MATHEMATICAL SCIENCES

WEYL'S CIRCLE AND POINT FOR TWO-PARAMETRIC SYSTEM OF STRUM-LIOVILLE DIFFERENTIAL EQUATION

Almammadov M.
Professor, doctor of Mathematical science Azerbaijan State University of Economics (UNEC)

Abstract
We introduce analogues of Weyl's circumferences and circles for the system of equations [1] with two parameters.

It is proved that the deficiency index of the two-parameter system (1) in the singular end $b=(b_1,b_2)$ is not less than one.

If one of two-parametric equations (1) has the case of limit circumference, then the deficiency index of the problem (1) is not less than two. But if both of the equations of the two-parameter system (1) has the case of limit circumferences, then the deficiency index of the problem (1) equals four.

Keywords: Circle and point, two –parameter equations, deficiency index, limit circumference, problem (1), differential equation, continuous, interval $(a_i,b_i),$ complex valued functions, not self-adjoint operators, discrete spectrum.

Introduction:
We consider a two –parameter system of Sturm-Liouville equations

$$\left.\begin{aligned} &-y''(x_1)+q_1(x_1)y_1(x_1)+[\lambda_1 a_{11}(x_1)+\lambda_2 a_{12}(x_1)]y_1(x_1)=0 \\ &-y''(x_2)+q_2(x_2)y_2(x_2)+[\lambda_1 a_{21}(x_2)+\lambda_2 a_{22}(x_2)]y_2(x_2)=0 \end{aligned}\right\} \tag{1}$$
$$-\infty \le a_i < x_i < b_i \le \infty \quad (i=1,2).$$

Assume that $a_{11}(x_1), a_{12}(x_1), a_{21}(x_2), a_{22}(x_2)$ and $q_1(x_1), q_2(x_2)$ are arbitrary complex-valued functions, continuous on the interval $(a_i,b_i),$ and $P_i=\operatorname{Re} q_i, r_i=\operatorname{Im} q_i \quad (i=1,2).$

We introduce analogues of G. Weyl's circumferences and circles for the system of equations (1) with two parameters.

G. Weyl's theory on limit circumference for Sturm-Liouville operators with real potentials (see [1], [2]) is extended also for the case of complex potentials as was noted by V.B. Lidsky [3] when constructing theory of not self-adjoint Sturm-Lioville operators with discrete spectrum.

We will use the denotations and reasonings of the works ([3],[4]).

Let us introduce the following function

$$S_i(x_i,\lambda)=\sum_{k=1}^{2}\operatorname{Im}(\lambda_k a_{ik}(x_i))+r_i(x_i),\ (i=1,2)$$

and consider the following two subsets of C^2:

$$\Lambda_i^+=\left\{\lambda\in C^2; S_i(x_i,\lambda)>0, x_i\in(a_i,b_i)\right\},$$
$$\Lambda_i^-=\left\{\lambda\in C^2; S_i(x_i,\lambda)<0, x_i\in(a_i,b_i)\right\}.$$

Introduce the basis $\varphi_i(x_i,\lambda),\theta_i(x_i,\lambda)$ of the space of solutions of the equation (1) satisfying the following initial conditions at the fixed point $\alpha_i\in(a_i,b_i)$:

$$\varphi_i(\alpha_i,\lambda)=1,\quad \varphi_i'(\alpha_i,\lambda)=0,$$
$$\theta_i(\alpha_i,\lambda)=0,\quad \theta_i'(\alpha_i,\lambda)=1.$$

Let $\lambda\in\Lambda_i^+\bigcup\Lambda_i^-.$

Then there exists the sequence $(b_i^{(m)})_{m\ge1}\subset(a_i,b_i)$ converging to b_i as $m\to\infty$ and posing the properties:

The solution of the form

$$\psi_i(x_i,\lambda)=\theta_i(x_i,\lambda)+l_i\varphi_i(x_i,\lambda)$$

of the equation (1) at the point $b_i^{(m)}$ satisfies the boundary condition

$$\operatorname{Im}\psi_i^{'}(x_i,\lambda,l_i)\overline{\psi_i}(x_i,\lambda,l_i)\Big|_{k_i=b_i(m)}=0, \tag{2}$$

(where l_i is still an arbitrary parameter) if and only if l_i lies on the circumference $C_{b_i^{(m|)}}(\lambda)$:

$$\int_{\alpha_i}^{b_i^{(m)}} S_i(x_i,\lambda)\left|\psi_i(x_i,\lambda,l_i)\right|^2 dx_i=\operatorname{Im} l_i \tag{3}$$

centered at the point

$$O_{b_i^{(m)}}(\lambda)=\Delta\left[\theta_i,\overline{\varphi}_i\right]_{x_i=b_i^{(m)}}\left(2i\int_{\alpha_i}^{b_i^{(m)}} S_i(x_i,\lambda)\left|\varphi_i(x_i,\lambda,l_i)\right|^2 dx_i\right)^{-1} \tag{4}$$

and with a radius equal to

$$R_{b_i^{(m)}}=\frac{1}{2}\left(\left|\int_{\alpha_i}^{b_i} S_i(x_i,\lambda)\left|\varphi_i(x_i,\lambda)\right|^2\right|dx_i\right)^{-1}. \tag{5}$$

With increasing *m* circumference, the circumferences $C_{b_i^{(m)}}(\lambda)$ are contracted and as $m\to\infty$ refract either to the limit point $m_{b_i^{(m)}}(\lambda)$, or to the limit circumference $C_{b_i^{(m)}}(\lambda)$.

The point $b_i^{(m)}$ lies on a closed limit circle

$$\overline{K_{b_i}(\lambda)}=K_{b_i}(\lambda)\bigcup C_{b_i}(\lambda)$$

or coincides with the limit point $m_{b_i}(\lambda)$ if and if the following relation is fulfilled:

$$\int_{\alpha_i}^{b_i}\left|S_i(x_i,\lambda)\right|\left|\theta_i(x_i,\lambda)+l_{b_i}(\lambda)\varphi_i(x_i,\lambda)\right|^2 dx_i\le\left|\operatorname{Im} l_{b_i}(\lambda)\right| \tag{6}$$

It is easy to see that the Wronskian

$$\Delta(\varphi_i,\theta_i)=\varphi_i(\alpha_i\lambda)\theta_i^{'}(\alpha_i\lambda)-\theta_i(\alpha_i\lambda)\varphi_i^{'}(\alpha_i\lambda)=1$$

and

$$\Delta(\overline{\varphi}_i,\varphi_i)\Big|_{x=\alpha_i}=0,\ \Delta(\overline{\theta_i+l_i\varphi_i},\theta_i+l_i\varphi_i)\Big|_{x_i=\alpha_i}=-2i\operatorname{Im} l_i.$$

Now, multiplying the equation (1) by $\overline{y}_i$, integrating the obtained relation in the interval $\left(\alpha_i,b_i^{(m)}\right)$ and isolating the imaginary part, we obtain:

$$\Delta\left(\overline{y}_i,y_i\right)\Big|_{x=b_i^{(m)}}=\Delta\left(\overline{y}_i,y_i\right)\Big|_{x=\alpha_i}+2i\int_{\alpha_i}^{b_i^{(m)}} S_i(x_i,\lambda)\left|y_i(x_i)\right|^2 dx_i. \tag{7}$$

We now note that the condition

$$\operatorname{Im}\psi_i^{'}\overline{\psi}_i\Big|_{x_i=b_i^{(m)}}=0$$

is equivalent to such a selection of the values of l_i for which the values of the linear fractional mapping

$$z=\frac{\theta_i^{'}+l_i\varphi_i^{'}}{\theta_i+l_i\varphi_i}\Bigg|_{x=b_i^{(m)}}$$

is real.

And this is also equivalent to the fact that the point l_i belongs to the image of the real line under the linear-fractional mapping

$$z - l_i(b_i^{(m)}, \lambda, z) = \left.\frac{\theta_i z - \theta_i}{-\varphi_i z + \varphi_i^{'}}\right|_{x=b_i^{(m)}} \tag{8}$$

in other words, to the belonging of the point l_i of the circumference

$$C_{b_i^{(m)}}(\lambda) = l_i(b_i^{(m)}, \lambda, R).$$

and this circumference has the equation

$$\Delta\left(\overline{\theta_i + l_i y_i} + \theta_i + l_i y_i\right)\Big|_{x_i=b_i^{(m)}} = 0.$$

The center of the circumference $C_{b_i^{(m)}}(\lambda)$ is the image of the point with the pole of the function

$$l_i\left(b_i^{(m)}, \lambda, z\right) = \left.\frac{\theta_i z - \theta_i^{'}}{-\varphi_i z + \varphi_i^{'}}\right|_{x_i=b_i^{(m)}}.$$

Therefore,

$$O_{b_i^{(m)}}(\lambda) = \left.\frac{\Delta(\theta_i - \overline{\varphi}_i)}{\Delta(\overline{\varphi_i}, \varphi_i)}\right|_{x_i=b_i^{(m)}}.$$

The length of the radius $R_{b_i^{(m)}}(\lambda)$ of the circumference $C_{b_i^{(m)}}(\lambda)$ is found by the formula

$$R_{b_i^{(m)}}(\lambda) = \left|l_i(b_i^{(m)}, \lambda, 0) - O_{\alpha_i}(\lambda)\right| = \left|\Delta(\overline{\varphi_i}, \varphi_i)^{-1}\right|_{x_i=b_i^{(m)}}$$

Let us find the equation of the circle $K_{b_i^{(m)}}(\lambda)$.

Obviously, the imaginary part of the pole of the linear fractional mapping (8) equals the expression

$$\frac{\left.\frac{1}{2i}\Delta(\overline{\varphi_i}, \varphi_i)\right|_{x_i=b_i^{(m)}}}{\left|\varphi_i(b_i^{(m)}, \lambda)\right|^2}.$$

If $\lambda \in \Lambda_i^+ \bigcup \Lambda_i^-$, then from formula (7)

$$\Delta(\overline{\varphi_i}, \varphi_i)\Big|_{x_i=b_i^{(m)}} \neq 0,$$

therefore $\varphi_i(b_i^{(m)}, \lambda) \neq 0$ and $\varphi_i^{'}(b_i^{(m)}, \lambda) \neq 0.$

By means of formula (7) we deduce that

$$\operatorname{Im} z = \frac{1}{\left|\varphi_i(b_i^{(m)}, \lambda)\right|^2} \int_{\alpha_i}^{b_i^{(m)}} S_i(x_i, \lambda)\left|\varphi_i(x_i, \lambda)\right|^2 dx_i.$$

Hence it is seen that if $\lambda \in \Lambda_i^+$, then the point l_i belongs to the circle $K_{b_i^{(m)}}(\lambda)$ if and only if $\operatorname{Im} z < 0,$ i.e.

$$\frac{1}{2i}\Delta\left(\overline{\theta_i + l_i y_i} + \theta_i + l_i y_i\right)\Big|_{x_i=b_i^{(m)}} < 0$$

and this in its turn is equivalent to the inequality

$$\int_{\alpha_i}^{b_i^{(m)}} S_i(x_i, \lambda)\left|\theta_i(x, \lambda) + l_i\varphi_i(x_i, \lambda)\right|^2 dx_i < \operatorname{Im} l_i. \tag{9}$$

If $\lambda \in \Lambda_i^+$, we obtain the inequality

$$\int_{\alpha_i}^{b_i^{(m)}} S_i(x_i, \lambda)\left|\theta_i(x, \lambda) + l_i\varphi_i(x_i, \lambda)\right|^2 dx_i > \operatorname{Im} l_i. \tag{10}$$

From formulas (9) and (10) we obtain the circle $K_{b_i^{(m)}}(\lambda)$ for al $\lambda \in \Lambda_i^+ \bigcup \Lambda_i^-$ in the form of

$$\int_{\alpha_i}^{b_i^{(m)}} |S_i(x_i,\lambda)| |\theta_i(x,\lambda)+l_i\varphi_i(x_i,\lambda)|^2 dx_i < |\operatorname{Im} l_i|. \qquad (11)$$

Hence, if is obvious that if $\lambda \in \Lambda_i^+$ then $\operatorname{Im} l_i > 0$ i.e. for all the circles $K_{b_i^{(m)}}(\lambda) m = 1,2,\ldots$ are on the upper (l) half-plane . And if $\lambda \in \Lambda_i^-$ then i.e. for al the circles $K_{b_i^{(m)}}(\lambda) m = 1,2,\ldots$ are on the lower (*l*) half-plane. ???

From the inequality (11) it is seen that the circles $K_{b_i^{(m)}}(\lambda)$ are and therefore refract to the limit point $m_{b_i}(\lambda)$, or to the limit circle $K_{b_i^{(m)}}(\lambda)$ (with limit circumference $C_{b_i}(\lambda)$.)

If $C_{b_i}(\lambda) = m_{b_i}(\lambda)$, then the relation (11) holds.

We now note that there exists the solution to the equation (1) of the form

$$\psi_i(x_i,\lambda) = \theta_i(x_i,\lambda) + l_i\varphi_i(x_i,\lambda)$$

that for each values of $\lambda \in \Lambda_i^+ \bigcup \Lambda_i^-$ belongs to the space $L^2([\alpha_i,b_i],|S_i|dx_i)$ of complex valued functions $y_i(x_i)$, summed with a square with respect to the weight $|S_i(x_i,\lambda)|$ in the vicinity of the point b_i :

$$\int_{\alpha_i}^{b_i} |S_i(x_i,\lambda)| |y_i(x_i)|^2 dx_i < \infty.$$

In the case of limit circumference, for each value of $\lambda \in \Lambda_i^+ \bigcup \Lambda_i^-$ all the solutions of the equation (1) belong to the space

$$L^2([\alpha_i,b_i],|S_i|dx_i).$$

Indeed, on one hand

$$\psi_i(x_i,\lambda) = \theta_i(x_i,\lambda) + l_{b_i}(\lambda)\varphi_i(x_i,\lambda) \in L^2([\alpha_i,b_i],|S_i|dx_i).$$

on the other hand, in the case of limit circumference we have

$$\lim_{m\to\infty} R_{b_i^{(m)}}(\lambda) \neq 0.$$

Therefore, from formula (5) we deduce that the solution $\varphi_i(x_i,\lambda)$ of the equation (1) belongs to the space $L^2([\alpha_i,b_i],|S_i|dx_i)$

In the case of limit point, for each value of $\lambda \in \Lambda_i^+ \bigcup \Lambda_i^-$ there exists only one linearly independent solution of the equation (1) belonging to the space $L^2([\alpha_i,b_i],|S_i|dx_i)$.

Thus, we have: either all the solutions of the equation (1) lie in the space

$$L^2([\alpha_i,b_i],|S_i|dx_i)$$

(the case of limit circumference), or only one linearly independent solution of (1) belongs to this space (the case of limit point)

We can show that (see V.B Lidsky [2] and G.A. Isayev [4], that for the constructed solution $\psi_i(x_i,\lambda) = \theta_i(x_i,\lambda) + l_{b_i}(\lambda)\varphi_i(x_i,\lambda), (l_{b_i} = l_{b_i}(\lambda) \in C_{b_i}(\lambda)$ or $l_{b_i} = m_{b_i}(\lambda)$) the following limit relation is valid:

$$\lim_{x_i\to b_i} \operatorname{Im}\left\{\overline{\psi_i(x_i,\lambda)}, \psi_i'(x_i,\lambda)\right\} = 0, \quad \lambda \in \Lambda_i^+ \bigcup \Lambda_i^-.$$

We now study all-possible cases of belonging and not belonging of he products $y_1(x_1,\lambda)y_2(x_2,\lambda)$ to the space $L^2(I_b, B(x)dx)$ i.e. determination of the condition

$$\int_{I_b} B(x) |y_1(x_i,\lambda), y_2(x_2,\lambda)|^2 dx < \infty, x_i \in [\alpha_i,b_i], I_b = [d_1,b_1][\alpha_2,b_2].$$

Here $y_1(x_1,\lambda)$ and $y_2(x_2,\lambda)$ one the solutions of the system (1)

$$B(x) = \det\{a_{ik}(x_i)\}_{i,k=1}^2$$

We call the amount of products $y_1(x_1,\lambda)y_2(x_2,\lambda)$ belongs to $L^2(I_b, B(x)dx)$ a deficiency index of two-parameter problem (1) in the singular end b (corresponding to the point λ)

Let $\lambda \notin R^2$ and $\operatorname{Im}\lambda_1 \neq 0.$ Then

$$\int\limits_{[\alpha,b^m]} B(x)\left|y_1(x_1,\lambda), y_2(x_2,\lambda)\right|^2 dx =$$

$$= \frac{1}{\operatorname{Im}\lambda_1}\int\limits_{\alpha_1}^{b_1^{(m)}} S_1(x_1,\lambda)\left|y_1(x_i,\lambda)\right|^2 dx_1 \cdot \int\limits_{\alpha_2}^{b_2^{(m)}} a_{22}(x_2)\left|y_2(x_2,\lambda)\right|^2 dx_2 -$$

$$-\frac{1}{\operatorname{Im}\lambda_1}\left(\int\limits_{\alpha_2}^{b_2^{(m)}} S_2(x_2,\lambda)\left|y_2(x_2,\lambda)\right|^2 dx_2\right)\int\limits_{\alpha_1}^{b_1^{(m)}} a_{12}(x_1)\left|y_1(x_1,\lambda)\right|^2 dx_1 -$$

$$-\frac{1}{\operatorname{Im}\lambda_1}\int\limits_{\alpha_1}^{b_1^{(m)}} r_1(x_1)\left|y_1(x_i,\lambda)\right|^2 dx_1 \cdot \int\limits_{\alpha_2}^{b_2^{(m)}} a_{22}(x_2)\left|y_2(x_2,\lambda)\right|^2 dx_2 +$$

$$+\frac{1}{\operatorname{Im}\lambda_1}\int\limits_{\alpha_2}^{b_2^{(m)}} r_2(x_2)\left|y_2(x_2,\lambda)\right|^2 dx_2 \int\limits_{\alpha_1}^{b_1^{(m)}} a_{12}(x_1)\left|y_1(x_1,\lambda)\right|^2 dx_1. \tag{12}$$

and

$$\int\limits_{[\alpha,b^m]} \widetilde{B}_1(x)\left|y_1(x_1,\lambda), y_2(x_2,\lambda)\right|^2 dx =$$

$$= \left(\int\limits_{\alpha_1}^{b_1^{(m)}} S_1(x_1,\lambda)\left|y_1(x_1,\lambda)\right|^2 dx_1 \cdot \int\limits_{\alpha_2}^{b_2^{(m)}} a_{22}(x_2)\left|y_2(x_2,\lambda)\right|^2 dx_2\right) - \tag{13}$$

$$-\left(\int\limits_{\alpha_2}^{b_2^{(m)}} S_2(x_2,\lambda)\left|y_2(x_2,\lambda)\right|^2 dx_2 \int\limits_{\alpha_1}^{b_1^{(m)}} a_{12}(x_1)\left|y_1(x_1,\lambda)\right|^2 dx_1\right).$$

Now, in the equality (12) assuming $y_1(x_1,\lambda) = \varphi_1(x_1,\lambda), \quad y_2(x_2,\lambda) = \psi_2(x_1,\lambda)$ and taking into account formulas (3) and (5), we obtain (considering $r = 1$):

$$\int\limits_{[\alpha,b^m]} B(x)\left|\varphi_1(x_1,\lambda), \psi_2(x_2,\lambda)\right|^2 dx =$$

$$= \pm\left(2\operatorname{Im}\lambda_1 R_{b_1^{(m)}}(\lambda)\right)^{-1}\int\limits_{\alpha_2}^{b_2^{(m)}} a_{22}(x_2)\left|\psi_2(x_2,\lambda)\right|^2 dx_2 -$$

$$-\left(\int\limits_{\alpha_1}^{b_1^{(m)}} a_{12}(x_1)\left|\varphi_1(x_1,\lambda)\right|^2 dx_1\right)\frac{2\operatorname{Im} l_2^{(m)}(\lambda)}{\operatorname{Im}\lambda_1} -$$

$$-\frac{1}{\operatorname{Im}\lambda_1}\int\limits_{\alpha_1}^{b_1^{(m)}} r_1(x_1)\left|\varphi_1(x_i,\lambda)\right|^2 dx_1 \cdot \int\limits_{\alpha_2}^{b_2^{(m)}} a_{22}(x_2)\left|\psi_2(x_2,\lambda)\right|^2 dx_2 +$$

$$+\frac{1}{\operatorname{Im}\lambda_1}\int\limits_{\alpha_1}^{b_1^{(m)}} a_{12}(x_1)\left|\varphi_1(x_1,\lambda)\right|^2 dx_1 \cdot \int\limits_{\alpha_2}^{b_1^{(m)}} r_2(x_2)\left|\psi_2(x_2,\lambda)\right|^2 dx_2. \tag{14}$$

Here $l_2^{(m)}(\lambda) \in C_{b_2^{(m)}}(\lambda),$ and the sign $"+"("-")$ corresponds to the case $\lambda \in \Lambda_i^+ \bigcup \left(\Lambda_i^-\right)$ In all the addends of the right hand side of the equality (13), the second factors have finite limits as $m \to \infty$ if only the following conditions are fulfilled:

$$\lambda = (\lambda_1, \lambda_2) \in (\Lambda_1^+ \bigcup \Lambda_1^-) \bigcap (\Lambda_2^+ \bigcup \Lambda_2^-), \quad \operatorname{Im}\lambda_1 \neq 0,$$

and

$$\lim_{x_2\to b_2}\sup\left|\frac{a_{22}(x_2)}{S_2(x_2,\lambda)}\right|<\infty,\lim_{x_2\to b_2}\sup\left|\frac{r_1(x_1)}{S_1(x_1,\lambda)}\right|<\infty.$$

Assume that for the first equation in two-parameter problem (1) we have the case a of limit circle, and furthermore, let

$$\lim_{x_1\to b_1}\sup\left|\frac{a_{12}(x_1)}{S_1(x_1,\lambda)}\right|<\infty,\lim_{x_1\to b_1}\sup\left|\frac{r_1(x_1)}{S_1(x_1,\lambda)}\right|<\infty.$$

Then all first factors of all addends of the right hand side of the equality (13) have finite limits as $m\to\infty$. Thus, subject to the mentioned conditions, the function $\varphi_1(x_1,\lambda)\psi_2(x_2,\lambda)\in L^2(I_b,B(x)dx),$ i.e. deficiency index of two-parameter problem (1) is not less than two.

In a similar way we can show that

$$\varphi_2(x_1,\lambda)\psi_1(x_2,\lambda)\in L^2(I_b,B(x)dx),$$

if for the second equation in two-parameter problem (1) we have the case of a limit circle and the coefficients of equations satisfy the same conditions, only the order in which there terms are presented, changes.

Thus, in the case of a limit circle for both equations the products

$\psi_1(x_1,\lambda)\psi_2(x_2,\lambda),\varphi_1(x_1,\lambda)\psi_2(x_2,\lambda)$ and $\psi_1(x_1,\lambda)\varphi_2(x_2,\lambda)$

belong to the space $L^2(I_b,B(x)dx)$. It is easy to see that on this case, the products also belong to the space $L^2(I_b,B(x)dx)$ i.e. the deficiency index of the problem (1) is precisely four $\varphi_1(x_1,\lambda)\varphi_2(x_2,\lambda)$.

Thus follows from the equality (12) for $y_i(x,\lambda)=\varphi_i(x_i,\lambda),i=1,2.$ On the other hand, it is easy to see that if only for one of the two-parameter system of equations (1) we have the case of a limit point, then the deficiency index of the problem (1) can be equal to two or four, as a result of dominating near the singular end $b=(b_1,b_2)$ of the addend in the right hand side of formula (12), or as a result of mutual paying if features of separate addends.

For example if for the first equation from the two-parameter system (1) we have the case of a limit point, circle, and for the second one we have the case of a limit point, then it is easy to see that the product $\psi_1(x_1,\lambda)\varphi_2(x_2,\lambda)$ does not belong to the space

$$L^2(I_b,B(x)dx)$$

subject to the conditions

$$\lim_{x_1\to b_1}\sup\frac{|a_{12}(x_1)|+|r_1(x_1)|}{|S_1(x_1,\lambda)|}<\infty,$$

$$\int_{\alpha_2}^{b_2}\max\{|a_{22}(x_1,\lambda)|,|r_2(x_2)|\}|\varphi_2(x_2,\lambda)|dx_2<\infty.$$

Summarizing all the arguments related to the system of equations (1) we arrive at the following statement

Theorem. Let

$$\lambda\in(\Lambda_1^+\bigcup\Lambda_1^-)\bigcap(\Lambda_2^+\bigcup\Lambda_2^-),\quad \lambda\notin R^2.$$

Assume that if $\lambda\notin R^2$, the following condition is fulfilled:

$$\lim_{x_1\to b_1}\sup\frac{|a_{i2}(x_i)|+|r_i(x_i)|}{|S_i(x_i,\lambda)|}<\infty,\quad i=1,2$$

but if $\lambda\notin R^2$, then

$$\lim_{x_1\to b_1}\frac{|a_{i1}(x_i)|+|r_i(x_i)|}{|S_i(x_i,\lambda)|}<\infty,\quad i=1,2.$$

Then the deficiency index of two-parameter problem (1) in the singular end $b=(b_1,b_2)$ is not less than a unit.

If for one of two-parameter equations (1) we have the case of a limit circumference, then the deficiency index of the problem (1) is not than two.

But if for the both equations of two-parameter system (1) we have the case of limit circumferencesm then the deficiency index of problem (1) is equal to four.

References

1. Coddington E.A., Levinson N. Theory of ordinary differential equations IL Moscow, 1958, 240—276

2. Levitan B.M., Sargsyan I.S. Introduction to spectral theory. Nauka, Moscow 1970, 144-152, 616-622.

3. Lidsky V.B., Strum-Lioville type not self-adjoint operator with a discrete spectrum. Trudy moskovsogo math. obshestva. 1960, vol.9, 45-79.

4. Isaev H.A., To theory of deficiency indices of multiparameter differential operators of Strum-Lioville type. Doklady AN SSSR 1981, vol 261, no.4, 788-791.

5. Naimark M.A., Linear differential operators. Nauka, Moscow, 1969, 157-178, 287-347.

6. Achiezer N.I. Glazman I.M. Theory of linear operators in Hilbert spaces. Nauka 1966, 352-390, 485-500.