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Adomian Decomposition Method to Solve the Second Order Ordinary Differential Equations with Constant Coefficient

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Abstract: **In this paper, we will display Adomian decomposition method for solving second-order ordinary differential equations with constant coefficient. The Adomian decomposition method (ADM) is a creative and effective method for exact solution. It is important to note that a lot of research work has been devoted to the application of the Adomian decomposition method to a wide class of linear and non-linear problems. Some examples were presented to show the ability of method for linear and non-linear ordinary differential equations.**

Keywords: **Adomian decomposition method; second-order ordinary differential equations.**

1. INTRODUCTION

In this paper, we consider the second order ordinary differential equation with constant coefficient of the form

$$r'' + ar' + br = h(x) + q(x, r), \qquad (1)$$

$$r(0) = A, r' = B$$

Where q (x, r) is nonlinear function, h(x) is given function and A, B, a, b are constants. The purpose of this paper to introduce a new differential operator to study the problem (1).

In resent year a large amount of research developed concerning Adomian decomposition method [1,5,7], and the related modification [6,8,9,11] to investing various scientific models. The method provides the solution as an infinite series in which each term can be easily determined.

2. ANALYSIS OF THE METHOD

Under the transformation a= m-2n and b= n(n-m) the equation (1) is transformed to

$$r'' + (m - 2n)r' + n(n - m)r = h(x) + q(x,r), \qquad (2)$$

where m, n are constants.

We propose the new differential operator, as blew

$$L(\cdot) = e^{nx}\frac{d}{dx}e^{-mx}\frac{d}{dx}e^{(m-n)x}(\cdot), \qquad (3)$$

So, the problem (2) can be written as,

$$Lr = h(x) + q(x,r)\cdot \qquad (4)$$

The inverse operator L^{-1} is therefore considered a two-fold integeral operator, as below,

$$L^{-1`}(\cdot) = e^{-(m-n)x}\int_0^x e^{mx}\int_0^x e^{-nx}\ (\cdot)\, dxdx\cdot \qquad (5)$$

Applying L^{-1} of (5) to the third terms $r'' + (m-2n)r' + n(n-m)r$ *of* Eq $\cdot$ (2)we find

$$L^{-1}(r'' + (m-2n)r' + n(n-m)r)$$

$$= e^{-(m-n)x}\int_0^x e^{mx}\int_0^x e^{-nx}(r'' + (m-2n)r' + n(n-m)r)\,dxdx$$

$$= e^{-(m-n)}\int_0^x e^{mx}(e^{-nx}r' - (n-m)e^{-nx}r - r'(0) + (n-m)r(0))dx$$

$$= r - e^{-(m-n)x}r(0) - \frac{r'(0)}{m}e^{nx} + \frac{r'(0)}{m}e^{-(m-n)x} + \frac{n-m}{m}r(0)e^{nx} - \frac{n-m}{m}r(0)e^{-(m-n)x}.$$

Operating with L^{-1} on (4), it follows

$$r(x) = e^{-(m-n)x}r(0) + \frac{r'(0)}{m}e^{nx} - \frac{r'(0)}{m}e^{-(m-n)x} - \frac{n-m}{m}r(0)e^{nx} + \frac{n-m}{m}r(0)e^{-(m-n)x} + L^{-1}h(x) + L^{-1}q(x,r) \quad . \quad (6)$$

The Adomian decomposition method introduce the solution r(x) and the non-linear function q(x, r) by infinite series

$$r(x) = \sum_{n=0}^{\infty} r_n(x), \qquad (7)$$

and

$$q(x,r) = \sum_{n=0}^{\infty} A_n, \qquad (8)$$

Where the components $r_n(x)$ of the solution r(x) will be determined recurrently specific algorithums were seen in [7,10] to formulate Adomian polynomials.The following algorithum:

$$A_0 = F(u),$$

$$A_1 = F'(u0)u1,$$

$$A_2 = F'(u0)u_2 + \frac{1}{2}F''(u0)u_{1,}^2$$

$$A_3 = F'(u0)u_3 + F''(u0)u_1u_2 + \frac{1}{3!}F'''(u0)u_{1,}^3 \qquad (9)$$

$$\vdots$$

Can be used construct Adomian polynomials, when F(u) is a nonlinear function.

By substituting (7) and (8) into (6),

$$\sum_{n=0}^{\infty} r_n = e^{-(m-n)x}r(0) + \frac{r'(0)}{m}e^{nx} - \frac{r'(0)}{m}e^{-(m-n)x} - \frac{n-m}{m}r(0)e^{nx} + \frac{n-m}{m}r(0)e^{-(m-n)x} + L^{-1}h(x) + L^{-1}\sum_{n=0}^{\infty} A_n \quad . \quad (10)$$

Through using Adomian decomposition method, the components $r_n(x)$ can be determined as

$$r_0 = e^{-(m-n)x}r(0) + \frac{r'(0)}{m}e^{nx} - \frac{r'(0)}{m}e^{-(m-n)x} - \frac{n-m}{m}r(0)e^{nx} + \frac{n-m}{m}r(0)e^{-(m-n)x} + L^{-1}h(x), \qquad (11)$$

$$r_{n+1} = L^{-1}A_{n,}\ n \geq 0,$$

Which gives

$$r_0 = e^{-(m-n)x}r(0) + \frac{r'(0)}{m}e^{nx} - \frac{r'(0)}{m}e^{-(m-n)x} - \frac{n-m}{m}r(0)e^{nx} + \frac{n-m}{m}r(0)e^{-(m-n)x} + L^{-1}h(x),$$

$$r_1 = L^{-1}A_{0,}$$

$$r_2 = L^{-1}A_{1,}$$

$$r_3 = L^{-1}A_2 \qquad (12)$$

.

.

From (9) and (12), we can determine the components $r_n(x)$, and hence the series solution of r(x) in (7) can be immediately obtained. For numerical purposes, the n-term approximant

$$\varphi_n = \sum_{n=0}^{n-1} r_{n,} \qquad (13)$$

Can be used to approximate the exact solution [11]. The approach presented above can be validated by testing it on a variety of several linear and nonlinear initial value problem.

3. NUMERICAL ILLUSTRATION

We first consider two type of the linear homogenous initial value problem

Type 1: $r'' - zr = 0 \cdot$ Where $z = 1,2,3,4,5,\cdots$

Type 2: $r'' + zr = 0 \cdot$ Where $z = 1,2,3,4,5,\cdots$

Example 1. We consider the type 1

$$r'' - zr = 0 \cdot \qquad (14) \quad \text{Where } z = 1,2,3,4,5,\cdots$$

$$r(0) = 1, r'(0) = 0 \cdot$$

We put m-2n=0 and n(n-m)=z

It follow that m=2n, $n = \pm\sqrt{z}$,

Substitution of $n = \sqrt{z}$ and $m = 2\sqrt{z}$ in Eq.(3) yields the operator

$$L(\cdot) = e^{x\sqrt{z}}\frac{d}{dx}e^{-2x\sqrt{z}}\frac{d}{dx}\,e^{x\sqrt{z}}(\cdot),$$

So

$$L^{-1}(\cdot) = e^{-x\sqrt{z}}\int_0^x e^{2x\sqrt{z}}\int_0^x e^{-x\sqrt{z}}(\cdot)\,dxdx \cdot$$

In an operator from Eq. (14) becomes

$$Lr = 0 \cdot \qquad (15)$$

Applying L^{-1} on both sides of (5) we find

$$L^{-1}Lr = 0,$$

And it implies,

$$r = e^{-x\sqrt{z}}r(0) + \frac{r'(0)}{2\sqrt{z}}e^{x\sqrt{z}} - \frac{r'(0)}{2\sqrt{z}}e^{-x\sqrt{z}} + \frac{1}{2}r(0)e^{x\sqrt{z}} - \frac{1}{2}r(0)e^{-x\sqrt{z}}$$

$$= \frac{1}{2}e^{x\sqrt{z}} + \frac{1}{2}e^{-x\sqrt{z}}$$

Example 2. We consider type 2

$$r'' + zr = 0\cdot \qquad (16) \qquad \text{Where } z = 1,2,3,4,5,\cdots$$

$$r(0) = 1, r'(0) = 0\cdot$$

We put $m - 2n = 0$ and $n(n - m) = z$,

It follows that $n = \pm i\sqrt{z}$, where $i = \sqrt{-1}$,

Substitution of $n = i\sqrt{z}$ and $m = 2i\sqrt{z}$ in $Eq\cdot(3)$ yield the operator

$$L(\cdot) = e^{ix\sqrt{z}}\frac{d}{dx}e^{-2ix\sqrt{z}}\frac{d}{dx}e^{ix\sqrt{z}}(\cdot)$$

So

$$L^{-1} = e^{-ix\sqrt{z}}\int_0^x e^{2ix\sqrt{z}}\int_0^x e^{-ix\sqrt{z}}(\cdot)dxdx\cdot$$

In an operator from, Eq. (16) becomes

$$Lr = 0\cdot \qquad (17)$$

ApplyingL^{-1} on both sides of(17) we find

$$L^{-1}Lr = 0,$$

And it implies,

$$r = e^{-ix\sqrt{z}}r(0) + \frac{r'(0)}{2i\sqrt{z}}e^{ix\sqrt{z}} - \frac{r'(0)}{2i\sqrt{z}}e^{-ix\sqrt{z}} + \frac{1}{2}r(0)e^{ix\sqrt{z}} - \frac{1}{2}r(0)e^{-ix\sqrt{z}}$$

$$= \frac{1}{2}e^{ix\sqrt{z}} + \frac{1}{2}e^{-ix\sqrt{z}}\cdot$$

From two types we observation that, the exact solution is easily obtained by this method.

Example 3. We consider the linear non-homogenous initial value problem:

$$r'' + 3r' - 10r = x, \qquad (18)$$

$$r(0) = 4, r'(0) = -2\cdot$$

We put $m - 2n = 3$ and $n(n - m) = -10$

It follow that $n = 2, n = -5$ *and* $m = \pm 7$,

Substitution of n=2 and m=7 in Eq. (3) yield the operator

$$L(\cdot) = e^{2x}\frac{d}{dx}e^{-7x}\frac{d}{dx}e^{5x}(\cdot),$$

So

$$L^{-1}(\cdot) = e^{-5x}\int_0^x e^{7x}\int_0^x e^{-2x}(\cdot)dxdx\cdot$$

In an operator from, Eq. (18) becomes

$$Lr = x\cdot \qquad (19)$$

Applying L^{-1} on both sides of (19) we find

$$L^{-1}Lr = e^{-5x}\int_0^x e^{7x}\int_0^x e^{-2x}(x)dxdx,$$

And it implies

$$r(x) = r(0)e^{-5x} + \frac{r'(0)}{7}e^{2x} - \frac{r'(0)}{7}e^{-5x} + \frac{5}{7}e^{2x}\,r(0) - \frac{5}{7}e^{-5x}\,r(0) + \frac{1}{28}e^{2x} - \frac{x}{10} - \frac{3}{100} - \frac{4}{700}e^{-5x}\cdot$$

It implies

$$r(x) = \frac{73}{28}e^{2x} + \frac{996}{700}e^{-5x} - \frac{x}{10} - \frac{3}{100}\cdot$$

So, the exact solution is easily obtained by this method.

Example 4. In this example will display if m=n the Eq. (3) becomes

$$r'' - mr' = h(x) + q(x,r), \qquad (20)$$

the differential operator in Eq. (3) becomes

$$L(\cdot) = e^{mx}\frac{d}{dx}e^{-mx}\frac{d}{dx}(\cdot),$$

If m=n=1, and h(x)=0, and q(x, r)=r^2 the Eq. (20) becomes

$$r'' - r' = r^2 \qquad (21)$$

$$r(0) = 1, r'(0) = 0$$

The equation (21) is nonlinear, the differential operator becomes

$$L(\cdot) = e^{x}\frac{d}{dx}e^{-x}\frac{d}{dx}(\cdot),$$

So

$$L^{-1}(\cdot) = \int_0^x e^{-x}\int_0^x e^{x}(\cdot)dxdx\cdot$$

According to (21) we have

$$Lr = r^2,$$

Proceeding as before we obtain

$$r_0 = r(0) + r'(0)e^{x} - r'(0) = 1$$

$$r_{n+1} = L^{-1}A_n, n \geq 0$$

When A_n's are Adomian polynomials of nonlinear term r^2 as below[8]

$$A_0 = r_{0,}^2$$

$$A_1 = 2r_0r_1,$$

$$A_2 = 2r_0r_2 + r_{1,}^2$$

then

$$r_0 = 1$$

$$r_1 = \frac{x^2}{2}$$

$$r_2 = \frac{x^4}{12}$$

$$r_3 = \frac{x^6}{72}$$

This means that the solution in a series form is given by

$$r(x) = r_0 + r_1 + r_2 + r_3 + \cdots = 1 + \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{72} + \cdots,$$

And in the closed form

$$r(x) = e^{x^2} \cdot$$

4. CONCLUSION

In this paper, we offered a new differential operator for solving second order ordinary differential equation with constant coefficient. The examples presented in this paper illustrated the quality of the Adomian decomposition method for finding the solution. In example 1 and 2 and 3 we got the exact solution. In example 4 the results were very closed to exact solution. This indicate that the method is very efficient to solve equations considered.

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