Note on the Analytical Continuation of Riemann Zeta Function

Keyang Ding
(keyangding@gmail.com)

An explicit expression of the analytical continuation of Riemann zeta function is derived from the original equation of Riemann:

$$\zeta(s)=\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\left\{\frac{1}{s(s-1)}+\sum_{k=1}^{\infty}\left[\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}-k+1\right)}+\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}-k+1\right)}\right]\sum_{n=1}^{\infty}\frac{e^{-n^2\pi}}{(n^2\pi)^k}\right\}$$

It indicates that Riemann zeta function has no zeroes on the critical line.

It is well known that the functional equation of Riemann zeta function[1]

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \tag{1}$$

was derived from the equation (2):

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\frac{1}{s(s-1)}+\int_1^{\infty}\left(x^{\frac{s}{2}-1}+x^{\frac{1-s}{2}-1}\right)\psi(x)dx \tag{2}$$

in which the function $\psi(x)$ is defined by

$$\psi(x)=\sum_{n=1}^{\infty}e^{-n^2\pi x} \tag{3}$$

The equation (2) is an original equation of Riemann. It is straightforward to rewrite it as the following

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\frac{1}{s(s-1)}+\int_0^{\infty}\left(e^{\frac{s}{2}}+e^{\frac{1-s}{2}}\right)\psi(e^x)dx \tag{4}$$

By substituting the equation (3) into the equation (4), that leads to the equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\frac{1}{s(s-1)}+\sum_{n=1}^{\infty}\int_0^{\infty}\left(e^{-(n^2\pi e^x-\frac{sx}{2})}+e^{-[n^2\pi e^x-\frac{(1-s)x}{2}]}\right)dx \tag{5}$$

and finishing the integration, one can obtain

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\frac{1}{s(s-1)}+\sum_{k=1}^{\infty}\left[\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}-k+1\right)}+\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}-k+1\right)}\right]\sum_{n=1}^{\infty}\frac{e^{-n^2\pi}}{(n^2\pi)^k} \tag{6}$$

Both equations (5) and (6) are identical to the equation (2). Solving ζ(s) from the equations (5) and (6) results in the explicit expressions (7) and (8) of the analytical continuation of the Riemann zeta function:

$$\zeta(s)=\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\left\{\frac{1}{s(s-1)}+\sum_{n=1}^{\infty}\int_0^{\infty}\left(e^{-(n^2\pi e^x-\frac{sx}{2})}+e^{-[n^2\pi e^x-\frac{(1-s)x}{2}]}\right)dx\right\} \tag{7.1}$$

or letting s = σ+it

$$\zeta(\sigma+it)=\frac{\pi^{\frac{\sigma+it}{2}}}{\Gamma\left(\frac{\sigma+it}{2}\right)}\left\{\frac{1}{(\sigma+it)(\sigma-1+it)}+\right.$$

$$+\sum_{n=1}^{\infty}\int_0^{\infty}\left(e^{-(n^2\pi e^x-\frac{\sigma x}{2})}+e^{-[n^2\pi e^x-\frac{(1-\sigma)x}{2}]}\right)\cos\left(\frac{1}{2}tx\right)dx+$$

$$\left.+i\sum_{n=1}^{\infty}\int_0^{\infty}\left(e^{-(n^2\pi e^x-\frac{\sigma x}{2})}-e^{-[n^2\pi e^x-\frac{(1-\sigma)x}{2}]}\right)\sin\left(\frac{1}{2}tx\right)dx\right\} \tag{7.2}$$

and

$$\zeta(s)=\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\left\{\frac{1}{s(s-1)}+\sum_{k=1}^{\infty}\left[\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}-k+1\right)}+\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}-k+1\right)}\right]\sum_{n=1}^{\infty}\frac{e^{-n^2\pi}}{(n^2\pi)^k}\right\} \tag{8}$$

Both expressions (7) and (8) fulfill the Riemann functional equation (1). Except a single pole at s=1, the equations (7) and (8) converge in the whole complex plane of s.

On the vertical line s=1/2+it, the equation (5) becomes:

$$\pi^{-\left(\frac{1}{4}+\frac{it}{2}\right)}\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\zeta\left(\frac{1}{2}+it\right)=\frac{-1}{t^2+\frac{1}{4}}+2\sum_{n=1}^{\infty}\int_0^{\infty}\left(e^{-(n^2\pi e^x-\frac{x}{4})}\right)cos\left(\frac{tx}{2}\right)dx \tag{9}$$

$$\pi^{-\left(\frac{1}{4}+\frac{it}{2}\right)}\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\zeta\left(\frac{1}{2}+it\right) = -\int_0^\infty \left\{e^{-\frac{x}{4}} - 2\sum_{n=1}^{\infty} e^{-\left(n^2\pi e^x - \frac{x}{4}\right)}\right\}\cos\left(\frac{tx}{2}\right)dx \qquad (10)$$

The zeroes of the integral at the right side of the equation (10) are the zeroes of Riemann zeta function on the vertical line.

This integral can be defined as the reduced Z function:

$$Z_r(t) = \int_0^\infty \left\{e^{-\frac{x}{4}} - 2\sum_{n=1}^{\infty} e^{-\left(n^2\pi e^x - \frac{x}{4}\right)}\right\}\cos\left(\frac{1}{2}tx\right)dx \qquad (11.1)$$

If introducing a real function z(x) of x in the region [0,∞]

$$z(x) = e^{-\frac{x}{4}} - 2\sum_{n=1}^{\infty} e^{-\left(n^2\pi e^x - \frac{x}{4}\right)} \qquad (12)$$

z(x) is a monotonically decaying function with x in the region [0,∞] and $Z_r(t)$ is the real part of Fourier transformation of z(x) in the region [0,∞].

It is obvious that $Z_r(t)$ is also a monotonically decaying function with t in the region [0,∞] and only when t→∞, $Z_r(t) \to 0$. Therefore, the equation (10) or (11) indicates that Riemann zeta function has no zeroes on the critical line. By substituting s=1/2+it into the equation (8), the explicit expression of $Z_r(t)$ in the form of series expansion can be obtained.

$$Z_r(t) = \frac{1}{t^2+\frac{1}{4}} - \sum_{k=1}^{\infty}\left(\frac{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)}{\Gamma\left(\frac{5}{4}-k+\frac{it}{2}\right)} + \frac{\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)}{\Gamma\left(\frac{5}{4}-k-\frac{it}{2}\right)}\right)\sum_{n=1}^{\infty}\frac{e^{-n^2\pi}}{(n^2\pi)^k} \qquad (11.2)$$

According to the definition of Z function

$$\zeta\left(\frac{1}{2}+it\right) = Z(t)e^{i\theta(t)} \qquad (13)$$

and substituting s=1/2+it into the equation (7), the explicit expression of Z function can be obtained.

$$Z(t) = -\left\{\frac{\pi^{\frac{1}{4}}}{\Gamma\left(\frac{1}{4}\right)}\sqrt{\frac{e^{\pi t}+e^{-\pi t}}{2}}\prod_{m=0}^{\infty}\frac{4m+3}{\sqrt{(4m+3)^2+4t^2}}\right\}\times$$

$$\times\int_0^{\infty}\left\{e^{-\frac{x}{4}}-2\sum_{n=1}^{\infty}e^{-\left(n^2\pi e^x-\frac{x}{4}\right)}\right\}\cos\left(\frac{1}{2}tx\right)dx \qquad (14)$$

Since $Z_r(t)$ has no zeroes, Z(t) in the equation (14) also has no zeroes. After replacing the integral in equation (14) by the right side of equation (11.2), the explicit expression of Z function in the form of series expansion can be obtained.

$$Z(t) = -\left\{\frac{\pi^{\frac{1}{4}}}{\Gamma\left(\frac{1}{4}\right)}\sqrt{\frac{e^{\pi t}+e^{-\pi t}}{2}}\prod_{m=0}^{\infty}\frac{4m+3}{\sqrt{(4m+3)^2+4t^2}}\right\}\times$$

$$\times\left\{\frac{1}{t^2+\frac{1}{4}}-\sum_{k=1}^{\infty}\left(\frac{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)}{\Gamma\left(\frac{5}{4}-k+\frac{it}{2}\right)}+\frac{\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)}{\Gamma\left(\frac{5}{4}-k-\frac{it}{2}\right)}\right)\sum_{n=1}^{\infty}\frac{e^{-n^2\pi}}{(n^2\pi)^k}\right\} \qquad (15)$$

Recently a recurrence formula[2] for the analytical continuation of Riemann zeta function was derived.

$$\zeta(s) = \frac{1}{1-2^{-s}-3^{-s}-6^{-s}}\left\{1+\sum_{k=1}^{\infty}2\times 6^{-s-2k}\left(\prod_{m=1}^{2k}\frac{1-s-m}{m}\right)\zeta(s+2k)\right\} \qquad (16)$$

By using Gamma function, the equation (16) can be rewritten as

$$\zeta(s) = \frac{1}{1-2^{-s}-3^{-s}-6^{-s}}\left\{1+\sum_{k=1}^{\infty}2\times 6^{-s-2k}\frac{\Gamma(1-s)}{\Gamma(1+2k)\Gamma(1-s-2k)}\zeta(s+2k)\right\} \qquad (17)$$

or

$$\zeta(s) = \frac{1}{1-2^{-s}-3^{-s}-6^{-s}}\left\{1+\sum_{k=1}^{\infty}2\times 6^{-s-2k}\frac{\Gamma(s+2k)}{\Gamma(s)\Gamma(1+2k)}\zeta(s+2k)\right\} \qquad (18)$$

Three equations in (16-18) are identical. Except a single pole at s=1, the equations (16-19) converge in the whole complex plane of s. But they do not obey the Riemann functional equation (1). The equations in (16-18) also indicate that

Riemann zeta function has no zeroes on the critical line. As the same as the equations in (7) and (8), the equations in (16-18) are identical to

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{19}$$

in the half complex plane Re(s)>1.

By using the equation (18) onto itself iteratively, the equation (18) becomes

$$\zeta(s) = \frac{1}{1-2^{-s}-3^{-s}-6^{-s}} \left\{ 1 + \sum_{n=1}^{\infty} \sum_{k_1,k_2,\cdots,k_n=1}^{\infty} \frac{\Gamma\left(s+2\sum_{i=1}^{n} k_i\right)}{\Gamma(s)\prod_{i=1}^{n}\Gamma(1+2k_i)} \times \right.$$

$$\left. \times \prod_{i=1}^{n} \frac{2\times 6^\wedge\left(-s-2\sum_{m=1}^{i} k_m\right)}{1-2^\wedge\left(-s-2\sum_{m=1}^{i} k_m\right)-3^\wedge\left(-s-2\sum_{m=1}^{i} k_m\right)-6^\wedge\left(-s-2\sum_{m=1}^{i} k_m\right)} \right\} \tag{20}$$

When s is zero or negative integers, the summation over k_i's in equation (20) will contain finite number of terms and the k_i's in the summation should satisfy

$$2\sum_{i=1}^{n} k_i \leq -s \qquad s = 0, -1, -2, -3, \cdots. \tag{21}$$

Some values of ζ(s) obtained from equation (20) are listed below

$$\left.\begin{aligned} &\zeta(0) = -\frac{1}{2} \\ &\zeta(-1) = -\frac{1}{10} \\ &\zeta(-2n) = 0 \qquad (n = 1, 2, 3, \cdots) \end{aligned}\right\} \tag{22}$$

ζ(-1) obtained from equations (20) is different from that obtained from equation (8). In fact, the values of ζ(s) at negative odd integers s=-(2n-1) (n=1, 2, 3,⋯) obtained from equations (20) and (8) respectively are different. Therefore, the analytical continuation is not unique.

References

[1] B. Riemann, "*Über die Anzahl der Primzahlen unter einer gegebenen Grösse*", Monatsberichte der Berliner Akademie, (1859)

[2] K. Ding, "*A Recurrence Formula of Riemann Zeta Function ζ(s)*", (non-peer reviewed), DOI: 10.5281/zenodo.4476915, (2021)