Unitarity Bounds

We know that one of the conditions to satisfy unitarity is that:

$$|Re(\mathcal{M}^J_{ij})| \leq \frac{1}{2} \tag{1}$$

Where if the initial and final total spins are the same and both initial and final states are a pair of identical particles:

$$\mathcal{M}^J_{ij}(s) = \frac{\beta_{ii}}{64\pi}\int_{-1}^{1} d\cos\theta P^J(\cos\theta)\mathcal{M}_{ij}(s,\cos\theta)\,. \tag{2}$$

Here $P^J(\cos\theta)$ are the Legendre polynomials. For this proof, we will take the zeroth order ($P^0(\cos\theta) = 1$).

Model Lagrangian

The Model Lagrangian interaction terms that are relevant for this proof are:

$$\begin{aligned}
\mathcal{L} = &-ib_5\Big(X^\dagger_\nu(\partial_\mu X^\nu) - (\partial_\mu X^{\dagger\nu})X_\nu\Big)V^\mu \\
&-b_{6R}\Big(X^\dagger_\nu(\partial^\nu X_\mu) + (\partial^\nu X^\dagger_\mu)X_\nu\Big)V^\mu \\
&-ib_{6I}\Big(X^\dagger_\nu(\partial^\nu X_\mu) - (\partial^\nu X^\dagger_\mu)X_\nu\Big)V^\mu \\
&-b_{7R}\epsilon_{\mu\nu\rho\sigma}\Big(X^{\dagger\mu}(\partial^\nu X^\rho) + X^\nu(\partial^\nu X^{\dagger\rho})\Big)V^\sigma \\
&-ib_{7I}\epsilon_{\mu\nu\rho\sigma}\Big(X^{\dagger\mu}(\partial^\nu X^\rho) - X^\nu(\partial^\nu X^{\dagger\rho})\Big)V^\sigma\,,
\end{aligned} \tag{3}$$

Where $X/X^\dagger$ is DM and its antiparticle, and V^μ is the mediator particle. This model has 5 couplings between DM and the mediator.

Feynmann Rules

The Feynmann Rules for this Lagrangian are introduced in arXiv:1810.01515. The vertex factor for the b_5 term is

$$-ib_5(p'_\mu + p_\mu)\,. \tag{4}$$

The vertex factor for the b_{6R} term is (where α is the index of the mediator, γ is the index for the particle entering the vertex, and β is the index of the particle leaving the vertex) is

$$b_{6R}(p'_\nu - p_\nu)\Big(g^{\nu\beta}\delta^\gamma_\mu + \delta^\beta_\mu g^{\nu\gamma}\Big)g^{\mu\alpha}\,, \tag{5}$$

where p' is the momentum of the particle exiting the vertex. The vertex factor for the b_{6I} term is

$$ib_{6I}(p'_\nu - p_\nu)\Big(g^{\nu\beta}\delta^\gamma_\mu - \delta^\beta_\mu g^{\nu\gamma}\Big)g^{\mu\alpha}\,. \tag{6}$$

The vertex factor for the b_{7R} term is

$$-b_{7R}\epsilon_{\mu\nu\rho\sigma}(p^\nu + p'^\nu)\,. \tag{7}$$

The vertex factor for the $b_{7,I}$ term is

$$-ib_{7I}\epsilon_{\mu\nu\rho\sigma}(p^\nu - p'^\nu)\,. \tag{8}$$

The propagator for the vector mediator is

$$-i\frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{m_V^2}}{k^2 - m_V^2}\,. \tag{9}$$

Each external spin-1 vector must have a polarisation $\epsilon_\mu/\epsilon^*_\mu$ depending on whether they are incoming or outgoing. Since these ϵ are real in the COM frame for the longitudinal polarisations, I will drop the conjugate signs for the remainder of this proof.

Notes and Notation

For these notes, I will use the signature:

$$(+,-,-,-) \tag{10}$$

We know that when summing over all polarizations:

$$\sum_{\lambda} \epsilon_i^{\mu} \epsilon_i^{*\nu} = -g^{\mu\nu} + \frac{p_i^{\mu} p_i^{\nu}}{m_i^2} \tag{11}$$

We also know:

$$\epsilon_{\mu}^{*} \epsilon^{\mu} = -1 \tag{12}$$

Therefore in the centre of Mass frame, the momentum of two incoming (p_1, p_2) and outgoing (p_3, p_4) DM particles of equal momenta are

$$\begin{aligned} p_1 &= (E, 0, 0, P) \\ p_2 &= (E, 0, 0, -P) \\ p_3 &= (E, P\sin\theta, 0, P\cos\theta) \\ p_4 &= (E, -P\sin\theta, 0, -P\cos\theta)\,, \end{aligned} \tag{13}$$

where $E^2 = P^2 + m_{DM}^2$, and P is the magnitude of the incoming momentum of each particle. From the two requirements that: $\epsilon_i \cdot p_i = 0$ and $\epsilon_i \cdot \epsilon_i = -1$, we can find that the longitudinal polarisations are

$$\begin{aligned} \epsilon_1 &= \frac{1}{m_{DM}}\Big(P, 0, 0, E\Big) \\ \epsilon_2 &= \frac{1}{m_{DM}}\Big(P, 0, 0, -E\Big) \\ \epsilon_3 &= \frac{1}{m_{DM}}\Big(P, E\sin\theta, 0, E\cos\theta\Big) \\ \epsilon_4 &= \frac{1}{m_{DM}}\Big(P, -E\sin\theta, 0, -E\cos\theta\Big)\,. \end{aligned} \tag{14}$$

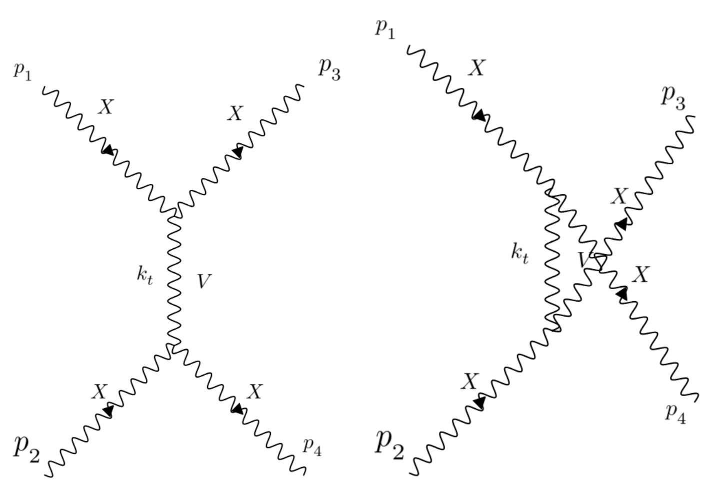
There are also two other transverse polarisations, these are of the form (for incoming particles)

$$\begin{aligned} \epsilon &= \Big(0, 1, 0, 0\Big) \\ \epsilon &= \Big(0, 0, 1, 0\Big)\,. \end{aligned} \tag{15}$$

These will not have the same dependence on P, and so when the incoming momentum is very high, these should be subdominant to the longitudinal polarisation. It is also worth noting that all momentum and polarisation terms commute.

Relevant Processes

For DM self-scattering, there are t-channel and u-channel diagrams. The u-channel diagrams are needed because of the indistinguishability of the two final particles.



[H] (16)

Since the u-channel process is simply a relabelling of the two outgoing states, from eqs (13) and (14) it can be seen that the u-channel amplitude will be the t-channel amplitude but with the transformation

$$\begin{aligned} \sin\theta &\to -\sin\theta \\ \cos\theta &\to -\cos\theta\,. \end{aligned} \tag{17}$$

Scattering Amplitudes

There are two vertices in each diagram with 5 vertex factors, and so there will be up to 25 different terms in each diagram

t-channel

We know that:

$$k_t = p_1 - p_3 = p_4 - p_2 = (0, -P\sin\theta, 0, P(1-\cos\theta)) \tag{18}$$

$$k_t^2 = -2P^2(1-\cos\theta) \tag{19}$$

The scattering amplitude of the t-channel diagram will be:

$$\begin{aligned} i\mathcal{M} = \Big[&- ib_5\epsilon_{3\tau}(p_{3\mu}+p_{1\mu})\epsilon_1^\tau \\ &+(b_{6R}+ib_{6I})\epsilon_{3\mu}(p_{3\alpha}-p_{1\alpha})\epsilon_1^\alpha + (b_{6R}-ib_{6I})\epsilon_3^\tau(p_{3\tau}-p_{1\tau})\epsilon_{1\mu} \\ &-\epsilon_3^\tau\epsilon_{\alpha\gamma\tau\mu}\Big((b_{7R}-ib_{7I})p_3^\gamma + (b_{7R}+ib_{7I})p_1^\gamma\Big)\epsilon_1^\alpha\Big] \\ &\times\Big(-i\frac{g^{\mu\nu}-\frac{k_t^\mu k_t^\nu}{m_V^2}}{k_t^2-m_V^2}\Big)\times \\ \Big[&- ib_5\epsilon_{2\sigma}(p_{4\nu}+p_{2\nu})\epsilon_4^\sigma \\ &+(b_{6R}+ib_{6I})\epsilon_{2\nu}(p_{2\sigma}-p_{4\sigma})\epsilon_4^\sigma + (b_{6R}-ib_{6I})\epsilon_2^\sigma(p_{2\sigma}-p_{4\sigma})\epsilon_{4\nu} \\ &-\epsilon_2^\sigma\epsilon_{\sigma\kappa\beta\nu}\Big((b_{7R}+ib_{7I})p_4^\kappa + (b_{7R}-ib_{7I})p_2^\kappa\Big)\epsilon_4^\beta\Big] \end{aligned} \tag{20}$$

There are three terms before the propagator and three afterwards. I will add up the individual contributions from each (using the notation (i, j) for ith term before the propagator and jth term after).

(1, 1):

The amplitude for this term is

$$\mathcal{M}_1 = \frac{(b_5)^2}{k_t^2-m_V^2}\Big[\epsilon_{3\tau}\epsilon_1^\tau\epsilon_{2\sigma}\epsilon_4^\sigma\Big](p_{3\mu}+p_{1\mu})\Big(g^{\mu\nu}-\frac{k_t^\mu k_t^\nu}{m_V^2}\Big)(p_{4\nu}+p_{2\nu})\,. \tag{21}$$

Substituting in the different polarisation and momenta gives

$$\mathcal{M}_1 = \frac{b_5^2}{m_{DM}^4\big(-2P^2(1-\cos\theta)-m_V^2\big)}\Big(P^2-(P^2+m_{DM}^2)\cos\theta\Big)^2\Big(6P^2+4m_{DM}^2+2P^2\cos\theta\Big)\,. \tag{22}$$

(1, 2):

The amplitude for this term is

$$\mathcal{M}_2 = \frac{ib_5}{k_t^2-m_V^2}(p_{3\mu}+p_{1\mu})\Big(g^{\mu\nu}-\frac{k_t^\mu k_t^\nu}{m_V^2}\Big)\Big((b_{6R}+ib_{6I})(p_{2\sigma}-p_{4\sigma})\Big[\epsilon_{3\tau}\epsilon_1^\tau\epsilon_{2\nu}\epsilon_4^\sigma\Big] + (b_{6R}-ib_{6I})(p_{2\sigma}-p_{4\sigma})\Big[\epsilon_{3\tau}\epsilon_1^\tau\epsilon_2^\sigma\epsilon_{4\nu}\Big]\Big)\,. \tag{23}$$

Since we know that $\epsilon_i\cdot p_i = 0$ and $(p_2\cdot\epsilon_4) = (p_4\cdot\epsilon_2)$, this can be simplified to

$$\mathcal{M}_2 = \frac{ib_5(p_2\cdot\epsilon_4)(\epsilon_3\cdot\epsilon_1)}{k_t^2-m_V^2}(p_{3\mu}+p_{1\mu})\Big(g^{\mu\nu}-\frac{k_t^\mu k_t^\nu}{m_V^2}\Big)\Big((b_{6R}+ib_{6I})\epsilon_{2\nu}-(b_{6R}-ib_{6I})\epsilon_{4\nu}\Big)\,. \tag{24}$$

Splitting into the two terms inside of the propagator ($\mathcal{M}_2 = \mathcal{M}_{21}+\mathcal{M}_{22}$)

$$\mathcal{M}_{21} = \frac{ib_5(p_2\cdot\epsilon_4)(\epsilon_3\cdot\epsilon_1)}{k_t^2-m_V^2}\Big((b_{6R}+ib_{6I})((p_3\cdot\epsilon_2)+(p_1\cdot\epsilon_2))-(b_{6R}-ib_{6I})((p_3\cdot\epsilon_4)+(p_1\cdot\epsilon_4))\Big)\,. \tag{25}$$

We know that $(p_3 \cdot \epsilon_2) + (p_1 \cdot \epsilon_2) = (p_3 \cdot \epsilon_4) + (p_1 \cdot \epsilon_4)$

$$\mathcal{M}_{21} = \frac{-2b_5 b_{6I}(p_2 \cdot \epsilon_4)(\epsilon_3 \cdot \epsilon_1)((p_3 \cdot \epsilon_2) + (p_1 \cdot \epsilon_2))}{k_t^2 - m_V^2}\,. \tag{26}$$

For this second term...

$$\mathcal{M}_{22} = \frac{-ib_5(p_2 \cdot \epsilon_4)(\epsilon_3 \cdot \epsilon_1)}{m_V^2(k_t^2 - m_V^2)}(p_{3\mu} + p_{1\mu})\Big(k_t^\mu k_t^\nu\Big)\Big((b_{6R} + ib_{6I})\epsilon_{2\nu} - (b_{6R} - ib_{6I})\epsilon_{4\nu}\Big) \tag{27}$$

However, since $(p_1 \cdot k_t) = -(p_3 \cdot k_t)$, this term vanishes. Therefore

$$\mathcal{M}_{2} = \frac{-2b_5 b_{6I}(p_2 \cdot \epsilon_4)(\epsilon_3 \cdot \epsilon_1)((p_3 \cdot \epsilon_2) + (p_1 \cdot \epsilon_2))}{k_t^2 - m_V^2}\,. \tag{28}$$

Substituting in the momenta and polarisations gives

$$\mathcal{M}_2 = \frac{-2b_5 b_{6I} E^2 P^2(1-\cos\theta)(P^2 - E^2\cos^2\theta)(3+\cos\theta)}{m_{DM}^4(-2P^2(1-\cos\theta) - m_V^2)}\,. \tag{29}$$

(1, 3)

The amplitude for this term is

$$\mathcal{M}_3 = \frac{ib_5}{k_t^2 - m_V^2}\Big[-\epsilon_{3\tau}(p_{3\mu} + p_{1\mu})\epsilon_1^\tau\Big]\Big(g^{\mu\nu} - \frac{k_t^\mu k_t^\nu}{m_V^2}\Big)\Big[-\epsilon_2^\sigma \epsilon_{\sigma\kappa\beta\nu}\Big((b_{7R} + ib_{7I})p_4^\kappa + (b_{7R} - ib_{7I})p_2^\kappa\Big)\epsilon_4^\beta\Big] \tag{30}$$

This term must vanish as there are no non-zero terms in any of the third components of the polarisations or momenta in the COM frame. When contracted with the Levi-Cevita symbol, these terms are all zero, and any term without these has a repeated index in the Levi-Civita symbol (for which it is zero).

$$\mathcal{M}_3 = 0 \tag{31}$$

(2, 1)

The amplitude for this term is

$$\mathcal{M}_4 = \frac{ib_5}{k_t^2 - m_V^2}\Big((b_{6R} + ib_{6I})\Big[\epsilon_{3\mu}\epsilon_1^\alpha \epsilon_{2\sigma}\epsilon_4^\sigma\Big](p_{3\alpha} - p_{1\alpha}) + (b_{6R} - ib_{6I})\Big[\epsilon_3^\tau \epsilon_{1\mu}\epsilon_{2\sigma}\epsilon_4^\sigma\Big](p_{3\tau} - p_{1\tau})\Big)\Big(g^{\mu\nu} - \frac{k_t^\mu k_t^\nu}{m_V^2}\Big)\Big(p_{4\nu} + p_{2\nu}\Big) \tag{32}$$

Through the same process as the **(1,2)** term, this amplitude becomes

$$\mathcal{M}_4 = \frac{-2b_{6I} b_5 E^2 P^2(P^2 - E^2\cos\theta)(1-\cos\theta)(3+\cos\theta)}{m_{DM}^4(-2P^2(1-\cos\theta) - m_V^2)} = \mathcal{M}_2\,, \tag{33}$$

which is identical to the **(1,2)** term.

(2, 2)

The amplitude for this term is

$$\begin{aligned}\mathcal{M}_5 = \frac{-1}{k_t^2 - m_V^2}&\Big[(b_{6R} + ib_{6I})\epsilon_{3\mu}(p_{3\alpha} - p_{1\alpha})\epsilon_1^\alpha + (b_{6R} - ib_{6I})\epsilon_3^\tau(p_{3\tau} - p_{1\tau})\epsilon_{1\mu}\Big]\\ &\times\Big(g^{\mu\nu} - \frac{k_t^\mu k_t^\nu}{m_V^2}\Big)\times\\ &\Big[(b_{6R} + ib_{6I})\epsilon_{2\nu}(p_{2\sigma} - p_{4\sigma})\epsilon_4^\sigma + (b_{6R} - ib_{6I})\epsilon_2^\sigma(p_{2\sigma} - p_{4\sigma})\epsilon_{4\nu}\Big]\end{aligned} \tag{34}$$

Splitting into the two terms inside of the propagator ($\mathcal{M}_5 = \mathcal{M}_{51} + \mathcal{M}_{52}$). This first term is

$$\begin{aligned}\mathcal{M}_{51} = \frac{-1}{k_t^2 - m_V^2}&\Big[(b_{6R} + ib_{6I})\epsilon_{3\mu}(p_{3\alpha} - p_{1\alpha})\epsilon_1^\alpha + (b_{6R} - ib_{6I})\epsilon_3^\tau(p_{3\tau} - p_{1\tau})\epsilon_{1\mu}\Big]\\ &\Big[(b_{6R} + ib_{6I})\epsilon_2^\mu(p_{2\sigma} - p_{4\sigma})\epsilon_4^\sigma + (b_{6R} - ib_{6I})\epsilon_2^\sigma(p_{2\sigma} - p_{4\sigma})\epsilon_4^\mu\Big],\end{aligned} \tag{35}$$

which becomes

$$\mathcal{M}_{51} = \frac{-2E^2P^2(1-\cos\theta)^2}{m_{DM}^4(-2P^2(1-\cos\theta)-m_V^2)}\Big[b_{6R}^2(E^2(\cos\theta-1)) - b_{6I}^2(2P^2+E^2(1+\cos\theta))\Big]\,, \tag{36}$$

while the second term becomes

$$\mathcal{M}_{52} = \frac{4b_{6R}^2E^4P^4(1-\cos\theta)^4}{m_{DM}^4m_V^2(-2P^2(1-\cos\theta)-m_V^2)}\,. \tag{37}$$

Therefore the $\mathcal{M}_5$ is

$$\mathcal{M}_5 = \frac{-2E^2P^2(1-\cos\theta)^2}{m_{DM}^4(-2P^2(1-\cos\theta)-m_V^2)}\Big[b_{6R}^2(E^2(\cos\theta-1)) - b_{6I}^2(2P^2+E^2(1+\cos\theta))\Big] + \frac{4b_{6R}^2E^4P^4(1-\cos\theta)^4}{m_{DM}^4m_V^2(-2P^2(1-\cos\theta)-m_V^2)}\,. \tag{38}$$

(2, 3)

The amplitude for this term is

$$\begin{aligned}\mathcal{M}_6 = \frac{-1}{k_t^2-m_V^2}\Big[(b_{6R}+ib_{6I})\epsilon_{3\mu}(p_{3\alpha}-p_{1\alpha})\epsilon_1^\alpha + (b_{6R}-ib_{6I})\epsilon_3^\tau(p_{3\tau}-p_{1\tau})\epsilon_{1\mu}\Big]\\ \times\Big(g^{\mu\nu}-\frac{k_t^\mu k_t^\nu}{m_V^2}\Big)\times\\ \Big[-\epsilon_2^\sigma\epsilon_{\sigma\kappa\beta\nu}\Big((b_{7R}+ib_{7I})p_4^\kappa + (b_{7R}-ib_{7I})p_2^\kappa\Big)\epsilon_4^\beta\Big]\end{aligned} \tag{39}$$

This term must vanish as there are no non-zero terms in any of the third components of the polarisations or momenta in the COM frame. When contracted with the Levi-Cevita symbol, these terms are all zero, and any term without these has a repeated index in the Levi-Civita symbol (for which it is zero).

$$\mathcal{M}_6 = 0 \tag{40}$$

(3, 1)

The amplitude for this term is

$$\begin{aligned}\mathcal{M}_7 = \frac{ib_5}{k_t^2-m_V^2}\Big[-\epsilon_3^\tau\epsilon_{\alpha\gamma\tau\mu}\Big((b_{7R}-ib_{7I})p_3^\gamma + (b_{7R}+ib_{7I})p_1^\gamma\Big)\epsilon_1^\alpha\Big]\\ \times\Big(g^{\mu\nu}-\frac{k_t^\mu k_t^\nu}{m_V^2}\Big)\times\\ \Big[\epsilon_{2\sigma}(p_{4\nu}+p_{2\nu})\epsilon_4^\sigma\Big]\end{aligned} \tag{41}$$

This term must vanish as there are no non-zero terms in any of the third components of the polarisations or momenta in the COM frame. When contracted with the Levi-Cevita symbol, these terms are all zero, and any term without these has a repeated index in the Levi-Civita symbol (for which it is zero).

$$\mathcal{M}_7 = 0 \tag{42}$$

(3, 2)

The amplitude for this term is

$$\begin{aligned}\mathcal{M}_8 = \frac{-1}{k_t^2-m_V^2}\Big[-\epsilon_3^\tau\epsilon_{\alpha\gamma\tau\mu}\Big((b_{7R}-ib_{7I})p_3^\gamma + (b_{7R}+ib_{7I})p_1^\gamma\Big)\epsilon_1^\alpha\Big]\\ \times\Big(g^{\mu\nu}-\frac{k_t^\mu k_t^\nu}{m_V^2}\Big)\times\\ \Big[(b_{6R}+ib_{6I})\epsilon_{2\nu}(p_{2\sigma}-p_{4\sigma})\epsilon_4^\sigma + (b_{6R}-ib_{6I})\epsilon_2^\sigma(p_{2\sigma}-p_{4\sigma})\epsilon_{4\nu}\Big]\end{aligned} \tag{43}$$

This term must vanish as there are no non-zero terms in any of the third components of the polarisations or momenta in the COM frame. When contracted with the Levi-Cevita symbol, these terms are all zero, and any term without these has a repeated index in the Levi-Civita symbol (for which it is zero).

$$\mathcal{M}_8 = 0 \tag{44}$$

(3, 3)

The amplitude for this term is

$$\mathcal{M}_9 = \frac{-1}{k_t^2 - m_V^2}\Big[-\epsilon_3^\tau \epsilon_{\alpha\gamma\tau\mu}\Big((b_{7R} - ib_{7I})p_3^\gamma + (b_{7R} + ib_{7I})p_1^\gamma\Big)\epsilon_1^\alpha\Big] \times\Big(g^{\mu\nu} - \frac{k_t^\mu k_t^\nu}{m_V^2}\Big)\times \Big[-\epsilon_2^\sigma \epsilon_{\sigma\kappa\beta\nu}\Big((b_{7R} + ib_{7I})p_4^\kappa + (b_{7R} - ib_{7I})p_2^\kappa\Big)\epsilon_4^\beta\Big] \tag{45}$$

We can see that this term must automatically be zero unless $\mu = 2$ and $\nu = 2$ (given the form of the momenta and polarisations in the C.O.M. frame).

$$\mathcal{M}_9 = \frac{-1}{k_t^2 - m_V^2}\Big[-\epsilon_3^\tau \epsilon_{\alpha\gamma\tau 2}\Big((b_{7R} - ib_{7I})p_3^\gamma + (b_{7R} + ib_{7I})p_1^\gamma\Big)\epsilon_1^\alpha\Big] \times\Big(g^{22} - \frac{k_t^2 k_t^2}{m_V^2}\Big)\times \Big[-\epsilon_2^\sigma \epsilon_{\sigma\kappa\beta 2}\Big((b_{7R} + ib_{7I})p_4^\kappa + (b_{7R} - ib_{7I})p_2^\kappa\Big)\epsilon_4^\beta\Big] \tag{46}$$

From the form of k_t we can see that the second term in the propagator is zero.

$$\mathcal{M}_9 = \frac{1}{k_t^2 - m_V^2}\Big[\epsilon_3^\tau \epsilon_1^\alpha \epsilon_2^\sigma \epsilon_4^\beta\Big]\Big[\epsilon_{\alpha\gamma\tau 2}\Big((b_{7R} - ib_{7I})p_3^\gamma + (b_{7R} + ib_{7I})p_1^\gamma\Big)\Big] \Big[\epsilon_{\sigma\kappa\beta 2}\Big((b_{7R} + ib_{7I})p_4^\kappa + (b_{7R} - ib_{7I})p_2^\kappa\Big)\Big] \tag{47}$$

Going through all combinations of the non-zero Levi-Cevita terms (16 of them):

$$\mathcal{M}_9 = \sum_{i=1,16} \mathcal{M}_{9i} \tag{48}$$

$\epsilon_{0132} = -1$, $\epsilon_{0132} = -1$:

$$\mathcal{M}_{91} = \frac{E^2 P^4 \sin^2\theta \cos^2\theta}{m_{DM}^4 (k_t^2 - m_V^2)}\Big(b_{7R}^2 + b_{7I}^2\Big) \tag{49}$$

$\epsilon_{0312} = 1$, $\epsilon_{0132} = -1$:

$$\mathcal{M}_{92} = \frac{-E^2 P^4 \sin^2\theta \cos\theta}{m_{DM}^4 (k_t^2 - m_V^2)}\Big[\Big((b_{7R} - ib_{7I})\cos\theta + (b_{7R} + ib_{7I})\Big)\Big(b_{7R} + ib_{7I}\Big) \tag{50}$$

$\epsilon_{3012} = -1$, $\epsilon_{0132} = -1$:

$$\mathcal{M}_{93} = \frac{E^4 P^2 \sin^2\theta \cos\theta}{m_{DM}^4 (k_t^2 - m_V^2)}\Big(2b_{7R}\Big)\Big(b_{7R} + ib_{7I}\Big) \tag{51}$$

$\epsilon_{3102} = 1$, $\epsilon_{0132} = -1$:

$$\mathcal{M}_{94} = \frac{-E^2 P^4 \sin^2\theta \cos\theta}{m_{DM}^4 (k_t^2 - m_V^2)}\Big(b_{7R}^2 + b_{7I}^2\Big) \tag{52}$$

$\epsilon_{0132} = -1$, $\epsilon_{0312} = 1$:

$$\mathcal{M}_{95} = \frac{-E^2 P^4 \sin^2\theta \cos\theta}{m_{DM}^4 (k_t^2 - m_V^2)}\Big(b_{7R} - ib_{7I}\Big)\Big[\Big((b_{7R} + ib_{7I})\cos\theta + (b_{7R} - ib_{7I})\Big)\Big] \tag{53}$$

$\epsilon_{0312} = 1$, $\epsilon_{0312} = 1$:

$$\mathcal{M}_{96} = \frac{E^2 P^4 \sin^2\theta}{m_{DM}^4(k_t^2 - m_V^2)} \Big[\Big((b_{7R} - ib_{7I})\cos\theta + (b_{7R} + ib_{7I})\Big)\Big]\Big[\Big((b_{7R} + ib_{7I})\cos\theta + (b_{7R} - ib_{7I})\Big)\Big] \tag{54}$$

$\epsilon_{3012} = -1$, $\epsilon_{0312} = 1$:

$$\mathcal{M}_{97} = \frac{-E^4 P^2 \sin^2\theta}{m_{DM}^4(k_t^2 - m_V^2)} \Big(2b_{7R}\Big)\Big[\Big((b_{7R} + ib_{7I})\cos\theta + (b_{7R} - ib_{7I})\Big)\Big] \tag{55}$$

$\epsilon_{3102} = 1$, $\epsilon_{0312} = 1$:

$$\mathcal{M}_{98} = \frac{E^2 P^4 \sin^2\theta}{m_{DM}^4(k_t^2 - m_V^2)} \Big[\Big((b_{7R} - ib_{7I})\Big)\Big]\Big[\Big((b_{7R} + ib_{7I})\cos\theta + (b_{7R} - ib_{7I})\Big)\Big] \tag{56}$$

$\epsilon_{0132} = -1$, $\epsilon_{3012} = -1$:

$$\mathcal{M}_{99} = \frac{E^4 P^2 \sin^2\theta\cos\theta}{m_{DM}^4(k_t^2 - m_V^2)} \Big(2b_{7R}(b_{7R} - ib_{7I})\Big) \tag{57}$$

$\epsilon_{0312} = 1$, $\epsilon_{3012} = -1$:

$$\mathcal{M}_{910} = \frac{-E^4 P^2 \sin^2\theta}{m_{DM}^4(k_t^2 - m_V^2)} \Big[\Big((b_{7R} - ib_{7I})\cos\theta + (b_{7R} + ib_{7I})\Big)\Big]\Big[\Big(2b_{7R}\Big)\Big] \tag{58}$$

$\epsilon_{3012} = -1$, $\epsilon_{3012} = -1$:

$$\mathcal{M}_{911} = \frac{E^6 \sin^2\theta}{m_{DM}^4(k_t^2 - m_V^2)} \Big(4b_{7R}^2\Big) \tag{59}$$

$\epsilon_{3102} = 1$, $\epsilon_{3012} = -1$:

$$\mathcal{M}_{912} = \frac{-E^4 P^2 \sin^2\theta}{m_{DM}^4(k_t^2 - m_V^2)} \Big(2b_{7R}(b_{7R} - ib_{7I})\Big) \tag{60}$$

$\epsilon_{0132} = -1$, $\epsilon_{3102} = 1$:

$$\mathcal{M}_{913} = \frac{-E^2 P^4 \sin^2\theta\cos\theta}{m_{DM}^4(k_t^2 - m_V^2)} \Big(b_{7R}^2 + b_{7I}^2\Big) \tag{61}$$

$\epsilon_{0312} = 1$, $\epsilon_{3102} = 1$:

$$\mathcal{M}_{914} = \frac{E^2 P^4 \sin^2\theta}{m_{DM}^4(k_t^2 - m_V^2)} \Big[\Big((b_{7R} - ib_{7I})\cos\theta + (b_{7R} + ib_{7I})\Big)\Big]\Big[\Big((b_{7R} + ib_{7I})\Big)\Big] \tag{62}$$

$\epsilon_{3012} = -1$, $\epsilon_{3102} = 1$:

$$\mathcal{M}_{915} = \frac{-E^4 P^2 \sin^2\theta}{m_{DM}^4(k_t^2 - m_V^2)} \Big(2b_{7R}(b_{7R} + ib_{7I})\Big) \tag{63}$$

$\epsilon_{3102} = 1$, $\epsilon_{3102} = 1$:

$$\mathcal{M}_{916} = \frac{E^2 P^4 \sin^2\theta}{m_{DM}^4(k_t^2 - m_V^2)} \Big(b_{7R}^2 + b_{7I}^2\Big) \tag{64}$$

Summing These together gives

$$
\begin{aligned}
\mathcal{M}_9 = \frac{\sin^2\theta}{m_{DM}^4(k_t^2 - m_V^2)}\Bigg[\\
P^4E^2\Big[(b_{7R}+ib_{7I})(b_{7R}-ib_{7I}) + (b_{7R}-ib_{7I})(b_{7R}-ib_{7I}) + (b_{7R}+ib_{7I})(b_{7R}+ib_{7I}) + b_{7R}^2 + b_{7I}^2\Big] \\
P^2E^4\Big[-2b_{7R}(b_{7R}-ib_{7I}) - 2b_{7R}(b_{7R}+ib_{7I}) - 2b_{7R}(b_{7R}-ib_{7I}) - 2b_{7R}(b_{7R}+ib_{7I})\Big] \\
E^6\Big[4b_{7R}^2\Big] \\
E^2P^4\cos\theta\Big[-(b_{7R}+ib_{7I})(b_{7R}+ib_{7I}) - (b_{7R}^2+b_{7I}^2) - (b_{7R}-ib_{7I})(b_{7R}-ib_{7I}) + (b_{7R}-ib_{7I})(b_{7R}-ib_{7I}) \\
+(b_{7R}+ib_{7I})(b_{7R}+ib_{7I}) + (b_{7R}-ib_{7I})(b_{7R}+ib_{7I}) - (b_{7R}^2+b_{7I}^2) + (b_{7R}-ib_{7I})(b_{7R}+ib_{7R})\Big] \\
E^4P^2\cos\theta\Big[2b_{7R}(b_{7R}+ib_{7I}) - 2b_{7R}(b_{7R}+ib_{7I}) + 2b_{7R}(b_{7R}-ib_{7I}) - 2b_{7R}(b_{7R}-ib_{7I})\Big] \\
E^2P^4\cos^2\theta\Big[b_{7R}^2 + b_{7I}^2 - (b_{7R}+ib_{7I})(b_{7R}-ib_{7I}) - (b_{7R}-ib_{7I})(b_{7R}+ib_{7I}) + (b_{7R}-ib_{7I})(b_{7R}+ib_{7I})\Big].\Bigg]
\end{aligned}
\tag{65}
$$

This simplifies greatly to

$$
\mathcal{M}_9 = \frac{4b_{7R}^2\sin^2\theta}{m_{DM}^4(-2P^2(1-\cos\theta) - m_V^2)}\Big[P^4E^2 - 2P^2E^4 + E^6\Big] \tag{66}
$$

The b_{7I} term cancels entirely.

Total t-channel Amplitude

Adding all of these together we get:

$$
\mathcal{M}_t = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_4 + \mathcal{M}_5 + \mathcal{M}_9 \tag{67}
$$

$$
\begin{aligned}
\mathcal{M}_t = \frac{1}{m_{DM}^4\big(-2P^2(1-\cos\theta) - m_V^2\big)}\Bigg[b_5^2\Big(P^2 - (P^2+m_{DM}^2)\cos\theta\Big)^2\Big(6P^2 + 4m_{DM}^2 + 2P^2\cos\theta\Big) \\
-4b_{6I}b_5E^2P^2(P^2 - E^2\cos\theta)(1-\cos\theta)(3+\cos\theta) \\
-2E^2P^2(1-\cos\theta)^2\Big[b_{6R}^2(E^2(\cos\theta-1)) - b_{6I}^2(2P^2 + E^2(1+\cos\theta))\Big] + \frac{4b_{6R}^2E^4P^4(1-\cos\theta)^4}{m_V^2} \\
+4b_{7R}^2\sin^2\theta\Big[P^4E^2 - 2P^2E^4 + E^6\Big]\Bigg]
\end{aligned}
\tag{68}
$$

u-channel

The u-channel scattering amplitude will simply be the t-channel one with two final states relabelled. For the amplitude this has the effect that:

$$
\cos\theta \to -\cos\theta \tag{69}
$$

$$
\sin\theta \to -\sin\theta \tag{70}
$$

The total amplitude for the u-channel process is therefore:

$$
\begin{aligned}
\mathcal{M}_u = \frac{1}{m_{DM}^4\big(-2P^2(1+\cos\theta) - m_V^2\big)}\Bigg[b_5^2\Big(P^2 + (P^2+m_{DM}^2)\cos\theta\Big)^2\Big(6P^2 + 4m_{DM}^2 - 2P^2\cos\theta\Big) \\
-4b_{6I}b_5E^2P^2(P^2 + E^2\cos\theta)(1+\cos\theta)(3-\cos\theta) \\
-2E^2P^2(1+\cos\theta)^2\Big[-b_{6R}^2(E^2(\cos\theta+1)) - b_{6I}^2(2P^2 + E^2(1-\cos\theta))\Big] + \frac{4b_{6R}^2E^4P^4(1+\cos\theta)^4}{m_V^2} \\
+4b_{7R}^2\sin^2\theta\Big[P^4E^2 - 2P^2E^4 + E^6\Big]\Bigg]
\end{aligned}
\tag{71}
$$

Total scattering amplitude

The total amplitude for this process is:

$$\mathcal{M} = \mathcal{M}_t + \mathcal{M}_u \tag{72}$$

$$\begin{aligned}
\mathcal{M} = \frac{1}{m_{DM}^4\big(-2P^2(1-\cos\theta)-m_V^2\big)}\Bigg[&b_5^2\Big(P^2-(P^2+m_{DM}^2)\cos\theta\Big)^2\Big(6P^2+4m_{DM}^2+2P^2\cos\theta\Big)\\
&-4b_{6I}b_5E^2P^2(P^2-E^2\cos\theta)(1-\cos\theta)(3+\cos\theta)\\
-2E^2P^2(1-\cos\theta)^2\Big[&b_{6R}^2(E^2(\cos\theta-1))-b_{6I}^2(2P^2+E^2(1+\cos\theta))\Big]+\frac{4b_{6R}^2E^4P^4(1-\cos\theta)^4}{m_V^2}\\
&+4b_{7R}^2\sin^2\theta\Big[P^4E^2-2P^2E^4+E^6\Big]\Bigg]\\
+\frac{1}{m_{DM}^4\big(-2P^2(1+\cos\theta)-m_V^2\big)}\Bigg[&b_5^2\Big(P^2+(P^2+m_{DM}^2)\cos\theta\Big)^2\Big(6P^2+4m_{DM}^2-2P^2\cos\theta\Big)\\
&-4b_{6I}b_5E^2P^2(P^2+E^2\cos\theta)(1+\cos\theta)(3-\cos\theta)\\
-2E^2P^2(1+\cos\theta)^2\Big[&-b_{6R}^2(E^2(\cos\theta+1))-b_{6I}^2(2P^2+E^2(1-\cos\theta))\Big]+\frac{4b_{6R}^2E^4P^4(1+\cos\theta)^4}{m_V^2}\\
&+4b_{7R}^2\sin^2\theta\Big[P^4E^2-2P^2E^4+E^6\Big]\Bigg]
\end{aligned} \tag{73}$$

Expressing this in terms of $s = 4E^2 = 4P^2 + 4m_{DM}^2$:

$$\begin{aligned}
\mathcal{M} = \frac{-1}{m_{DM}^4\big((\frac{s}{2}-2m_{DM}^2)(1-\cos\theta)+m_V^2\big)}\Bigg[&b_5^2\Big(\frac{s}{4}-m_{DM}^2-\frac{s}{4}\cos\theta\Big)^2\Big(\frac{3s}{2}-2m_{DM}^2+(\frac{s}{2}-2m_{DM}^2)\cos\theta\Big)\\
&-b_{6I}b_5s(\frac{s}{4}-m_{DM}^2)(\frac{s}{4}-m_{DM}^2-\frac{s}{4}\cos\theta)(1-\cos\theta)(3+\cos\theta)\\
-\frac{s}{2}(\frac{s}{4}-m_{DM}^2)(1-\cos\theta)^2\Big[&b_{6R}^2\frac{s}{4}(\cos\theta-1)-b_{6I}^2(\frac{3s}{4}-2m_{DM}^2+\frac{s}{4}\cos\theta)\Big]+\frac{b_{6R}^2s^2(\frac{s}{4}-m_{DM}^2)^2(1-\cos\theta)^4}{4m_V^2}\\
&+b_{7R}^2\sin^2\theta\Big[sm_{DM}^4\Big]\Bigg]\\
+\frac{-1}{m_{DM}^4\big((\frac{s}{2}-2m_{DM}^2)(1+\cos\theta)+m_V^2\big)}\Bigg[&b_5^2\Big(\frac{s}{4}-m_{DM}^2+\frac{s}{4}\cos\theta\Big)^2\Big(\frac{3s}{2}-2m_{DM}^2-(\frac{s}{2}-2m_{DM}^2)\cos\theta\Big)\\
&-b_{6I}b_5s(\frac{s}{4}-m_{DM}^2)(\frac{s}{4}-m_{DM}^2+\frac{s}{4}\cos\theta)(1+\cos\theta)(3-\cos\theta)\\
-\frac{s}{2}(\frac{s}{4}-m_{DM}^2)(1+\cos\theta)^2\Big[&-b_{6R}^2\frac{s}{4}(\cos\theta+1)-b_{6I}^2(\frac{3s}{4}-2m_{DM}^2-\frac{s}{4}\cos\theta)\Big]+\frac{b_{6R}^2s^2(\frac{s}{4}-m_{DM}^2)^2(1+\cos\theta)^4}{4m_V^2}\\
&+b_{7R}^2\sin^2\theta\Big[sm_{DM}^4\Big]\Bigg]
\end{aligned} \tag{74}$$

Unitarity Bound

The next step is to plug this amplitude into the equation at zeroth order:

$$\mathcal{M}_{ij}^0(s) = \frac{\beta_{ii}}{64\pi}\int_{-1}^{1} d\cos\theta\,\mathcal{M}(s,\cos\theta) \tag{75}$$

For ease in performing the integral with Mathematica, I will write it out in terms of different factors that are constant in θ. Pulling some terms to the left-hand side, I will write it in the form:

$$-\frac{64\pi m_{DM}^4}{\beta_{ii}}\mathcal{M}_{ij}^0(s) = term1 + term2 + term3 + term4 \tag{76}$$

Where

$$\begin{aligned}
term1+term2+term3+term4 = \int_{-1}^{1} d\cos\theta \frac{1}{A(1-\cos\theta)+B}\Bigg[& C\Big(D-E\cos\theta\Big)^2\Big(F+A\cos\theta\Big) \\
& +H(D-E\cos\theta)(1-\cos\theta)(3+\cos\theta) \\
& +K(1-\cos\theta)^2\Big[L(\cos\theta-1)+M(N+E\cos\theta)\Big]+P(1-\cos\theta)^4 \\
& +Q\sin^2\theta\Bigg] \\
+\frac{1}{A(1+\cos\theta)+B}\Bigg[& C\Big(D+E\cos\theta\Big)^2\Big(F-A\cos\theta\Big) \\
& +H(D+E\cos\theta)(1+\cos\theta)(3-\cos\theta) \\
& +K(1+\cos\theta)^2\Big[-L(\cos\theta+1)+M(N-E\cos\theta)\Big]+P(1+\cos\theta)^4 \\
& +Q\sin^2\theta\Bigg],
\end{aligned} \tag{77}$$

and

$$\begin{aligned}
A &= \frac{s}{2}-2m_{DM}^2 \\
B &= m_V^2 \\
C &= b_5^2 \\
D &= \frac{s}{4}-m_{DM}^2 \\
E &= \frac{s}{4} \\
F &= \frac{3s}{2}-2m_{DM}^2 \\
H &= -b_{6I}b_5 s(\frac{s}{4}-m_{DM}^2) \\
K &= -\frac{s}{2}(\frac{s}{4}-m_{DM}^2) \\
L &= b_{6R}^2\frac{s}{4} \\
M &= -b_{6I}^2 \\
N &= \frac{3s}{4}-2m_{DM}^2 \\
P &= \frac{b_{6R}^2 s^2(\frac{s}{4}-m_{DM}^2)^2}{4m_V^2} \\
Q &= \Big(b_{7R}^2\Big[sm_{DM}^4\Big]\Big).
\end{aligned} \tag{78}$$

Performing this integral on each term gives:

Term 1

$$\begin{aligned}
term1 = \int_{-1}^{1} d\cos\theta \frac{1}{A(1-\cos\theta)+B}&\Bigg[C\Big(D-E\cos\theta\Big)^2\Big(F+A\cos\theta\Big)\Bigg] \\
+\frac{1}{A(1+\cos\theta)+B}&\Bigg[C\Big(D+E\cos\theta\Big)^2\Big(F-A\cos\theta\Big)\Bigg]
\end{aligned} \tag{79}$$

$$\begin{aligned}
term1 = \Bigg[\frac{-1}{3A^3}C(&2Ax(A^2(3D^2-6DE+E^2(x^2+3))+3AE(-2BD+2BE-2DF+EF)+3BE^2(B+F)) \\
&+3(A+B+F)(A(D-E)-BE)^2 log(A(x-1)-B) \\
&-3(A+B+F)(A(E-D)+BE)^2 log(Ax+A+B))\Bigg]_{-1}^{1}
\end{aligned} \tag{80}$$

Evaluating the integration limits

$$term1 = \frac{-1}{3A^3}C\Bigg[4A(A^2(3D^2-6DE+4E^2)+3AE(-2BD+2BE-2DF+EF)+3BE^2(B+F)) \\ +6(A+B+F)(A(D-E)-BE)^2log(\frac{B}{2A+B})\Bigg] \tag{81}$$

Substituting back in the terms

$$term1 = \frac{-1}{3(\frac{s}{2}-2m_{DM}^2)^3}b_5^2\Bigg[(2s-8m_{DM}^2)((\frac{s}{2}-2m_{DM}^2)^2(3(\frac{s}{4}-m_{DM}^2)^2-\frac{3s}{2}(\frac{s}{4}-m_{DM}^2)+\frac{s^2}{4}) \\ +\frac{3s}{4}(\frac{s}{2}-2m_{DM}^2)(-2m_V^2(\frac{s}{4}-m_{DM}^2)+\frac{m_V^2 s}{2}-(\frac{s}{2}-2m_{DM}^2)(\frac{3s}{2}-2m_{DM}^2)+\frac{s}{4}(\frac{3s}{2}-2m_{DM}^2)) \\ +\frac{3m_V^2 s^2}{16}(m_V^2+\frac{3s}{2}-2m_{DM}^2)) \\ +6(2s-4m_{DM}^2+m_V^2)((\frac{s}{2}-2m_{DM}^2)((\frac{s}{4}-m_{DM}^2)-\frac{s}{4})-\frac{m_V^2 s}{4})^2log(\frac{m_V^2}{s-4m_{DM}^2+m_V^2})\Bigg] \tag{82}$$

Term 2

$$term2 = \int_{-1}^{1} d\cos\theta \frac{1}{A(1-\cos\theta)+B}\Bigg[H(D-E\cos\theta)(1-\cos\theta)(3+\cos\theta)\Bigg] \\ +\frac{1}{A(1+\cos\theta)+B}\Bigg[H(D+E\cos\theta)(1+\cos\theta)(3-\cos\theta)\Bigg] \tag{83}$$

$$term2 = \Bigg[\frac{H}{3A^4}(-2Ax(A^2(Ex^2-9D)-3AB(D-4E)+3B^2E) \\ -3B(4A+B)(A(E-D)+BE)log(A(x-1)-B) \\ +3B(4A+B)(A(E-D)+BE)log(Ax+A+B))\Bigg]_{-1}^{1} \tag{84}$$

Evaluating the Integration limits

$$term2 = \frac{H}{3A^4}\Bigg[-4A(A^2(E-9D)-3AB(D-4E)+3B^2E) \\ +6B(4A+B)(A(E-D)+BE)log(\frac{2A+B}{B})\Bigg] \tag{85}$$

Substituting the terms in

$$term2 = \frac{-b_{6I}b_5 s(\frac{s}{4}-m_{DM}^2)}{3(\frac{s}{2}-2m_{DM}^2)^4}\Bigg[-(2s-8m_{DM}^2)((\frac{s}{2}-2m_{DM}^2)^2(-2s+9m_{DM}^2)-3m_V^2(\frac{s}{2}-2m_{DM}^2)(-\frac{3s}{4}-m_{DM}^2)+\frac{3m_V^4 s}{4}) \\ +6m_V^2(2s-8m_{DM}^2+m_V^2)(m_{DM}^2(\frac{s}{2}-2m_{DM}^2)+\frac{m_V^2 s}{4})log(\frac{s-4m_{DM}^2+m_V^2}{m_V^2})\Bigg] \tag{86}$$

Term 3

$$term3 = \int_{-1}^{1} d\cos\theta \frac{1}{A(1-\cos\theta)+B}\Bigg[K(1-\cos\theta)^2\Big[L(\cos\theta-1)+M(N+E\cos\theta)\Big]\Bigg] \\ +\frac{1}{A(1+\cos\theta)+B}\Bigg[K(1+\cos\theta)^2\Big[-L(\cos\theta+1)+M(N-E\cos\theta)\Big]\Bigg] \tag{87}$$

$$term3 = \left[\frac{-K}{3A^4}(2Ax(A^2(EMx^2 + L(x^2+3) - 3MN) - 3AB(L - MN) + 3B^2(EM + L)) \right.$$
$$+3B^2 log(A(-x) + A + B)(AM(E + N) + B(EM + L))$$
$$\left. -3B^2 log(Ax + A + B)(AM(E + N) + B(EM + L))) \right]_{-1}^{1} \tag{88}$$

Evaluating the Integration limits

$$term3 = \frac{-K}{3A^4}\left[4A(A^2(EM + 4L - 3MN) - 3AB(L - MN) + 3B^2(EM + L)) \right.$$
$$\left. +6B^2 log(\frac{B}{2A + B})(AM(E + N) + B(EM + L)) \right] \tag{89}$$

Substituting in the terms

$$term3 = \frac{\frac{s}{2}(\frac{s}{4} - m_{DM}^2)}{3(\frac{s}{2} - 2m_{DM}^2)^4}\left[(2s - 8m_{DM}^2)((\frac{s}{2} - 2m_{DM}^2)^2(-\frac{b_{6I}^2 s}{4} + b_{6R}^2 s + 3b_{6I}^2(\frac{3s}{4} - 2m_{DM}^2)) \right.$$
$$-3m_V^2(\frac{s}{2} - 2m_{DM}^2)(b_{6R}^2\frac{s}{4} + b_{6I}^2(\frac{3s}{4} - 2m_{DM}^2)) + \frac{3m_V^4 s}{4}(b_{6R}^2 - b_{6I}^2))$$
$$\left. +6m_V^4 log(\frac{m_V^2}{s - 4m_{DM}^2 + m_V^2})(-b_{6I}^2(\frac{s}{2} - 2m_{DM}^2)(s - 2m_{DM}^2) + \frac{m_V^2 s}{4}(b_{6R}^2 - b_{6I}^2)) \right] \tag{90}$$

Term 4

$$term4 = \int_{-1}^{1} d\cos\theta \frac{1}{A(1-\cos\theta) + B}\left[P(1-\cos\theta)^4 + Q\sin^2\theta \right]$$
$$+\frac{1}{A(1+\cos\theta) + B}\left[P(1+\cos\theta)^4 + Q\sin^2\theta \right] \tag{91}$$

$$term4 = \left[\frac{1}{12A^5}(-12B(-2A^3Q - A^2BQ + B^3P)log(A(x-1) - B) + 12B(-2A^3Q - A^2BQ + B^3P)log(Ax + A + B) \right.$$
$$\left. +A(3A^3(8Px^3 + 8Px + P + 8Qx + 6Q) - 4A^2B(2Px^3 + 6Px + P - 6Qx - 3Q) + 6AB^2P(4x+1) - 12B^3P(2x+1))) \right]_{-1}^{1} \tag{92}$$

Evaluating the Integration limits

$$term4 = \frac{1}{12A^5}\left[24B(-2A^3Q - A^2BQ + B^3P)log(\frac{2A + B}{B}) \right.$$
$$\left. +A(3A^3(32P + 16Q) - 4A^2B(16P - 12Q) + 48AB^2P - 48B^3P) \right] \tag{93}$$

Substituting the terms back in

$$term4 = \frac{1}{12(\frac{s}{2}-2m_{DM}^2)^5}\Bigg[24m_V^2(-2(\frac{s}{2}-2m_{DM}^2)^3\Big(b_{7R}^2\Big[sm_{DM}^4\Big]\Big)$$
$$-m_V^2(\frac{s}{2}-2m_{DM}^2)^2\Big(4b_{7R}^2\Big[sm_{DM}^4\Big]\Big)+\frac{b_{6R}^2m_V^4s^2(\frac{s}{4}-m_{DM}^2)^2}{4})log(\frac{s-4m_{DM}^2+m_V^2}{m_V^2})$$
$$+(\frac{s}{2}-2m_{DM}^2)(3(\frac{s}{2}-2m_{DM}^2)^3(\frac{8b_{6R}^2s^2(\frac{s}{4}-m_{DM}^2)^2}{m_V^2}$$
$$+16\Big(b_{7R}^2\Big[sm_{DM}^4\Big]\Big)) \tag{94}$$
$$-4m_V^2(\frac{s}{2}-2m_{DM}^2)^2(\frac{4b_{6R}^2s^2(\frac{s}{4}-m_{DM}^2)^2}{m_V^2}-12\Big(b_{7R}^2\Big[sm_{DM}^4\Big]\Big))$$
$$+12b_{6R}^2m_V^2s^2(\frac{s}{2}-2m_{DM}^2)(\frac{s}{4}-m_{DM}^2)^2-12b_{6R}^2m_V^4s^2(\frac{s}{4}-m_{DM}^2)^2)\Bigg]$$

Final Bound

Adding all four of these terms back together...

$$\mathcal{M}_{ij}^0(s) = \frac{b_5^2\beta_{ii}}{192\pi m_{DM}^4(\frac{s}{2}-2m_{DM}^2)^3}\Bigg[(2s-8m_{DM}^2)((\frac{s}{2}-2m_{DM}^2)^2(3(\frac{s}{4}-m_{DM}^2)^2-\frac{3s}{2}(\frac{s}{4}-m_{DM}^2)+\frac{s^2}{4})$$
$$+\frac{3s}{4}(\frac{s}{2}-2m_{DM}^2)(-2m_V^2(\frac{s}{4}-m_{DM}^2)+\frac{m_V^2s}{2}-(\frac{s}{2}-2m_{DM}^2)(\frac{3s}{2}-2m_{DM}^2)+\frac{s}{4}(\frac{3s}{2}-2m_{DM}^2))+\frac{3m_V^2s^2}{16}(m_V^2+\frac{3s}{2}-2m_{DM}^2))$$
$$+6(2s-4m_{DM}^2+m_V^2)((\frac{s}{2}-2m_{DM}^2)((\frac{s}{4}-m_{DM}^2)-\frac{s}{4})-\frac{m_V^2s}{4})^2log(\frac{m_V^2}{s-4m_{DM}^2+m_V^2})\Bigg]$$
$$+\frac{b_{6I}b_5\beta_{ii}s}{384\pi m_{DM}^4(\frac{s}{2}-2m_{DM}^2)^3}\Bigg[-(2s-8m_{DM}^2)((\frac{s}{2}-2m_{DM}^2)^2(-2s+9m_{DM}^2)-3m_V^2(\frac{s}{2}-2m_{DM}^2)(-\frac{3s}{4}-m_{DM}^2)+\frac{3m_V^4s}{4})$$
$$+6m_V^2(2s-8m_{DM}^2+m_V^2)(m_{DM}^2(\frac{s}{2}-2m_{DM}^2)+\frac{m_V^2s}{4})log(\frac{s-4m_{DM}^2+m_V^2}{m_V^2})\Bigg]$$
$$-\frac{\beta_{ii}s}{768\pi m_{DM}^4(\frac{s}{2}-2m_{DM}^2)^3}\Bigg[(2s-8m_{DM}^2)((\frac{s}{2}-2m_{DM}^2)^2(-\frac{b_{6I}^2s}{4}+b_{6R}^2s+3b_{6I}^2(\frac{3s}{4}-2m_{DM}^2))$$
$$-3m_V^2(\frac{s}{2}-2m_{DM}^2)(b_{6R}^2\frac{s}{4}+b_{6I}^2(\frac{3s}{4}-2m_{DM}^2))+\frac{3m_V^4s}{4}(b_{6R}^2-b_{6I}^2))$$
$$+6m_V^4log(\frac{m_V^2}{s-4m_{DM}^2+m_V^2})(-b_{6I}^2(\frac{s}{2}-2m_{DM}^2)(s-2m_{DM}^2)+\frac{m_V^2s}{4}(b_{6R}^2-b_{6I}^2))\Bigg]$$
$$-\frac{\beta_{ii}}{768\pi m_{DM}^4(\frac{s}{2}-2m_{DM}^2)^5}\Bigg[24m_V^2(-2(\frac{s}{2}-2m_{DM}^2)^3\Big(b_{7R}^2\Big[sm_{DM}^4\Big]\Big)$$
$$-m_V^2(\frac{s}{2}-2m_{DM}^2)^2\Big(b_{7R}^2\Big[sm_{DM}^4\Big]\Big)+\frac{b_{6R}^2m_V^4s^2(\frac{s}{4}-m_{DM}^2)^2}{4})log(\frac{s-4m_{DM}^2+m_V^2}{m_V^2})$$
$$+(\frac{s}{2}-2m_{DM}^2)(3(\frac{s}{2}-2m_{DM}^2)^3(\frac{8b_{6R}^2s^2(\frac{s}{4}-m_{DM}^2)^2}{m_V^2}$$
$$+16\Big(b_{7R}^2\Big[sm_{DM}^4\Big]\Big))$$
$$-4m_V^2(\frac{s}{2}-2m_{DM}^2)^2(\frac{4b_{6R}^2s^2(\frac{s}{4}-m_{DM}^2)^2}{m_V^2}-12\Big(b_{7R}^2\Big[sm_{DM}^4\Big]\Big))$$
$$+12b_{6R}^2m_V^2s^2(\frac{s}{2}-2m_{DM}^2)(\frac{s}{4}-m_{DM}^2)^2-12b_{6R}^2m_V^4s^2(\frac{s}{4}-m_{DM}^2)^2)\Bigg] \tag{95}$$

Rearranging in terms of the different couplings

$$\mathcal{M}^0_{ij}(s) = \frac{b_5^2\beta_{ii}}{192\pi m_{DM}^4(\frac{s}{2}-2m_{DM}^2)^3}\Bigg[(2s-8m_{DM}^2)((\frac{s}{2}-2m_{DM}^2)^2(3(\frac{s}{4}-m_{DM}^2)^2-\frac{3s}{2}(\frac{s}{4}-m_{DM}^2)+\frac{s^2}{4})$$
$$+\frac{3s}{4}(\frac{s}{2}-2m_{DM}^2)(-2m_V^2(\frac{s}{4}-m_{DM}^2)+\frac{m_V^2 s}{2}-(\frac{s}{2}-2m_{DM}^2)(\frac{3s}{2}-2m_{DM}^2)+\frac{s}{4}(\frac{3s}{2}-2m_{DM}^2))+\frac{3m_V^2 s^2}{16}(m_V^2+\frac{3s}{2}-2m_{DM}^2))$$
$$+6(2s-4m_{DM}^2+m_V^2)((\frac{s}{2}-2m_{DM}^2)((\frac{s}{4}-m_{DM}^2)-\frac{s}{4})-\frac{m_V^2 s}{4})^2 log(\frac{m_V^2}{s-4m_{DM}^2+m_V^2})\Bigg]$$
$$+\frac{b_{6I}b_5\beta_{ii}s}{384\pi m_{DM}^4(\frac{s}{2}-2m_{DM}^2)^3}\Bigg[-(2s-8m_{DM}^2)((\frac{s}{2}-2m_{DM}^2)^2(-2s+9m_{DM}^2)-3m_V^2(\frac{s}{2}-2m_{DM}^2)(-\frac{3s}{4}-m_{DM}^2)+\frac{3m_V^4 s}{4})$$
$$+6m_V^2(2s-8m_{DM}^2+m_V^2)(m_{DM}^2(\frac{s}{2}-2m_{DM}^2)+\frac{m_V^2 s}{4})log(\frac{s-4m_{DM}^2+m_V^2}{m_V^2})\Bigg]$$
$$-\frac{b_{6R}^2\beta_{ii}s}{768\pi m_{DM}^4(\frac{s}{2}-2m_{DM}^2)^3}\Bigg[(2s-8m_{DM}^2)(s(\frac{s}{2}-2m_{DM}^2)^2-3\frac{m_V^2 s}{4}(\frac{s}{2}-2m_{DM}^2)+\frac{3m_V^4 s}{4})+\frac{3m_V^6 s}{2}log(\frac{m_V^2}{s-4m_{DM}^2+m_V^2})\Bigg]$$
$$-\frac{b_{6R}^2\beta_{ii}}{768\pi m_{DM}^4(\frac{s}{2}-2m_{DM}^2)^3}\Bigg[6m_V^6 s^2 log(\frac{s-4m_{DM}^2+m_V^2}{m_V^2})$$
$$+\frac{6s^2}{m_V^2}(\frac{s}{2}-2m_{DM}^2)^4-4s^2(\frac{s}{2}-2m_{DM}^2)^3+12m_V^2 s^2(\frac{s}{4}-m_{DM}^2)^2-6m_V^4 s^2(\frac{s}{4}-m_{DM}^2)\Bigg]$$
$$-\frac{b_{6I}^2\beta_{ii}s}{768\pi m_{DM}^4(\frac{s}{2}-2m_{DM}^2)^3}\Bigg[(2s-8m_{DM}^2)((\frac{s}{2}-2m_{DM}^2)^2(-\frac{s}{4}+3(\frac{3s}{4}-2m_{DM}^2))-3m_V^2(\frac{s}{2}-2m_{DM}^2)(\frac{3s}{4}-2m_{DM}^2)-\frac{3m_V^4 s}{4})$$
$$-6m_V^4 log(\frac{m_V^2}{s-4m_{DM}^2+m_V^2})((\frac{s}{2}-2m_{DM}^2)(s-2m_{DM}^2)+\frac{m_V^2 s}{4})\Bigg]$$
$$-\frac{b_{7R}^2\beta_{ii}s}{32\pi(\frac{s}{2}-2m_{DM}^2)^3}\Bigg[m_V^2(-s+4m_{DM}^2-m_V^2)log(\frac{s-4m_{DM}^2+m_V^2}{m_V^2})+2(\frac{s}{2}-2m_{DM}^2)(\frac{s}{2}-2m_{DM}^2+m_V^2)\Bigg] \tag{96}$$

Finally, substituting back in the kinematic factor, the unitarity bound becomes...

$$\Bigg| \frac{b_5^2\sqrt{\frac{s-4m_{DM}^2}{s}}}{192\pi m_{DM}^4(\frac{s}{2}-2m_{DM}^2)^3}\Bigg[(2s-8m_{DM}^2)((\frac{s}{2}-2m_{DM}^2)^2(3(\frac{s}{4}-m_{DM}^2)^2-\frac{3s}{2}(\frac{s}{4}-m_{DM}^2)+\frac{s^2}{4})$$
$$+\frac{3s}{4}(\frac{s}{2}-2m_{DM}^2)(-2m_V^2(\frac{s}{4}-m_{DM}^2)+\frac{m_V^2 s}{2}-(\frac{s}{2}-2m_{DM}^2)(\frac{3s}{2}-2m_{DM}^2)+\frac{s}{4}(\frac{3s}{2}-2m_{DM}^2))+\frac{3m_V^2 s^2}{16}(m_V^2+\frac{3s}{2}-2m_{DM}^2))$$
$$+6(2s-4m_{DM}^2+m_V^2)((\frac{s}{2}-2m_{DM}^2)((\frac{s}{4}-m_{DM}^2)-\frac{s}{4})-\frac{m_V^2 s}{4})^2 log(\frac{m_V^2}{s-4m_{DM}^2+m_V^2})\Bigg]$$
$$+\frac{b_{6I}b_5\sqrt{\frac{s-4m_{DM}^2}{s}}s}{384\pi m_{DM}^4(\frac{s}{2}-2m_{DM}^2)^3}\Bigg[-(2s-8m_{DM}^2)((\frac{s}{2}-2m_{DM}^2)^2(-2s+9m_{DM}^2)-3m_V^2(\frac{s}{2}-2m_{DM}^2)(-\frac{3s}{4}-m_{DM}^2)+\frac{3m_V^4 s}{4})$$
$$+6m_V^2(2s-8m_{DM}^2+m_V^2)(m_{DM}^2(\frac{s}{2}-2m_{DM}^2)+\frac{m_V^2 s}{4})log(\frac{s-4m_{DM}^2+m_V^2}{m_V^2})\Bigg]$$
$$-\frac{b_{6R}^2\sqrt{\frac{s-4m_{DM}^2}{s}}s}{768\pi m_{DM}^4(\frac{s}{2}-2m_{DM}^2)^3}\Bigg[(2s-8m_{DM}^2)(s(\frac{s}{2}-2m_{DM}^2)^2-3\frac{m_V^2 s}{4}(\frac{s}{2}-2m_{DM}^2)+\frac{3m_V^4 s}{4})+\frac{3m_V^6 s}{2}log(\frac{m_V^2}{s-4m_{DM}^2+m_V^2})\Bigg]$$
$$-\frac{b_{6R}^2\sqrt{\frac{s-4m_{DM}^2}{s}}}{768\pi m_{DM}^4(\frac{s}{2}-2m_{DM}^2)^3}\Bigg[6m_V^6 s^2 log(\frac{s-4m_{DM}^2+m_V^2}{m_V^2})$$
$$+\frac{6s^2}{m_V^2}(\frac{s}{2}-2m_{DM}^2)^4-4s^2(\frac{s}{2}-2m_{DM}^2)^3+12m_V^2 s^2(\frac{s}{4}-m_{DM}^2)^2-6m_V^4 s^2(\frac{s}{4}-m_{DM}^2)\Bigg]$$
$$-\frac{b_{6I}^2\sqrt{\frac{s-4m_{DM}^2}{s}}s}{768\pi m_{DM}^4(\frac{s}{2}-2m_{DM}^2)^3}\Bigg[(2s-8m_{DM}^2)((\frac{s}{2}-2m_{DM}^2)^2(-\frac{s}{4}+3(\frac{3s}{4}-2m_{DM}^2))-3m_V^2(\frac{s}{2}-2m_{DM}^2)(\frac{3s}{4}-2m_{DM}^2)-\frac{3m_V^4 s}{4})$$
$$-6m_V^4 log(\frac{m_V^2}{s-4m_{DM}^2+m_V^2})((\frac{s}{2}-2m_{DM}^2)(s-2m_{DM}^2)+\frac{m_V^2 s}{4})\Bigg]$$
$$-\frac{b_{7R}^2\sqrt{\frac{s-4m_{DM}^2}{s}}s}{32\pi(\frac{s}{2}-2m_{DM}^2)^3}\Bigg[m_V^2(-s+4m_{DM}^2-m_V^2)log(\frac{s-4m_{DM}^2+m_V^2}{m_V^2})+2(\frac{s}{2}-2m_{DM}^2)(\frac{s}{2}-2m_{DM}^2+m_V^2)\Bigg]\Bigg| \leq \frac{1}{2} \tag{97}$$

Rearranging...

$$
\begin{aligned}
&\frac{1}{32\pi m_{\rm DM}^4(\frac{s}{2}-2m_{\rm DM}^2)^3}\sqrt{\frac{s-4m_{\rm DM}^2}{s}}\Bigg|\frac{b_5^2}{3}\Bigg[2\Big((\frac{s}{2}-2m_{\rm DM}^2)^2(\frac{s^2}{4}(m_{\rm DM}^2-s)+\frac{3m_{\rm DM}^2 s}{4}(3s-4m_{\rm DM}^2)\\
&\quad+3m_{\rm DM}^4(\frac{s}{2}-2m_{\rm DM}^2))+\frac{3m_{\rm V}^2 s}{8}(s-4m_{\rm DM}^2)(\frac{3s^2}{8}+\frac{1}{2}m_{\rm DM}^2 s-4m_{\rm DM}^4)+\frac{3m_{\rm V}^4 s^2}{32}(s-4m_{\rm DM}^2)\Big)\\
&\quad-3(\frac{m_{\rm V}^2 s}{4}+m_{\rm DM}^2(\frac{s}{2}-2m_{\rm DM}^2))^2\big(2s-4m_{\rm DM}^2+m_{\rm V}^2\big)\ln\Big(\frac{s-4m_{\rm DM}^2+m_{\rm V}^2}{m_{\rm V}^2}\Big)\Bigg]\\
&+\frac{b_5{\rm Im}(b_6)s}{12}\Bigg[-(2s-8m_{\rm DM}^2)\Big((\frac{s}{2}-2m_{\rm DM}^2)^2(-2s+9m_{\rm DM}^2)-3m_{\rm V}^2(\frac{s}{2}-2m_{\rm DM}^2)(-\frac{3s}{4}-m_{\rm DM}^2)+\frac{3m_{\rm V}^4 s}{4}\Big)\\
&\quad+6m_{\rm V}^2\big(2s-8m_{\rm DM}^2+m_{\rm V}^2\big)\big(m_{\rm DM}^2(\frac{s}{2}-2m_{\rm DM}^2)+\frac{m_{\rm V}^2 s}{4}\big)\ln\Big(\frac{s-4m_{\rm DM}^2+m_{\rm V}^2}{m_{\rm V}^2}\Big)\Bigg]\\
&-\frac{{\rm Re}(b_6)^2}{24}\Bigg[s(2s-8m_{\rm DM}^2)\Big(s(\frac{s}{2}-2m_{\rm DM}^2)^2-3\frac{m_{\rm V}^2 s}{4}(\frac{s}{2}-2m_{\rm DM}^2)+\frac{3m_{\rm V}^4 s}{4}\Big)+\frac{9m_{\rm V}^6 s^2}{2}\ln\Big(\frac{s-4m_{\rm DM}^2+m_{\rm V}^2}{m_{\rm V}^2}\Big)\\
&\quad+\frac{6s^2}{m_{\rm V}^2}\big(\frac{s}{2}-2m_{\rm DM}^2\big)^4-4s^2\big(\frac{s}{2}-2m_{\rm DM}^2\big)^3+12m_{\rm V}^2 s^2\big(\frac{s}{4}-m_{\rm DM}^2\big)^2-6m_{\rm V}^4 s^2\big(\frac{s}{4}-m_{\rm DM}^2\big)\Bigg]\\
&-\frac{{\rm Im}(b_6)^2 s}{24}\Bigg[(2s-8m_{\rm DM}^2)\Big((\frac{s}{2}-2m_{\rm DM}^2)^2(2s-6m_{\rm DM}^2)-3m_{\rm V}^2(\frac{s}{2}-2m_{\rm DM}^2)(\frac{3s}{4}-2m_{\rm DM}^2)-\frac{3m_{\rm V}^4 s}{4}\Big)\\
&\quad+6m_{\rm V}^4\big((\frac{s}{2}-2m_{\rm DM}^2)(s-2m_{\rm DM}^2)+\frac{m_{\rm V}^2 s}{4}\big)\ln\Big(\frac{s-4m_{\rm DM}^2+m_{\rm V}^2}{m_{\rm V}^2}\Big)\Bigg]\\
&-{\rm Re}(b_7)^2 m_{\rm DM}^4 s\Bigg[2(\frac{s}{2}-2m_{\rm DM}^2)\big(\frac{s}{2}-2m_{\rm DM}^2+m_{\rm V}^2\big)+m_{\rm V}^2\big(-s+4m_{\rm DM}^2-m_{\rm V}^2\big)\ln\Big(\frac{s-4m_{\rm DM}^2+m_{\rm V}^2}{m_{\rm V}^2}\Big)\Bigg]\Bigg|\le\frac{1}{2},
\end{aligned}
\tag{98}
$$

In the limit of large s, this becomes:

$$
\left|-\frac{b_5^2 s^2}{96\pi m_{DM}^4}+\frac{b_{6I}b_5 s^2}{48\pi m_{DM}^4}-\frac{b_{6R}^2 s^3}{256\pi m_V^2 m_{DM}^4}-\frac{b_{6I}^2 s^2}{96\pi m_{DM}^4}-\frac{b_{7R}^2}{8\pi}\right|\le\frac{1}{2}
\tag{99}
$$

The b_{7R} term is independent of s, and if all other couplings are zero, this is simply $b_{7R}\le\sqrt{4\pi}$. The b_{6R} term is the only one that is dependent on the mediator mass in the limit.