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Brouwer fixed point theorem in strictly star-shaped sets

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Dedicated to the memory of Professor Kazimierz Gobel (1940 - 2022)

Abstract

In this note, we show that the Brouwer fixed point theorem in open strictly star-shaped sets is equivalent to a number of results closely related to the Euclidean spaces.

Keywords: Brouwer fixed point, Star-shaped, retraction, contractible, homotopy groups
2010 MSC: 47H10, 47H40, 54H25

1. Introduction

Brouwer's fixed-point theorem assures that every continuous from the closed ball into itself in Euclidean space has a fixed point. First studied by Poincaré in 1887 [15] and by Bohl in 1904 [2] in the context of ordinaries differential equations [19], then by Hadamard in 1910 and Brouwer in 1911 [3, 8].
The most standard an popular proofs of the Brouwer fixed point theorem, can be found it is worth mentioning those using:

- degree topological, see [4].
- homology or homotopy functors, see [7].
- combinatorial proofs based on Knaster-Kuratowski-Mazurkiewicz(KKM) principle see. e.g. [11].

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- Constructive proofs, analytical tools and and the Weierstrass theorem see e.g., [13, 12].

The equivalent formulations and generalizations of the theorem we can be found in the paper by Park [14]. In game theory, the existence of equilibrium was uniformly obtained by the application of a fixed point theorem. In fact, Nash [16, 17] shows the existence of equilibria for noncooperative static games as a direct consequence of Brouwer or Kakutani [10] theorems.

The purpose of this note is to, give the proof of Brouwer fixed point theorem, in some class of nonconvex sets.

This note is organized as follows. In Section 2, we introduce all the background material needed such as strictly star convex set, and homotopy group. Section 3 is devoted to establishing the prove of Brouwer fixed point theorem on star convex sets via homotopy group.

2. Preliminaries

Definition 2.1. An nonempty subset U of $\mathbb{R}^n$ is said to be star-shaped with respect to some $x_0 \in U$, if for all $x \in \overline{U}$, we have

$$[x_0, x[= \{(1-t)x_0 + tx : t \in [0,1]\} \subset U.$$

Definition 2.2. An open bounded neighborhood $U \subset \mathbb{R}^n$ of the origin, is strictly star-shaped with respect to the origin if U is star-shaped with respect 0 and for any $x \in \partial V$, we have

$$\{\lambda x : \lambda > 0\} \cap \partial U = \{x\}.$$

Remark 2.3. If U is a bounded open neighborhood of the origin, the following strict inclusions hold (see [18, Proposition 1]):

$$Convex \subset Strictly\ Star - shaped \subset Star - shaped.$$

For any strictly star-shaped open and bounded neighborhood U of the origin, we can define the Minkowski function $\mu_U : \mathbb{R}^n \to [0, \infty)$ by

$$\mu_U(x) = \inf\{\lambda > 0 : x \in \lambda U\}.$$

Proposition 2.4. *[1, 5] Let $U \subset \mathbb{R}^n$ be a strictly star-shaped with respect to 0. Then, we have the following properties:*

- $\mu_U(0)$, $\mu_U(x) > 0$ *for each* $x \in \mathbb{R}^n \backslash \{0\}$.
- $\mu_U(tx) = t\mu_U(x)$, *for all* $x \in X$ *and* $t \geq 0$.
- $\mu_U(x) > 0$ *for any* $x \in \mathbb{R}^n$.
- $U = \{x : \mu_U(x) < 1\}, \overline{U} = \{x : \mu_U(x) \leq 1\}$, *and* $\partial U = \{x : \mu_U(x) = 1\}$.
- $\mu_U : \mathbb{R}^n \to \mathbb{R}$ *is continuous.*

Remark 2.5. For strictly star set, the Minkowski function can equivalently be defined as: for $x \neq 0$, $\mu_U(x)$ is the unique positive number such that $\frac{x}{\mu_U(x)} \in U$.

Proposition 2.6. *[5] Let $U \subset \mathbb{R}^n$ be an bounded open neighborhood of zero. If U is strictly star-shaped, then the mapping $r_U : \mathbb{R}^n \to \overline{U}$ given by*

$$r_U(x) = \begin{cases} \frac{x}{\mu_U(x)}, & x \in \mathbb{R}^n \setminus \overline{U}, \\ x\ , & x \in \overline{U}, \end{cases}$$

is a continuous retract of $\mathbb{R}^n$ into closure of U.

Proposition 2.7. *[5] Let $U \subset \mathbb{R}^{n+1}$ be an bounded open neighborhood of zero and $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is a sphere. If U is strictly star-shaped, then the mapping $h : \mathbb{S}^n \to \partial U$ given by*

$$h(x) = \frac{x}{\mu_U(x)}, \qquad x \in \mathbb{S}^n,$$

is a homemorphism.

Let X be a topological space and x_0 and $I^n = [0,1] \times \ldots \times [0,1] \subset \mathbb{R}^n,\ I = [0,1]$, where

$$\partial I^n = \{(t_1, \ldots, t_n) \in I^n : \text{at least one of the } t_i = 1, 0\}.$$

We now consider the set

$$\Omega(X, x_0) = \{f : (I^n, \partial I^n) \to (X, x_0) : f \text{ is continuous and } f(\partial I^n) = x_0\}.$$

Definition 2.8. Let f, g be two continuous functions $\Omega(X, x_0)$. Then f, g are called to be homotopic related to ∂I^n denoted by $f \simeq g_{|_{rel \partial I^n}}$ if there exists a continuous map $H : I^n \times I \to X$ such that

$$H(t_1, \ldots, t_n, 0) = f(t_1, \ldots, t_n), \quad H(t_1, \ldots, t_n, 1) = g(t_1, \ldots, t_n), \quad \text{for all } (t_1, \ldots, t_n) \in I^n$$

and

$$H(t_1, \ldots, t_n, s) = x_0, \quad \text{for all } (t_1, \ldots, t_n) \in I^n \text{ and } s \in I.$$

We shall use the equivariant homotopy functors $\pi_n, n = 1, 2, \ldots$

Corollary 2.9. *The relative homotopy relation is an equivalence relation.*

Definition 2.10. For each $n \in \mathbb{N}$ we define the group of homotopy $\pi_n(X, x_0)$ by

$$\pi_n(X, x_0) = \{[\alpha] : \alpha \in \Omega(X, x_0)\} = \Omega(X, x_0) / \simeq_{|_{rel \partial I^n}}$$

with respect to the product $[\alpha], [\beta] \in \pi_n(X, x_0)$ given by

$$[\alpha][\beta] = [\alpha.\beta],$$

where

$$(\alpha.\beta)(t_1, \ldots, t_n) = \begin{cases} \alpha(2t_1, t_2, \ldots, t_n) & \text{if } 0 \leq t_1 \leq \frac{1}{2},\ (t_2, \ldots, t_n) \in I^{n-1}, \\ \beta(2t_1 - 1, t_2, \ldots, t_n)) & \text{if } \frac{1}{2} \leq t_1 \leq 1,\ (t_2, \ldots, t_n) \in I^{n-1}. \end{cases}$$

Proposition 2.11. *[9] Let X, Y are two topological spaces, and $f : (X, x_0) \to (Y, y_0)$ be a continuous map with $f(x_0) = y_0$. Then, f induces group homomorphisms $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ given by*

$$f_*[\alpha] = [\alpha \circ f], \quad [\alpha] \in \pi_n(X, x_0).$$

Remark 2.12. Induced homomorphisms satisfy the following properties:

- $(f \circ g)_* = f_* \circ g_*$.
- $(Id_X)_* = Id_{\pi_n(X, x_0)}$.

Proposition 2.13. *[9] For all $n \in \mathbb{N}$, we have $\pi_n(\mathbb{S}^n)$ is a isomorphism to $\mathbb{Z}$.*

Definition 2.14. A set $A \in X$ is said a contractible space provided there exists a continuous $H : A \times [0,1] \to A$ and $x_0 \in A$ such that

(a) $H(x, 0) = x$, for every $x \in A$,

(b) $H(x, 1) = x_0$, for every $x \in A$,

i.e. if the identity map $A \longrightarrow A$ is homotopic to a constant map (A is homotopically equivalent to a point).

Remark 2.15. If $A \subset \mathbb{R}^n$, is a star set, then A is contractible. Also the class of contractible sets is much larger than the class of closed star sets.

Corollary 2.16. *Every contractile space X, has a trivial homotpy group.*

3. Brouwer fixed point theorem on strictly star-shapes set

In this section, we deduce various Brouwer fixed point theorems on strictly star-shaped set, various non-retract theorems, and the non-contractibility of a bored of some class start set. Those are shown to be all equivalent to the Brouwer theorem and, consequently, we would have our second classical circular tour which starts and ends with the Brouwer theorem.

Proposition 3.1. *Let C be a open bounded and strictly star-shaped subset of $\mathbb{R}^{n+1}$ with $0 \in C$. Then ∂C is not retract of C.*

Proof. Suppose there exists a retraction $r : C \to \partial C$ and from proposition $h : \mathbb{S}^{n-1} \to \partial C$ is a homeomorphism and $i : \partial C \to C$ is continuous injection. Construct the following mapping

$$h^{-1} \circ r \circ i \circ h : \mathbb{S}^n \xrightarrow{h} \partial C \xrightarrow{i} C \xrightarrow{r} \partial C \xrightarrow{h^{-1}} \mathbb{S}^n,$$

$$(h^{-1} \circ r \circ i \circ h)(x) = Id_{\mathbb{S}^n}(x), \quad x \in \mathbb{S}^n.$$

Therefore,

$$(h^{-1} \circ r \circ i \circ h)_* = h_*^{-1} \circ r_* \circ i_* \circ h_* : \pi_n(\mathbb{S}^n) \xrightarrow{h_*} \pi_n(\partial C) \xrightarrow{i_*} \pi_n(C) \xrightarrow{r_*} \pi_n(\partial C) \xrightarrow{h_*^{-1}} \pi_n(\mathbb{S}^n),$$

is a isomorphism. Since C is a contractible space then $\pi_n(C) = \{0\}$ and from proposition 2.13, we get $\pi_n(\mathbb{S}^n) = \pi_n(\partial C) = \mathbb{Z}$. Since h_*^{-1}, r_*, i_*, h_* are homomorphisms and $(h^{-1} \circ r \circ i \circ h)_*$ is a isomorphisms, hence $\pi_n(\mathbb{S}^n) = \{0\}$, which is a contradiction. □

Theorem 3.2. *Let $C \subset \mathbb{R}^{n+1}$ be a open bounded and strictly star-shaped with respect $0 \in C$ and $f : \overline{C} \to \overline{C}$ is continuous function, then there is $x \in \overline{C}$ such that $x = f(x)$.*

Proof. Suppose that $f(x) \neq x$ for all $x \in \overline{C}$, we define the mapping $g : \overline{C} \to \mathbb{S}^n$ by

$$g(x) = \frac{x - f(x)}{\|x - f(x)\|}, \quad x \in \overline{C}.$$

It clear that g is a continuous and $\overline{C} \subset 2\overline{C}$, then, we can extend the mapping g to the set $2\overline{C}$ as follows

$$g_*(x) = \begin{cases} g(x), & x \in \overline{C}, \\ (2 - \mu_C(x)) g\left(h\left(\frac{x}{\|x\|}\right)\right), & x \in 2\overline{C} \backslash \overline{C}. \end{cases}$$

Since, for $x \in \partial C$, we have $\mu_C(x) = 1$ and $h\left(\frac{x}{\|x\|}\right) = x$, so

$$(2 - \mu_C(x)) g\left(h\left(\frac{x}{\|x\|}\right)\right) = g(x),$$

hence, it is easy to check that g is a continuous function on C. Now, we define the continuous application $\widetilde{g} : \overline{C} \to \mathbb{R}^{n+1}$ by

$$\widetilde{g}(x) = \frac{g_*(2x)}{2}, \quad x \in \overline{C}.$$

For $x \in \overline{C}$, then

- If $2x \in \overline{C}$, then $\widetilde{g}(x) = \frac{g(2x)}{2} + \frac{1}{2} 0 \in \overline{C}$.

- If $2x \in 2\overline{C}\backslash\overline{C}$, thus
$$\widetilde{g}(x) = (1-\mu_C(x))g\left(h\left(\frac{x}{\|x\|}\right)\right) + \mu_C(x)0 \in \overline{C}.$$

This implies that
$$\widetilde{g}(\overline{C}) \subset \overline{C}.$$

Now, we will show that for all $x \in \overline{C}, \widetilde{g}(x) \neq x$. Assume that there exists $x \in \overline{C}$ such that $\widetilde{g}(x) = x$. It follows that, if $2x \in \overline{C}$, we get
$$g(2x) = 2x \Longrightarrow f(2x) = 2x,$$

which is contradiction.
If $2x \notin \overline{C}$ we have
$$(2-\mu_C(2x))g\left(h\left(\frac{x}{\|x\|}\right)\right) = 2x \Longrightarrow (2-\mu_C(2x))\mu_C\left(g\left(h\left(\frac{x}{\|x\|}\right)\right)\right) = \mu_C(2x).$$

We shall consider the following three cases:

Case 1: If $\mu_C\left(g\left(h\left(\frac{x}{\|x\|}\right)\right)\right) = 1$, we get
$$\mu_C(2x) = 1 \Longrightarrow 2x \in \overline{C},$$
this is a contradiction with $2x \notin \overline{C}$.

Case 2: If $0 < \mu_C\left(g\left(h\left(\frac{x}{\|x\|}\right)\right)\right) < 1$, we obtain
$$\mu_C(2x) = \frac{2\mu_C\left(g\left(h\left(\frac{x}{\|x\|}\right)\right)\right)}{1+\mu_C\left(g\left(h\left(\frac{x}{\|x\|}\right)\right)\right)} \leq 1 \Longrightarrow 2x \in \overline{C},$$
which,is contradiction.

Case 3: If $\mu_C\left(g\left(h\left(\frac{x}{\|x\|}\right)\right)\right) = 0$, we have
$$\mu_C(2x) = 0 \Longrightarrow 2x = 0 \in \overline{C},$$
this is impossible.

- We shows that $\widetilde{g}(\partial C) = \{0\}$. Let $x \in \partial C$, then $2x \notin \overline{C}$. By contradiction, assume $2x \in \overline{C}$, then, we have
$$\mu_C(2x) \leq 1 \Longrightarrow 1 = \mu_C(x) \leq \frac{1}{2},$$

which is a contradiction. Hence, we obtain
$$\widetilde{g}(x) = (2-2\mu_C(x))g\left(h\left(\frac{x}{\|x\|}\right)\right) = 0.$$

Consequently the retraction $r : \overline{C} \to \overline{C}$ given by
$$r(x) = h\left(\frac{x-\widetilde{g}(x)}{\|x-\widetilde{g}(x)\|}\right).$$

Clear that R is a continuous mapping and for every $x \in \partial C$, we get
$$r(x) = h\left(\frac{x}{\|x\|}\right), \ \mu_C(x) = 1 \Longrightarrow R(x) = x.$$

Thus r is a retraction of $\overline{C}$ onto ∂C, contradicting Proposition 3.1. $\square$

Theorem 3.3. *Let C be a open bounded and strictly star-shaped subset of $\mathbb{R}^n$ with $0 \in C$. Every continuous map $f : \overline{C} \to \mathbb{R}^n$ satisfied at least one of the following properties:*

(i) there $x \in \overline{C}$ such that $f(x) = x$; or

(ii) there exists $x \in \partial C$ such that $\lambda x = f(x)$ with $\lambda > 1$.

Proof. Using proposition 3.1 for define a map $r_C \circ f : \overline{C} \to \overline{C}$ by

$$(r_C \circ f)(x) = \begin{cases} \frac{f(x)}{\mu_C(f(x))}, & f(x) \in \mathbb{R}^n \setminus \overline{C}, \\ \\ f(x), & f(x) \in \overline{C}. \end{cases}$$

Then, by theorem 3.2, there exists $x \in \overline{C}$ such that

$$(r_C \circ f)(x) = x.$$

If $f(x) \in \overline{C}$, then $f(x) = x$.
If $f(x) \in \mathbb{R}^n \setminus \overline{C}$, we get

$$\frac{f(x)}{\mu_C(f(x))} = x,$$

this implies

$$\lambda f(x) = x, \ \lambda = \frac{1}{\mu_C(f(x))} > 1, \ x \in \partial C.$$

□

Proposition 3.4. *Let C be a open bounded and strictly star-shaped subset of $\mathbb{R}^{n+1}$ with $0 \in C$ and $f : \overline{C} \to \mathbb{R}^{n+1}$ be a continuous function such that*

$$f(x) \notin \{\lambda x : \lambda > 0\} \quad \textit{for all} x \in \partial C.$$

Then there exists $x_0 \in \overline{C}$ such that $f(x_0) = 0$.

Proof. Assume that, $f(x) \neq 0$, for every $x \in \overline{C}$. We define $\widetilde{f} : \overline{C} \to \overline{C}$ as follows

$$\widetilde{f}(x) = \frac{f(x)}{\mu_C(f(x))}, \quad x \in \overline{C},$$

is a continuous function. From Theorem 3.2, $\widetilde{f}$ has at least one fixed point ?$\bar{x} \in \overline{C}$. Then $\bar{x} = \frac{f(\bar{x})}{\mu_C(f(\bar{x}))}$ with $\mu_C(\bar{x}) = 1$, hence $\bar{x} \in \partial C$, contradicting the hypotheses. □

We are often interested to prove the not contractible set of star-shaped sets.

Theorem 3.5. *Let C be a open bounded and strictly star-shaped subset of $\mathbb{R}^{n+1}$ with $0 \in C$. Then, ∂C is not contractible.*

Proof. Assume by contradiction that ∂C is contractile, then there exists $x_* \in \partial C$ and homotopy $H : \partial C \times [0, 1] \to \partial C$, such that

$$H(x, 0) = x_*, \ H(x, 1) = x, \quad \text{fo all } x \in \partial C.$$

Define now the following map $r : C \to C$ by

$$r(x) = \begin{cases} x_*, & 0 \leq \mu_C(x) \leq \frac{1}{2}, \\ \\ H\left(\frac{x}{\mu(x)}, 2\mu_C(x) - 1\right), & \frac{1}{2} \leq \mu_C(x) \leq 1. \end{cases}$$

For $\mu_C(x) = \frac{1}{2}$, we have $H\left(\frac{x}{\mu(x)}, 2\mu_C(x) - 1\right) = H(x, 0) = x_*$, then r is continuous. On the other hand, for each $x \in \partial C$ we have

$$r(x) = H(x, 1) = x.$$

This contradicts Proposition 3.1.

□

From Theorem 3.5, we can deduce the prove of Theorem 3.2:

Proof. Assume that $f(x) \neq x$ for all $x \in \overline{C}$. Define a map $H : \partial C \times [0, 1] \to \partial C$ by

$$H(x, t) = h\left(\frac{(1-t)x - \widetilde{g}((1-t)x)}{\|(1-t)x - \widetilde{g}((1-t)x)\|}\right), \quad x \in \partial C,$$

where $\widetilde{g}$, as defined in the proof of Theorem 3.2. H is continuous and for any $t \in \{0, 1\}$ we observe that

$$H(x, 0) = x, \quad H(x, 1) = h\left(\frac{-\widetilde{g}(0)}{\|\widetilde{g}(0)\|}\right), \quad \text{for all } x \in \partial C,$$

contradicts the not contractible theorem 3.5. □

Theorem 3.6. *Let C be a open bounded and strictly star-shaped subset of $\mathbb{R}^{n+1}$ with $0 \in C$. Then the following (equivalent) assertions hold*

1) ∂C is not contractible.

2) Every continuous map $f : \overline{C} \to \mathbb{R}^{n+1}$ satisfies on to conditions:
a) there $x \in \overline{C}$ such that $f(x) = x$; or
b) there exists $x \in \partial C$ such that $x = \lambda f(x)$ with $\lambda \in (0, 1)$.

3) Brouwer fixed point theorem holds true on $\overline{C}$.

4) ∂C is not retract of C.

Proof. 1) $\Longrightarrow$ 2). Assume that 2) is not true, then for all $x \in C$, $f(x) \neq x$ and $x \neq \lambda f(x)$ for all $\lambda \in (0, 1)$. Now we define $H : \partial C \times [0, 1] \to \partial C$ by

$$H(x, \lambda) = \frac{(1-\lambda)x - f((1-\lambda)x)}{\mu_C((1-\lambda)x - f((1-\lambda)x))}, \quad (x, \lambda) \in \partial C \times [0, 1].$$

It is clear that $\mu_C(x - \lambda f(x)) \neq 0$, $\mu_C((1-\lambda)x - f((1-\lambda)x)) \neq 0$ for every $x \in \partial C$ and H is a continuou function. Moreover, we have

$$H(x, 0) = \frac{x - f(x)}{\mu_C(x - f(x))}, \quad H(x, 1) = \frac{-f(0)}{\mu_C(-f(0))}, \quad x \in \partial C.$$

Hence the identity map $Id_{\partial C}$ is a homotopy to constant application $\frac{-f(0)}{\mu_C(-f(0))}$. It contradicts to the assumption 1).

2) $\Longrightarrow$ 3). If there exists a continuous map $f : \overline{C} \to \overline{C}$ such that for every $x \in C$, we have $f(x) \neq x$, then there exist $x_* \in \partial C$ and $\lambda \in (0, 1)$ such that $x_* = \lambda f(x_*)$, thus $\mu_C(x_*) = \lambda \mu_C(f(x_*))$. So,

$$1 = \lambda \mu_C(f(x_*)) \leq \lambda < 1.$$

which is a contradiction.

3) $\Longrightarrow$ 4). Assume, by contradiction that there is a continuous function $r : \overline{C} \to \partial C$ such that $r(x) = x$, $x \in \partial C$. Hence the mapping $i \circ r : \overline{C} \xrightarrow{r} \partial C \xrightarrow{i} \overline{C}$ is a continuous. Then the continuous map $\eta = -i \circ r : \overline{C} \to \overline{C}$ without fixed point.

So 4) $\Longrightarrow$ 1) and the proof is completed. □

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