies on those involving central quadrinomial coefficients.

We introduce our first congruence which is similar to the Wolstenholme congruence.

Theorem 1. *Let $p \geq 5$. Then*

$$\kappa_p \equiv 4 + 8p^2 \left(\frac{-1}{p}\right) E_{p-3} \pmod{p^3}, \tag{5}$$

where E_n is the n-th Euler number (A122045, OEIS [21]).

The second congruence is an analogue to the Morley congruence (2) modulo squares of primes.

Theorem 2. *Let $p \geq 5$. Then*

$$\kappa_{(p-1)/2} \equiv \left(\frac{-2}{p}\right) + p\left(q_p(2)\left(\frac{7}{2}\left(\frac{-2}{p}\right) - 3\left(\frac{-1}{p}\right)\right) - 2\left(\frac{2}{p}\right) A_p\right) \pmod{p^2}, \tag{6}$$

where

$$A_p := \frac{(-1)^{(p-1)/2} P_p - (-8)^{(p-1)/2}}{p},$$

in which $(P_n)_n$ is the Pell sequence (A000129, OEIS [21]).

The next congruence is inspired by the Glaisher congruence (3).

Theorem 3. *Let $p \geq 5$ and n be a positive integer. Then*

$$\binom{np-1}{p-1}_3 \equiv \frac{1}{2}\left(\left(\frac{-1}{p}\right) + 1\right) + pq_p(2)\frac{n}{4}\left(5\left(\frac{-1}{p}\right) + 3\right) \pmod{p^2}.$$

The rest of the paper is devoted to the proofs of the theorems stated above.

2. Proof of Theorem 1

The following lemmas are needed to prove Theorem 1.

Lemma 1. *Let n and k be nonnegative integers. Then*

$$\binom{n}{k}_3 = \sum_{j=0}^{\min(n,k)} \binom{n}{j}\binom{3n-2j}{k-j}(-2)^j. \tag{7}$$

Proof. Let $g(t) := 1 + t + t^2 + t^3$. One observes that

$$g(t) = (1+t)^3 - 2t(1+t),$$

and consequently

$$\begin{aligned}\binom{n}{k}_3 &= \left[t^k\right]\left((1+t)^3 - 2t\,(1+t)\right)^n \\ &= \sum_{j=0}^{n}\binom{n}{j}(-2)^j\left[t^{k-j}\right](1+t)^{3(n-j)+j} \\ &= \sum_{j=0}^{n}\binom{n}{j}(-2)^j\binom{3n-2j}{k-j}.\end{aligned}$$

□

Lemma 2. *If s and k be integers, where $0 \le k \le p-1$, then*

$$\binom{sp-1}{k} \equiv (-1)^k\,(1 - spH_k) \pmod{p^2},$$

where H_n is the n-th harmonic number given by $H_n := \sum_{j=1}^{n} 1/j$ for $n \ge 1$ and $H_0 = 0$.

Proof. We have

$$\binom{sp-1}{k} = \prod_{j=1}^{k}\frac{sp-j}{j} = (-1)^k\prod_{j=1}^{k}\left(1 - \frac{sp}{j}\right),$$

and thus

$$\binom{sp-1}{k} \equiv (-1)^k\,(1 - spH_k) \pmod{p^2},$$

as claimed. □

Lemma 3. *Let $0 \le k \le p-2$ be an integer. Then*

$$H_{p-1-k} \equiv H_k \pmod{p}. \tag{8}$$

Proof. We have

$$H_{p-1-k} = \sum_{j=1}^{p-1-k}\frac{1}{j} = \sum_{j=1}^{p-1}\frac{1}{j} - \sum_{j=1}^{k}\frac{1}{p-k-1+j}.$$

For any integer x coprime with p, we have $1/(p+x) \equiv 1/x \pmod{p}$. Then

$$H_{p-1-k} \equiv H_{p-1} + \sum_{j=1}^{k}\frac{1}{k+1-j} \pmod{p}.$$

Knowing that $H_{p-1} \equiv 0 \pmod{p^2}$ (see [24]), we obtain (8). □

Lemma 4. *Let n, m, k, i be nonnegative integers with $0 \le k, i \le p-1$. We have*

$$\binom{np+2k}{mp+k} \equiv \binom{n}{m}\binom{2k}{k}\left(1+np\left(H_{2k}-H_k\right)\right) \pmod{p^2}. \tag{9}$$

If $i > k$, then

$$\binom{np+k}{mp+i} \equiv (n-m)\,p\binom{n}{m}\frac{(-1)^{i-k-1}}{\binom{i}{k}(i-k)} \pmod{p^2}. \tag{10}$$

Proof. We have

$$\binom{np+2k}{mp+k} = \binom{np}{mp}\frac{(np+1)(np+2)\cdots(np+2k)}{(mp+1)\cdots(mp+k)\left((n-m)p+1\right)\cdots\left((n-m)p+k\right)}.$$

Set

$$f(x) := \frac{(nx+1)(nx+2)\cdots(nx+2k)}{(mx+1)\cdots(mx+k)\left((n-m)x+1\right)\cdots\left((n-m)x+k\right)}.$$

Note that

$$f(0) = \binom{2k}{k}.$$

Also, we have

$$\log(f(x)) = \sum_{j=1}^{2k}\log(nx+j) - \sum_{j=1}^{k}\log(mx+j) - \sum_{j=1}^{k}\log((n-m)x+j),$$

by deriving both sides with respect to x we get

$$\frac{f'(x)}{f(x)} = \sum_{j=1}^{2k}\frac{n}{nx+j} - \sum_{j=1}^{k}\frac{m}{mx+j} - \sum_{j=1}^{k}\frac{n-m}{(n-m)x+j},$$

and thus

$$f'(0) = \binom{2k}{k}n\left(H_{2k}-H_k\right).$$

The Ljunggren congruence (4) and the taylor expansion of $f(p)$ yields (9). Similarly, we obtain (10). □

Lemma 5 ([20])**.** *Let $p \ge 5$. Then*

$$\left(\sum_{k=1}^{p-1}\binom{2k}{k}\frac{1}{2^k k}\right)^2 \equiv 4\sum_{k=1}^{p-1}\binom{2k}{k}\frac{1}{2^k k}\left(H_{2k-1}-H_k\right) \pmod{p}, \tag{11}$$

$$\sum_{k=1}^{p-1}\binom{2k}{k}\frac{1}{2^k k}\equiv q_p(2)\pmod{p}, \tag{12}$$

$$\sum_{k=1}^{p-1}\binom{2k}{k}\frac{1}{2^k k^2}\equiv\left(\frac{-1}{p}\right)E_{p-3}-\frac{q_p^2(2)}{2}\pmod{p}, \tag{13}$$

$$\sum_{k=1}^{p-1}\binom{2k}{k}\frac{H_k}{2^k k}\equiv\left(\frac{-1}{p}\right)E_{p-3}\pmod{p}. \tag{14}$$

Lemma 6. *Let $p\geq 5$. Then*

$$\sum_{k=1}^{p-1}\binom{2k}{k}\frac{1}{2^k k}\left(H_{2k}-H_k\right)\equiv\frac{1}{2}\left(\frac{-1}{p}\right)E_{p-3}\pmod{p} \tag{15}$$

and

$$\sum_{k=1}^{p-1}\binom{2k}{k}\frac{1}{2^k k}H_{k-1}\equiv\frac{1}{2}q_p^2(2)\pmod{p}. \tag{16}$$

Proof. We have

$$\sum_{k=1}^{p-1}\binom{2k}{k}\frac{1}{2^k k}\left(H_{2k}-H_k\right)=\sum_{k=1}^{p-1}\binom{2k}{k}\frac{1}{2^k k}\left(H_{2k-1}-H_k\right)+\frac{1}{2}\sum_{k=1}^{p-1}\binom{2k}{k}\frac{1}{2^k k^2}.$$

Applying (11), (12) and (13), we get the desired (15). From (14) and (13) one obtains (16). □

Proof of Theorem 1. Let $p\geq 5$. From (7) set $n=2p$ and $k=3p$, as κ_p is the coefficient of x^{3p} in the expansion of $\left(1+x+x^2+x^3\right)^{2p}$, we have

$$\kappa_p=\sum_{j=0}^{2p}\binom{2p}{j}\binom{6p-2j}{3p-j}(-2)^j.$$

Suppose that $k=3p-j$, then

$$\begin{aligned}\kappa_p&=\sum_{k=p}^{3p}\binom{2p}{3p-k}\binom{2k}{k}(-2)^{3p-k}\\&=\sum_{k=p}^{2p-1}\binom{2p}{3p-k}\binom{2k}{k}(-2)^{3p-k}+\sum_{k=2p}^{3p-1}\binom{2p}{3p-k}\binom{2k}{k}(-2)^{3p-k}+\binom{6p}{3p}\\&=\sum_{k=1}^{p-1}\binom{2p}{k}\binom{2k+2p}{k+p}(-2)^{2p-k}+\sum_{k=1}^{p-1}\binom{2p}{p-k}\binom{2k+4p}{k+2p}(-2)^{p-k}\\&\quad+\binom{2p}{p}\binom{4p}{2p}(-2)^p+\binom{2p}{p}(-2)^{2p}+\binom{6p}{3p}.\end{aligned}$$

Now, let

$$\Sigma_1 := \sum_{k=1}^{p-1} \binom{2p}{k} \binom{2k+2p}{k+p} (-2)^{2p-k}$$

and

$$\Sigma_2 := \sum_{k=1}^{p-1} \binom{2p}{p-k} \binom{2k+4p}{k+2p} (-2)^{p-k}.$$

From Lemma 2, for $1 \le k \le p-1$, we have

$$\binom{2p}{k} = \frac{2p}{k} \binom{2p-1}{k-1} \equiv \frac{2p}{k} (-1)^{k-1} (1 - 2pH_{k-1}) \pmod{p^3}, \tag{17}$$

and applying (9) and (17), we get

$$\begin{aligned}
\Sigma_1 \equiv\ & 4p \sum_{k=1}^{p-1} \binom{2k}{k} \frac{(-2)^{2p-k}}{k} (-1)^{k-1} + 8p^2 \sum_{k=1}^{p-1} \binom{2k}{k} \frac{(-2)^{2p-k}}{k} (-1)^{k-1} (H_{2k} - H_k) \\
& - 8p^2 \sum_{k=1}^{p-1} \binom{2k}{k} \frac{(-2)^{2p-k}}{k} (-1)^{k-1} H_{k-1} \pmod{p^3} \\
\equiv & - 2^{2p+2} p \sum_{k=1}^{p-1} \binom{2k}{k} \frac{1}{2^k k} - 2^{2p+3} p^2 \sum_{k=1}^{p-1} \binom{2k}{k} \frac{1}{2^k k} (H_{2k} - H_k) \\
& + 2^{2p+3} p^2 \sum_{k=1}^{p-1} \binom{2k}{k} \frac{1}{2^k k} H_{k-1} \pmod{p^3}.
\end{aligned}$$

In view of the Lehmer congruence [14],

$$H_{(p-1)/2} \equiv -2q_p(2) + pq_p(2)^2 \pmod{p^2}, \tag{18}$$

and Mao [15, Theorem 1.1], we have

$$\sum_{k=1}^{p-1} \binom{2k}{k} \frac{1}{2^k k} \equiv q_p(2) - p\frac{q_p(2)^2}{2} \pmod{p^2}. \tag{19}$$

Fermat's Little Theorem states that $b^{p-1} \equiv 1 \pmod{p}$ for all integer b coprime with p. Using this fact, and in view of Lemma 6 and (19), we conclude that

$$\Sigma_1 \equiv -2^{2p+2} pq_p(2) + p^2 \left(24q_p(2)^2 - 16 \left(\frac{-1}{p}\right) E_{p-3} \right) \pmod{p^3}. \tag{20}$$

Again, by (9) and (17), we find that

$$\begin{aligned}\Sigma_2 \equiv &-2p\binom{4}{2}\sum_{k=1}^{p-1}\binom{2k}{k}\frac{2^{p-k}\left(1-2pH_{p-k-1}\right)\left(1+4p\left(H_{2k}-H_k\right)\right)}{p-k} \pmod{p^3}\\ \equiv &-12p\sum_{k=1}^{p-1}\binom{2k}{k}\frac{2^{p-k}}{p-k}-48p^2\sum_{k=1}^{p-1}\binom{2k}{k}\frac{2^{p-k}}{p-k}\left(H_{2k}-H_k\right)\\ &+24p^2\sum_{k=1}^{p-1}\binom{2k}{k}\frac{2^{p-k}}{p-k}H_{p-k-1} \pmod{p^3}.\end{aligned}$$

Since, for $k=1,2,\ldots,p-1$, we have

$$\frac{1}{p-k}\equiv -\frac{1}{k}-p\frac{1}{k^2} \pmod{p^2}.$$

It follows that

$$\begin{aligned}\sum_{k=1}^{p-1}\binom{2k}{k}\frac{2^{p-k}}{p-k} &\equiv -\sum_{k=1}^{p-1}\binom{2k}{k}2^{p-k}\left(\frac{1}{k}+p\frac{1}{k^2}\right) \pmod{p^2}\\ &= -\sum_{k=1}^{p-1}\binom{2k}{k}\frac{2^{p-k}}{k}-p\sum_{k=1}^{p-1}\binom{2k}{k}\frac{2^{p-k}}{k^2} \pmod{p^2}.\end{aligned}$$

Now, using (19) and (13), we obtain

$$\sum_{k=1}^{p-1}\binom{2k}{k}\frac{2^{p-k}}{p-k}\equiv -2^pq_p(2)-2p\left(\left(\frac{-1}{p}\right)E_{p-3}-q_p(2)^2\right) \pmod{p^2}.$$

Furthermore, by (15), we find

$$\begin{aligned}\sum_{k=1}^{p-1}\binom{2k}{k}\frac{2^{p-k}}{p-k}\left(H_{2k}-H_k\right) &\equiv -\sum_{k=1}^{p-1}\binom{2k}{k}\frac{2^{p-k}}{k}\left(H_{2k}-H_k\right) \pmod{p}\\ &\equiv -\left(\frac{-1}{p}\right)E_{p-3} \pmod{p}.\end{aligned}$$

Applying (8) and by (14), we get

$$\begin{aligned}\sum_{k=1}^{p-1}\binom{2k}{k}\frac{2^{p-k}}{p-k}H_{p-k-1} &\equiv -2\sum_{k=1}^{p-1}\binom{2k}{k}\frac{2^{-k}}{k}H_k \pmod{p}\\ &\equiv -2\left(\frac{-1}{p}\right)E_{p-3} \pmod{p}.\end{aligned}$$

Therefore,

$$\Sigma_2 \equiv p12 \cdot 2^p q_p(2) + 24p^2 \left(\left(\frac{-1}{p} \right) E_{p-3} - q_p(2)^2 \right) \pmod{p^3}. \tag{21}$$

Using the Ljunggren congruence (4) and combining (20) and (21), yields that

$$\kappa_p \equiv 2^{2p+1} - 3 \cdot 2^{p+2} + 20 + pq_p(2) \left(12 \cdot 2^p - 2^{2p+2} \right) + 8p^2 \left(\frac{-1}{p} \right) E_{p-3} \pmod{p^3}.$$

Finally, using the fact that $2^{p-1} = 1 + pq_p(2)$, we complete the proof of (5). □

3. Proof of Theorem 2

Let us start with the following lemma.

Lemma 7. *Let $p \geq 5$ with $1 \leq k \leq (p-1)/2$ be an integer. Then*

$$\binom{p-1+2k}{(p-1)/2+k} \equiv p \frac{4^{2k}(-1)^{(p-1)/2}}{2k\binom{2k}{k}} \pmod{p^2}.$$

Proof. We have

$$\binom{p-1+2k}{(p-1)/2+k} = \binom{p-1}{(p-1)/2} \frac{\prod_{i=1}^{2k}(p-1+i)}{\prod_{i=1}^{k}((p-1)/2+i)^2} = p\binom{p-1}{(p-1)/2} \frac{2^{2k}\prod_{i=1}^{2k-1}(p+i)}{\prod_{i=1}^{k}(p+2i-1)^2}.$$

We can readily get

$$\frac{\prod_{i=1}^{2k-1}(p+i)}{\prod_{i=1}^{k}(p+2i-1)^2} \equiv \frac{2^{2k}}{2k\binom{2k}{k}} \pmod{p}.$$

Finally, by the Morley congruence (2), the result is obtained. □

Lemma 8. *Let a be an integer coprime with p. Then*

$$a^{(p-1)/2} \equiv \left(\frac{a}{p} \right) \left(1 + \frac{1}{2} pq_p(a) - \frac{1}{8} p^2 q_p(a)^2 \right) \pmod{p^3}. \tag{22}$$

Proof. See, [17, Lemma 4.1]. □

Remark 1. It is well-known that the Legendre symbol satisfies:

$$\left(\frac{a}{p} \right) \equiv a^{(p-1)/2} \pmod{p}.$$

Lemma 9. *If $0 \le k \le (p-1)/2$ be an integer, then*
$$\binom{(p-1)/2+k}{2k} \equiv \frac{1}{(-16)^k}\binom{2k}{k} \pmod{p^2}.$$

Proof. We have
$$\begin{aligned}\binom{(p-1)/2+k}{2k} &= \frac{\prod_{j=1}^{k}\left(p^2-(2j-1)^2\right)}{4^k(2k)!}\\ &\equiv \frac{\prod_{j=1}^{k}\left(-(2j-1)^2\right)}{4^k(2k)!} \pmod{p^2}\\ &= \frac{1}{(-16)^k}\binom{2k}{k} \pmod{p^2}.\end{aligned}$$

□

Lemma 10. *Let $0 \le k \le (p-3)/2$ be an integer. Then*
$$H_{(p+1)/2+k} \equiv -2q_p(2) + 2H_{2k+1} - H_k \pmod{p}.$$

Proof. We have
$$H_{(p+1)/2+k} = H_{(p-1)/2} + \sum_{j=(p+1)/2}^{(p+1)/2+k}\frac{1}{j} = H_{(p-1)/2} + \sum_{j=0}^{k}\frac{1}{j+(p+1)/2}.$$
By (18), we get
$$\begin{aligned}H_{(p+1)/2+k} &\equiv -2q_p(2) + 2\sum_{j=0}^{k}\frac{1}{2j+1} \pmod{p}\\ &= -2q_p(2) + 2H_{2k+1} - H_k \pmod{p},\end{aligned}$$
as claimed. □

Lemma 11. *We have*
$$\sum_{k=0}^{(p-1)/2}\binom{2k}{k}2^{-k} \equiv (-1)^{(p-1)/2}(1+2p\chi_p) \pmod{p^2},$$
in which $\chi_p := P_{p-\left(\frac{2}{p}\right)}/p$ is known as the Pell quotient.

Proof. From Lemma 9, we get
$$\begin{aligned}\sum_{k=0}^{(p-1)/2}\binom{2k}{k}2^{-k} &\equiv \sum_{k=0}^{(p-1)/2}\binom{(p-1)/2+k}{2k}(-8)^k \pmod{p^2}\\ &= \sum_{k=0}^{(p-1)/2}\binom{p-1-k}{k}(-8)^{(p-1)/2-k} \pmod{p^2}.\end{aligned}$$

Set

$$a_n := (-1)^n \sum_{k=0}^{n} \binom{2n-k}{k} (-8)^{n-k},$$

where n is a nonnegative integer. One can observe that $(a_n)_n$ is A002315 in OEIS [21], which satisfies

$$a_n = \sum_{k=0}^{n} \binom{2n+1}{2k} 2^k.$$

Set $n = (p-1)/2$, we obtain

$$\begin{aligned}\sum_{k=0}^{(p-1)/2} \binom{2k}{k} 2^{-k} &\equiv (-1)^{(p-1)/2} \sum_{k=0}^{(p-1)/2} \binom{p}{2k} 2^k \pmod{p^2}\\ &= (-1)^{(p-1)/2} \left(1 + p \sum_{k=1}^{(p-1)/2} \binom{p-1}{2k-1} \frac{2^k}{2k}\right) \pmod{p^2}.\end{aligned}$$

Applying Lemma 2, we arrive at

$$\sum_{k=0}^{(p-1)/2} \binom{2k}{k} 2^{-k} \equiv (-1)^{(p-1)/2} \left(1 - p \sum_{k=1}^{(p-1)/2} \frac{2^k}{2k}\right) \pmod{p^2}.$$

Using the following congruence [22, Corollary 1]:

$$\sum_{k=1}^{(p-1)/2} \frac{2^k}{k} \equiv -4 \frac{P_{p-\left(\frac{2}{p}\right)}}{p} \pmod{p},$$

the proof is done. □

Lemma 12. *We have*

$$\sum_{k=0}^{(p-3)/2} \binom{2k}{k} 2^{(p-1)/2-k} H_k \equiv q_p(2) \left(2(-1)^{(p-1)/2} - (-2)^{(p-1)/2}\right) \pmod{p}. \quad (23)$$

Proof. Since $\binom{2k}{k} \equiv 0 \pmod{p}$, for $k = (p+1)/2, \ldots, p-1$, we get

$$\begin{aligned}\sum_{k=0}^{(p-3)/2} \binom{2k}{k} 2^{(p-1)/2-k} H_k \equiv\ & 2^{(p-1)/2} \sum_{k=0}^{p-1} \binom{2k}{k} 2^{-k} H_k\\ & - \binom{p-1}{(p-1)/2} H_{(p-1)/2} \pmod{p}.\end{aligned}$$

From [20, Theorem 1], for $x = 1/2$, we have

$$\sum_{k=0}^{p-1} \binom{2k}{k} 2^{-k} H_k \equiv -q_p(2)(-1)^{(p-1)/2} \pmod{p},$$

and by the Morley congruence (2) and (18), this yields the desired (23). □

Proof of Theorem 2. Let $p \geq 5$. From (7), we have

$$\kappa_{(p-1)/2} = \sum_{j=0}^{p-1} \binom{p-1}{j} \binom{3(p-1)-2j}{3(p-1)/2-j} (-2)^j .$$

Letting $k = 3(p-1)/2 - j$, we get

$$\begin{aligned}
\kappa_{(p-1)/2} &= \sum_{k=(p-1)/2}^{3(p-1)/2} \binom{p-1}{3(p-1)/2-k} \binom{2k}{k} (-2)^{3(p-1)/2-k} \\
&= \sum_{k=1}^{p-1} \binom{p-1}{k} \binom{p-1+2k}{(p-1)/2+k} (-2)^{p-1-k} + \binom{p-1}{(p-1)/2} (-2)^{p-1} .
\end{aligned}$$

Set

$$\Omega_1 := \sum_{k=1}^{(p-1)/2} \binom{p-1}{k} \binom{p-1+2k}{(p-1)/2+k} (-2)^{p-1-k}$$

and

$$\Omega_2 := \sum_{k=0}^{(p-3)/2} \binom{p-1}{(p+1)/2+k} \binom{2k+2p}{k+p} (-2)^{(p-3)/2-k} .$$

Applying Lemma 7 and Lemma 2, we obtain

$$\begin{aligned}
\Omega_1 &\equiv p(-1)^{(p-1)/2} \sum_{k=1}^{(p-1)/2} (-1)^k \frac{4^{2k}}{2k\binom{2k}{k}} (-2)^{p-1-k} \pmod{p^2} \\
&= p(-1)^{(p-1)/2} \sum_{k=1}^{(p-1)/2} \frac{2^{3k}}{2k\binom{2k}{k}} \pmod{p^2}.
\end{aligned}$$

For $k = 0, 1, \ldots, (p-1)/2$, we have

$$\binom{(p-1)/2}{k} \equiv \binom{-1/2}{k} = \frac{1}{(-4)^k} \binom{2k}{k} \pmod{p}.$$

It follows that

$$\sum_{k=1}^{(p-1)/2} \frac{2^{3k}}{2k\binom{2k}{k}} \equiv \sum_{k=1}^{(p-1)/2} \frac{(-2)^k}{2k\binom{(p-1)/2}{k}} \pmod{p}.$$

Note that $k\binom{n}{k} = n\binom{n-1}{k-1}$ and by [12, (2.4)], we get

$$\sum_{k=1}^{(p-1)/2} \frac{2^{3k}}{2k\binom{2k}{k}} \equiv 2^{(p-1)/2} \left(\sum_{k=1}^{(p-1)/2} \frac{1}{2^k k} + \sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k} \right) \pmod{p}.$$

Considering the following two congruences (see [22]):

$$\sum_{k=1}^{(p-1)/2} \frac{1}{2^k k} \equiv -2\chi_p + q_p(2) \pmod{p}$$

and

$$\sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k} \equiv -q_p(2) \pmod{p},$$

we obtain

$$\sum_{k=1}^{(p-1)/2} \frac{2^{3k}}{2k\binom{2k}{k}} \equiv -2^{(p+1)/2}\chi_p \pmod{p}. \tag{24}$$

Thus,

$$\Omega_1 \equiv p\,(-2)^{(p+1)/2}\,\chi_p \pmod{p^2}. \tag{25}$$

Now, by (9) and Lemma 2, we obtain

$$\begin{aligned}\Omega_2 \equiv 2 &\sum_{k=0}^{(p-3)/2} \binom{2k}{k} (-1)^{(p+1)/2+k} \left(1 - pH_{(p+1)/2+k}\right) \times \\ &(1 + 2p\,(H_{2k} - H_k))\,(-2)^{(p-3)/2-k} \pmod{p^2} \\ \equiv &\sum_{k=0}^{(p-3)/2} \binom{2k}{k} 2^{(p-1)/2-k} + 2p \sum_{k=0}^{(p-3)/2} \binom{2k}{k} 2^{(p-1)/2-k}\,(H_{2k} - H_k) \\ &- p \sum_{k=0}^{(p-3)/2} \binom{2k}{k} 2^{(p-1)/2-k} H_{(p+1)/2+k} \pmod{p^2}.\end{aligned}$$

From Lemma 10, we find

$$\sum_{k=0}^{(p-3)/2}\binom{2k}{k}2^{(p-1)/2-k}H_{(p+1)/2+k}$$
$$\equiv -2q_p(2)\sum_{k=0}^{(p-3)/2}\binom{2k}{k}2^{(p-1)/2-k}+\sum_{k=0}^{(p-3)/2}\binom{2k}{k}2^{(p-1)/2-k}\left(H_{2k}-H_k\right)$$
$$+\sum_{k=0}^{(p-3)/2}\binom{2k}{k}2^{(p-1)/2-k}\left(H_{2k}+\frac{2}{2k+1}\right)\pmod{p},$$

and hence

$$\Omega_2\equiv\sum_{k=0}^{(p-3)/2}\binom{2k}{k}2^{(p-1)/2-k}-p\sum_{k=0}^{(p-3)/2}\binom{2k}{k}2^{(p-1)/2-k}H_k$$
$$+2pq_p(2)\sum_{k=0}^{(p-3)/2}\binom{2k}{k}2^{(p-1)/2-k}-2p\sum_{k=0}^{(p-3)/2}\binom{2k}{k}\frac{2^{(p-1)/2-k}}{2k+1}\pmod{p^2}.$$

From Lemma 11 and by the Morley congruence (2), we get

$$\sum_{k=0}^{(p-3)/2}\binom{2k}{k}2^{(p-1)/2-k}\equiv(-2)^{(p-1)/2}\left(1+2p\chi_p\right)-(-1)^{(p-1)/2}4^{p-1}\pmod{p^2},$$

which implies that

$$\sum_{k=0}^{(p-3)/2}\binom{2k}{k}2^{(p-1)/2-k}\equiv(-2)^{(p-1)/2}-(-1)^{(p-1)/2}\pmod{p}.$$

From [13, Theorem 2], we have

$$\sum_{k=0}^{(p-3)/2}\binom{2k}{k}\frac{2^{(p-1)/2-k}}{2k+1}\equiv 2^{(p-1)/2}A_p\pmod{p},$$

and by (23), we get

$$\Omega_2\equiv(-2)^{(p-1)/2}-(-1)^{(p-1)/2}4^{p-1}+p\left(q_p(2)\left(3(-2)^{(p-1)/2}-4(-1)^{(p-1)/2}\right)\right.$$
$$\left.-(-2)^{(p+1)/2}\chi_p-2^{(p+1)/2}A_p\right)\pmod{p^2}. \tag{26}$$

Combining (25) and (26) and using the Morley congruence (2), we conclude that

$$\kappa_{(p-1)/2}\equiv(-2)^{(p-1)/2}-(-1)^{(p-1)/2}4^{p-1}\left(1-2^{p-1}\right)$$
$$+p\left(q_p(2)\left(3(-2)^{(p-1)/2}-4(-1)^{(p-1)/2}\right)-2^{(p+1)/2}A_p\right)\pmod{p^2}.$$

Knowing that $1-2^{p-1}=-pq_p(2)$, and by (22), we finally obtain (6). □

4. Proof of Theorem 3

In order to prove Theorem 3, the following lemma is needed.

Lemma 13. *We have*

$$\sum_{k=0}^{(p-1)/2} \binom{2k}{k} \frac{H_{2k}}{2^k} \equiv \left(\frac{-1}{p}\right) \left(2\chi_p - \frac{1}{2} q_p(2)\right) \pmod{p}. \tag{27}$$

Proof. From Lemma 2, we have

$$p \sum_{k=0}^{(p-1)/2} \binom{2k}{k} 2^{-k} H_{2k} \equiv \sum_{k=0}^{(p-1)/2} \binom{2k}{k} 2^{-k} - \sum_{k=0}^{(p-1)/2} \binom{p-1}{2k} \binom{2k}{k} 2^{-k} \pmod{p^2}.$$

Set

$$b := \sum_{k=0}^{(p-1)/2} \binom{p-1}{2k} \binom{2k}{k} 2^{-k} = \sum_{k=0}^{(p-1)/2} \binom{p-1}{k} \binom{p-1-k}{k} 2^{-k}.$$

Again, by Lemma 2, we get

$$b \equiv \sum_{k=0}^{(p-1)/2} \binom{p-1-k}{k} (-2)^{-k} - p \sum_{k=0}^{(p-1)/2} \binom{p-1-k}{k} (-2)^{-k} H_k \pmod{p^2}.$$

We have

$$\binom{p-1-k}{k} \equiv \binom{-1-k}{k} \equiv \binom{2k}{k} (-1)^k \pmod{p},$$

which gives

$$\sum_{k=0}^{(p-1)/2} \binom{p-1-k}{k} (-1)^k H_k 2^{-k} \equiv \sum_{k=0}^{(p-1)/2} \binom{2k}{k} H_k 2^{-k} \pmod{p},$$

and from Lemma 12, we get

$$\sum_{k=0}^{(p-1)/2} \binom{p-1-k}{k} (-1)^k H_k 2^{-k} \equiv -\left(\frac{-1}{p}\right) q_p(2) \pmod{p}.$$

In view of Lemma 9, we find

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \binom{p-1-k}{k} (-1)^k 2^{-k} &= \sum_{k=0}^{(p-1)/2} \binom{(p-1)/2+k}{2k} (-2)^{(p-1)/2+k} \\ &\equiv (-2)^{(p-1)/2} \sum_{k=0}^{(p-1)/2} \binom{2k}{k} \frac{1}{8^k} \pmod{p^2}. \end{aligned}$$

Using [13, Theorem 2], for $t=1/8$, we get

$$\sum_{k=0}^{(p-1)/2}\binom{2k}{k}\frac{1}{8^k}\equiv\left(\frac{2}{p}\right)\pmod{p^2},$$

and hence

$$\sum_{k=0}^{(p-1)/2}\binom{p-1-k}{k}(-1)^k\,2^{-k}\equiv(-2)^{(p-1)/2}\left(\frac{2}{p}\right)\pmod{p^2}.$$

By (22), we obtain

$$\sum_{k=0}^{(p-1)/2}\binom{p-1-k}{k}(-1)^k\,2^{-k}\equiv\left(\frac{-1}{p}\right)\left(1-\frac{1}{2}pq_p\left(2\right)\right)\pmod{p^2}.$$

Thus,

$$b\equiv\left(\frac{-1}{p}\right)\left(1+\frac{1}{2}pq_p\left(2\right)\right)\pmod{p^2}.$$

Finally, using Lemma 11, we arrive at (27). □

Proof of Theorem 3. From (7), we have

$$\begin{aligned}\binom{np-1}{p-1}_3 &=\sum_{j=0}^{p-1}\binom{np-1}{j}\binom{3(np-1)-2j}{p-1-j}(-2)^j\\ &=\sum_{j=0}^{p-1}\binom{np-1}{p-1-j}\binom{(3n-1)\,p+2j-1}{j}(-2)^{p-1-j}\\ &=\sum_{j=1}^{(p-1)/2}\binom{np-1}{p-1-j}\binom{(3n-1)\,p+2j-1}{j}(-2)^{p-1-j}\\ &\quad+\sum_{j=(p+1)/2}^{p-1}\binom{np-1}{p-1-j}\binom{(3n-1)\,p+2j-1}{j}(-2)^{p-1-j}\\ &\quad+\binom{np-1}{p-1}(-2)^{p-1}.\end{aligned}$$

Set

$$\Theta_1:=\sum_{j=1}^{(p-1)/2}\binom{np-1}{p-1-j}\binom{(3n-2)\,p+2j-1}{j}(-2)^{p-1-j}.$$

Observe that

$$\binom{(3n-2)\,p+2j-1}{j} = \frac{(3n-2)\,p+2j}{(3n-2)\,p+j}\binom{(3n-2)\,p+2j}{j}.$$

In view of (9), we have

$$\binom{(3n-2)\,p+2j-1}{j} \equiv \frac{1}{2}\binom{2j}{j}\left(1+(3n-2)\,p\left(H_{2j-1}-H_{j-1}\right)\right) \pmod{p^2}.$$

Applying Lemma 2 and (8), we get

$$\begin{aligned}\binom{np-1}{p-1-j} &\equiv (-1)^{p-1-j}\left(1-npH_{p-1-j}\right) \pmod{p^2}\\ &\equiv (-1)^{j}\left(1-npH_{j}\right) \pmod{p^2}.\end{aligned}$$

Which implies that

$$\begin{aligned}\Theta_1 \equiv 2^{p-2}\sum_{j=1}^{(p-1)/2}\binom{2j}{j}\frac{1}{2^j}+\frac{1}{2}p\Bigg((3n-2)\sum_{j=1}^{(p-1)/2}\binom{2j}{j}\frac{\left(H_{2j-1}-H_{j-1}\right)}{2^j}\\ -n\sum_{j=1}^{(p-1)/2}\binom{2j}{j}\frac{H_j}{2^j}\Bigg) \pmod{p^2}.\end{aligned} \tag{28}$$

In view of Lemma 12 and (12), we find

$$\begin{aligned}\sum_{j=1}^{(p-1)/2}\binom{2j}{j}\frac{H_{j-1}}{2^j} &= \sum_{j=1}^{(p-1)/2}\binom{2j}{j}\frac{H_j}{2^j}-\sum_{j=1}^{(p-1)/2}\binom{2j}{j}\frac{1}{j2^j}\\ &\equiv -q_p\left(2\right)\left(\left(\frac{-1}{p}\right)+1\right) \pmod{p}.\end{aligned}$$

Also, by (27) and (12), we obtain

$$\begin{aligned}\sum_{j=1}^{(p-1)/2}\binom{2j}{j}\frac{H_{2j-1}}{2^j} &= \sum_{j=1}^{(p-1)/2}\binom{2j}{j}\frac{H_{2j}}{2^j}-\frac{1}{2}\sum_{j=1}^{(p-1)/2}\binom{2j}{j}\frac{1}{j2^j}\\ &\equiv \left(\frac{-1}{p}\right)\left(2\chi_p-\frac{1}{2}q_p\left(2\right)\right)-\frac{1}{2}q_p\left(2\right) \pmod{p}.\end{aligned}$$

Hence,

$$\sum_{j=1}^{(p-1)/2}\binom{2j}{j}\frac{\left(H_{2j-1}-H_{j-1}\right)}{2^j} \equiv \left(\frac{-1}{p}\right)\left(2\chi_p+\frac{1}{2}q_p\left(2\right)\right)+\frac{1}{2}q_p\left(2\right) \pmod{p}.$$

Putting the above result into (28) and using Lemma 11 and Lemma 12 with some simplifications, we conclude that

$$\Theta_1 \equiv \frac{1}{2}\left(\left(\frac{-1}{p}\right)-1\right)+p\left(\frac{5n}{4}\left(\frac{-1}{p}\right)q_p(2)+(3n-1)\left(\frac{-1}{p}\right)\chi_p + \frac{3n-4}{4}q_p(2)\right) \pmod{p^2}. \tag{29}$$

Set

$$\Theta_2 := \sum_{j=(p+1)/2}^{p-1}\binom{np-1}{p-1-j}\binom{(3n-2)p+2j-1}{j}(-2)^{p-1-j},$$

or equivalently

$$\Theta_2 = \sum_{j=1}^{(p-1)/2}\binom{np-1}{(p-1)/2-j}\binom{(3n-1)p+2j-2}{(p-1)/2+j}(-2)^{(p-1)/2-j}.$$

In view of (10), we have

$$\binom{(3n-1)p+2j-2}{(p-1)/2+j} \equiv p(3n-1)\frac{(-1)^{(p-1)/2-j-1}}{\binom{(p-1)/2+j}{2j-2}((p-1)/2-j+2)} \pmod{p^2},$$

furthermore, observing that

$$\binom{(p-1)/2+j}{2j-2} = \binom{(p-1)/2+j}{2j}\frac{2j(2j-1)}{((p-1)/2-j+1)((p-1)/2-j+1)},$$

and from Lemma 9, we find

$$\binom{(3n-1)p+2j-2}{(p-1)/2+j} \equiv p(3n-1)\left(\frac{-1}{p}\right)\frac{4^{2j-1}}{j\binom{2j}{j}} \pmod{p^2}.$$

It follows that

$$\Theta_2 \equiv p(3n-1)\left(\frac{-2}{p}\right)\sum_{j=1}^{(p-1)/2}\frac{2^{3j-2}}{j\binom{2j}{j}} \pmod{p^2}.$$

From (24), we conclude that

$$\Theta_2 \equiv -p(3n-1)\left(\frac{-1}{p}\right)\chi_p \pmod{p^2}. \tag{30}$$

In view of Glaisher's congruence (3), we have

$$\binom{np-1}{p-1}(-2)^{p-1} \equiv 1+pq_p(2) \pmod{p^2}.$$

Combining the above congruence with (29) and (30) completes the proof of Theorem 3. □

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