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ON THE DEGREE OF REGULARITY OF GENERALIZED VAN DER WAERDEN TRIPLES

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Abstract

Let a and b be positive integers such that $a \leq b$ and $(a,b) \neq (1,1)$. We prove that there exists a 6-coloring of the positive integers that does not contain a monochromatic (a,b)-triple, that is, a triple (x,y,z) of positive integers such that $y = ax+d$ and $z = bx+2d$ for some positive integer d. This confirms a conjecture of Landman and Robertson.

1. Introduction

In 1916, Schur [13] proved that for every finite coloring of the positive integers there is a monochromatic solution to $x+y=z$. In 1927, van der Waerden [15] proved that every finite coloring of the positive integers contains arbitrarily long monochromatic arithmetic progressions. Rado's 1933 thesis [12] was a seminal work in Ramsey theory, generalizing the earlier theorems of Schur and van der Waerden. Rado called a linear homogenous equation $a_1x_1 + \ldots + a_nx_n = 0$ (a_i's are nonzero integers) *r-regular* if every r-coloring of $\mathbb{N}$ contains a monochromatic solution to that equation. An equation is *regular* if it is r-regular for all positive integers r. Rado's theorem for a linear homogeneous equation states that an equation is regular if and only if a non-empty subset of a_i's sums to 0. Rado also made a conjecture [12] that further differentiates between those linear homogeneous equations that are regular and those that are not.

Conjecture 1 *(Rado's Boundedness Conjecture, 1933) For every positive integer n, there exists an integer $k := k(n)$ such that every linear homogeneous equation $a_1x_1 + \ldots + a_nx_n = 0$ that is k-regular must be regular as well.*

This outstanding conjecture has remained open except in the trivial cases ($n = 1, 2$) until recently, when the first author and Kleitman settled the first nontrivial case $n = 3$ [4], [9]. They proved that $k(3) \leq 24$.

Van der Waerden's theorem has been strengthened and generalized in numerous other ways [1], [2], [6], [7], [8], [11], [14]. In this note, we consider one of the generalizations, proposed by Landman and Robertson in [10].

Let a and b be positive integers such that $a \leq b$. A triple (x, y, z) of positive integers is called an (a, b)-triple if there exists a positive integer d such that $y = ax+d$ and $z = bx+2d$. The *degree of regularity* of an (a, b)-triple, denoted by $\mathrm{dor}(a, b)$, is the largest positive integer r, if it exists, such that for every r-coloring of the positive integers there is a monochromatic (a, b)-triple. If no such r exists, that is, for every finite coloring of the positive integers there is a monochromatic (a, b)-triple, then set $\mathrm{dor}(a, b) = \infty$. Note that van der Waerden's theorem for 3-term arithmetic progressions is equivalent to $\mathrm{dor}(1, 1) = \infty$.

Landman and Robertson proved that $\mathrm{dor}(a, b) = 1$ if and only if $b = 2a$. They also showed that $\mathrm{dor}(a, 2a - 1) = 2$ for $a \geq 2$. For small values of a and b, they provide few additional results:

$$\mathrm{dor}(1, 3) \leq 3,\ \mathrm{dor}(2, 2) \leq 5,\ \mathrm{dor}(2, 5) \leq 3,\ \mathrm{dor}(2, 6) \leq 3,$$

$$\mathrm{dor}(3, 3) \leq 5,\ \mathrm{dor}(3, 4) \leq 5,\ \mathrm{dor}(3, 8) \leq 3,\ \mathrm{dor}(3, 9) \leq 3.$$

Finally, they conjectured that if $(a, b) \neq (1, 1)$, then $dor(a, b)$ is finite [10], [11].

We confirm and further strengthen their conjecture.

Theorem 1 *If $(a, b) \neq (1, 1)$, then $dor(a, b) < 6$.*

Our proof that $\mathrm{dor}(a, b)$ is finite uses Rado's theorem for a homogenous linear equation. Proving a specific upper bound of 6, regardless of parameters a and b, relies on the above mentioned proof of Fox and Kleitman [4].

2. Proof of Theorem 1

First, notice that if (x, y, z) is an (a, b)-triple, then (x, y, z) satisfies the equation

$$-(b - 2a)x - 2y + z = 0.$$

By Rado's theorem [12], this equation is regular if and only if $b \in \{2a - 2, 2a - 1, 2a + 1\}$. Therefore, if $b - 2a \notin \{-2, -1, 1\}$, then $\mathrm{dor}(a, b)$ is finite. Moreover, since $k(3) \leq 24$ [4], if $b - 2a \notin \{-2, -1, 2\}$, then $\mathrm{dor}(a, b) \leq 23$. As mentioned in the introduction, Landman and Robertson [10] proved that $\mathrm{dor}(a, 2a - 1) = 2$ for $a \geq 2$.

For the remaining cases, we use Lemma 1, which is stated and proved next.

Lemma 1 *Let α and β be real numbers such that $1 < \alpha < \beta$. Set $r = \lceil \log_\alpha \beta \rceil$. Then every r-coloring of the positive integers contains integers x and y of the same color with $\alpha x \le y \le \beta x$. Moreover, there is an $(r+1)$-coloring of the positive integers that contains no integers x and y of the same color with $\alpha x \le y \le \beta x$.*

Proof. Consider a coloring of $\mathbb{N}$ without x and y of the same color with $\alpha x \le y \le \beta x$. Since $r = \lceil \log_\alpha \beta \rceil$, then $\alpha^{r-1} < \beta$. Let $x_1 > \sum_{k=0}^{r-2} \alpha^k / (\beta - \alpha^{r-1})$ be a positive integer. For $i > 1$, set $x_{i+1} = \lceil \alpha x_i \rceil$. We have $\alpha x_i \le x_{i+1} < \alpha x_i + 1$. Repeatedly using the inequality $x_{i+1} < \alpha x_i + 1$, we obtain $x_r < \alpha^{r-1} x_1 + \sum_{k=0}^{r-2} \alpha^k$. Since we appropriately chose x_1, the last inequality yields $x_r < \beta x_1$. Hence, $\alpha x_i \le x_j \le \beta x_i$ for $1 \le i < j \le r$, so $x_1, \ldots, x_r$ must all have different colors. Therefore, the number of colors is at least $r+1$.

Next, we construct a coloring of the positive integers by the elements of $\mathbb{Z}_{r+1}$ such that there do not exist x and y of the same color with $\alpha x \le y \le \beta x$. For every nonnegative integer n, integers in the interval $[\alpha^n, \alpha^{n+1})$ receive color $n \pmod{r+1}$. Within each interval, every pair of integers x and y have the same color, but $y < \alpha x$. For monochromatic x and y from different intervals, with $y > x$, we have $y > \alpha^r x \ge \beta x$. Therefore, this $(r+1)$-coloring of the integers has no monochromatic x and y such that $\alpha x \le y \le \beta x$. □

Now, we continue with the proof of Theorem 1. We have two cases.

Case 1. $b = 2a + 1$.

In this case, we have $y = ax + d$ and $z = (2a+1)x + 2d$. Therefore, $2y < z < (\frac{2a+1}{a})y$. Using Lemma 1 and noting $a \ge 1$, we obtain

$$\operatorname{dor}(a, 2a+1) \le \lceil \log_2(2 + \frac{1}{a}) \rceil = 2.$$

Hence, for all positive integers a, we have $\operatorname{dor}(a, 2a+1) = 2$.

Case 2. $b = 2a - 2$.

Since b must be a positive integer, then $a \ge 2$. As mentioned in the introduction, Landman and Robertson [10] proved that $\operatorname{dor}(2,2) \le 5$. If $a > 2$, then $y = ax + d$ and $z = (2a-2)x + 2d$. So, $(\frac{2a-2}{a})y < z < 2y$. Using Lemma 1 and $a \ge 3$, we obtain

$$\operatorname{dor}(a, 2a-2) \le \lceil \log_{2-\frac{2}{a}} 2 \rceil.$$

We have $2 - \frac{2}{a} > \sqrt{2}$ when $a > 3$. Therefore, $2 \le \operatorname{dor}(a, 2a-2) \le 3$ for $a = 3$ and $\operatorname{dor}(a, 2a-2) = 2$ for $a > 3$.

At this stage, we have $\operatorname{dor}(a,b) < 24$, whenever $(a,b) \ne (1,1)$. Next, we improve the upper bound using some sophisticated tools from the paper of Fox and Kleitman [4]. For the sake of completeness and clarity, we repeat some of their analysis that applies in our context. We need the following bit of notation.

Definition: Let p be a prime number. For every integer n, let $v_p(n)$ denote the largest power of p that divides n. If $n = 0$, let $v_p(n) = +\infty$.

Notice that $v_p(m_1m_2) = v_p(m_1) + v_p(m_2)$ for every prime p, and integers m_1 and m_2. The following straightforward lemma (Lemma 3 in [4]) gives basic properties of the function v_p, which we will repeatedly use.

Lemma 2 *If m_1, m_2, m_3 are integers with $v_p(m_1) \le v_p(m_2) \le v_p(m_3)$ and $v_p(m_1) < v_p(m_1+m_2+m_3)$, then $v_p(m_1) = v_p(m_2)$. Furthermore, if $v_p(m_3) < v_p(m_1+m_2+m_3)$, then also $v_p(m_1+m_2) = v_p(m_3)$.*

Recall that if (x, y, z) is an (a, b)-triple, then (x, y, z) satisfies the equation $(b - 2a)x + 2y - z = 0$. Let $t_x = b-2a$, $t_y = 2$, and $t_z = -1$ denote the coefficients of x, y, and z in this equation, respectively. We have three cases to consider, depending on t_x.

Case A. t_x is a multiple of 4.

Clearly, we have $v_2(t_x) > v_2(t_y) = 1 > v_2(t_z) = 0$. Let $S = \{v_2(t_x), v_2(t_y), v_2(t_x)-v_2(t_y)\}$, and let $\Gamma(S)$ be the undirected Cayley graph of the group $(\mathbb{Z}, +)$ with generators being the elements of S. Since every vertex of $\Gamma(S)$ has degree $2|S|$, there exists a proper (greedy) $(|S|+1)$-coloring χ' of its vertices. This result is "folklore" and we refer the reader to Lemma 2 in [4] for details. Now, define $\chi(n) = \chi'(v_2(n))$, for every $n \in \mathbb{N}$. We claim that in the 4-coloring χ of $\mathbb{N}$ there are no x, y, and z, all of the same color and $v_2(t_xx + t_yy + t_zz) > \min\{v_2(t_xx), v_2(t_yy), v_2(t_zz)\}$. Indeed, otherwise (by Lemma 2) we have $v_2(t_xx) = v_2(t_yy)$; or $v_2(t_xx) = v_2(t_zz)$; or $v_2(t_yy) = v_2(t_zz)$. This implies $v_2(y) - v_2(x) = v_2(t_x) - v_2(t_y)$; or $v_2(z)-v_2(x) = v_2(t_x)$; or $v_2(z)-v_2(y) = v_2(t_y)$. However, this contradicts that χ' is a proper coloring of $\Gamma(S)$ and $v_2(x)$, $v_2(y)$, and $v_2(z)$ are all of the same color.

Since $v_2(0) = +\infty$, by definition, and since there are no x, y, z, all of the same color and $v_2(t_xx + t_yy + t_zz) > \min\{v_2(t_xx), v_2(t_yy), v_2(t_zz)\}$, then, in particular, there are no monochromatic solutions to $t_xx+t_yy+t_zz = 0$ (i.e. $(b-2a)x+2y-z = 0$) in χ. Case A is equivalent to Lemma 4 (with $p = 2$) in [4].

Case B. t_x has an odd prime factor p.[1]

In this case we have $v_p(t_x) > v_p(t_y) = v_p(t_z) = 0$. Let $d = v_p(t_x)$. We construct a 6-coloring χ that is a product of a 2-coloring χ_1 and a 3-coloring χ_2. For $n \in \mathbb{N}$ define $\chi_1(n) \equiv \lfloor \frac{v_p(n)}{d} \rfloor \pmod 2$. The coloring $\chi_1(n)$ colors intervals of v_p values of length d, open on one side, periodically in 2 colors with period 2. Let Γ be the undirected Cayley graph on $\mathbb{Z}_p \setminus \{0\}$ such that (u, v) is an edge of Γ if and only if $u - 2v \equiv 0 \pmod p$ or $2u - v \equiv 0 \pmod p$. Since every vertex of Γ has degree 2, there exists a proper 3-coloring $\chi_2' : V(\Gamma) \to \{0, 1, 2\}$. For $n \in \mathbb{N}$ define $\chi_2(n) = \chi_2'(m \bmod p)$, where $n = mp^{v_p(n)}$. Finally, for $n \in \mathbb{N}$ define $\chi(n) = (\chi_1(n), \chi_2(n))$.

[1]Note that Cases A and B overlap. However, it is only important that Cases A, B, and C cover all the possibilities.

We claim that in the 6-coloring χ of $\mathbb{N}$ there are no x, y, and z, all of the same color and $v_p(t_x x+t_y y+t_z z) > \max\{v_p(t_x x), v_p(t_y y), v_p(t_z z)\}$. Indeed, otherwise (by Lemma 2) we have $v_p(t_x x) = v_p(t_y y) \le v_p(t_z z)$; or $v_p(t_x x) = v_p(t_z z) \le v_p(t_y y)$; or $v_p(t_y y) = v_p(t_z z) \le v_p(t_x x)$. If $v_p(t_x x) = v_p(t_y y)$, then $d + v_p(x) = v_p(y)$, hence, $\chi_1(x) \ne \chi_1(y)$, which contradicts $\chi(x) = \chi(y)$. If $v_p(t_x x) = v_p(t_z z)$, then $d + v_p(x) = v_p(z)$, hence, $\chi_1(x) \ne \chi_1(z)$, which contradicts $\chi(x) = \chi(z)$. So, assume that $v_p(t_y y) = v_p(t_z z) \le v_p(t_x x)$. Recalling our coefficients, we obtain $v_p(y) = v_p(z) \le d + v_p(x)$. By (the second part of) Lemma 2, we also have $v_p(2y-z) = d+v_p(x)$. Let e denote the common value of $v_p(y)$ and $v_p(z)$. Let $y = y'p^e$ and $z = z'p^e$. Since $\chi_2(y) = \chi_2(z)$, then $\chi_2'(y' \bmod p) = \chi_2'(z' \bmod p)$, hence, $2y' - z' \not\equiv 0 \pmod p$. However, this implies $v_p(2y - z) = e$, so $v_p(y) = v_p(z) = e = d + v_p(x)$. It follows from here that $\chi_1(x)$ is different from $\chi_1(y)$ and $\chi_1(z)$, which contradicts $\chi(x) = \chi(y) = \chi(z)$.

Since $v_p(0) = +\infty$ and there are no x, y, z, all of the same color and $v_p(t_x x+t_y y+t_z z) > \max\{v_p(t_x x), v_p(t_y y), v_p(t_z z)\}$, then, in particular, there are no monochromatic solutions to $t_x x + t_y y + t_z z = 0$ (i.e. $(b - 2a)x + 2y - z = 0$) in χ. Case B is essentially equivalent to Lemma 6 (with $s = 1$) in [4].

Notice that one can define χ_2 to be a 2-coloring in the proof above, as long as the order of 2 mod p is even.

Case C. $t_x \in \{-2, -1, 1, 2\}$

Case $t_x = -1$ is taken care of in [10], as mentioned before, while cases $t_x = 1$ and $t_x = -2$ correspond to Cases 1 and 2, respectively. The only remaining case is $t_x = 2$.[2] In this case, we have $y = ax + d$ and $z = (2a + 2)x + 2d$. Therefore, $2y < z < 4y$. Using Lemma 1, we obtain $\text{dor}(a, 2a + 2) \le \lceil \log_2 4 \rceil = 2$. Hence, for all positive integers a, we have $\text{dor}(a, 2a + 2) = 2$. □

3. Concluding remarks

As a consequence of our proof, we obtained

$$\text{dor}(1,3) = 2,\ \text{dor}(2,5) = 2,\ \text{dor}(2,6) = 2,$$

$$\text{dor}(3,3) \le 3,\ \text{dor}(3,4) \le 3,\ \text{dor}(3,8) = 2.$$

These results improve the corresponding entries in the table provided by Landman and Robertson [10] for small values of a and b.

After submission, we learned that Frantzikinakis, Landman, and Robertson [5] independently showed that $\text{dor}(a, b)$ is finite unless $(a, b) = (1, 1)$.

[2] The equation is not regular in this case, however this possibility is not covered by Cases A and B of the improved upper bound analysis.

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