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Generalized Ćirić's contraction in quasi-metric spaces

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Abstract

Inspired by a work from P. Kumam, N.V. Dung and K. Sitytithakerngkiet [Filomat 29 (2015), 1549–1556], we investigate the problem of obtaining fixed point results for generalizations of Ćirić's contraction in the realm of quasi-metric spaces.

Keywords: Ćirić's contraction; quasi-metric space; complete; fixed point.
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1. Introduction and preliminaries

Since Ćirić obtained in [4, Theorem 1] his famous fixed point theorem in terms of the so-called quasi-contractions, many authors have extended and improved this theorem in several directions and contexts (see e.g. [1, 2, 3, 8, 9, 10, 13, 16, 17, 19] and the references therein). Here, we will focus our attention on the generalizations of Ćirić's theorem for quasi-metric spaces. This approach is not new; in fact, old contributions in this setting can be found in [8, 9, 19]. However, more recent investigations, mainly those conducted in [13, 17], have encouraged us to carry out the study presented in this note.

Let us recall that although the notion of a quasi-metric (for the T_1 case) was introduced by Wilson [23] in 1931, the systematized study of the topological properties of these spaces and their relations with other topological structures begins with the article [11] by Kelly. Since then, many authors have contributed to the development of the theory of quasi-metric spaces (see e.g. [6, 14, 5] and the references therein). Moreover, several authors have applied (non-T_1) quasi-metric spaces and some related structures to successfully modelling, among others, various fundamental procedures that appear in theoretical computer science (see e.g. [7, 15, 20, 21, 22]).

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In the sequel, by $\mathbb{R}, \mathbb{R}^+, \mathbb{N}$ and ω we will denote the sets of real numbers, the set of non-negative real numbers, the set of positive integer numbers and the set of non-negative integer numbers, respectively.

A quasi-metric on a set X is a function $d : X \times X \to \mathbb{R}^+$ verifying the following conditions for all $x, y, z \in X$:

(qm1) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$;

(qm2) $d(x, z) \leq d(x, y) + d(y, z)$.

If the quasi-metric d satisfies

(qm1') $d(x, y) = 0 \Leftrightarrow x = y$;

for all $x, y \in X$, we say that d is a T_1 quasi-metric on X.

A (T_1) quasi-metric space is a pair (X, d) where X is a set and d is a (T_1) quasi-metric on X.

If (X, d) is a quasi-metric space, the function $d^s : X \times X \to \mathbb{R}^+$ given by $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ for all $x, y \in X$, is a metric on X.

Let (X, d) be a quasi-metric space. For each $x \in X$ and $\varepsilon > 0$ put $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$. Then, the family $\{B_d(x, \varepsilon) : x \in X, \varepsilon > 0\}$ is a base for a T_0 topology τ_d on X. If d is T_1, then τ_d is T_1. We say that (X, d) is a Hausdorff quasi-metric space if τ_d is a Hausdorff (or T_2) topology.

It is interesting to underline that a sequence $(x_n)_{n\in\omega}$ in X is τ_d-convergent to $x \in X$ if and only if $d(x, x_n) \to 0$ as $n \to \infty$.

In our context, a quasi-metric space (X, d) is complete provided every Cauchy sequence in the metric space (X, d^s) is τ_d-convergent.

We finish this section by reminding a fundamental example of a complete quasi-metric space.

Example 1.1. Let d be the quasi-metric on $\mathbb{R}$ given by $d(x, y) = \max\{x - y, 0\}$ for all $x, y \in \mathbb{R}$. Then $(\mathbb{R}, d)$ is a complete (non-T_1) quasi-metric space (observe that d^s is the Euclidean metric on $\mathbb{R}$).

2. The results

As we indicated in Section 1, Ćirić proved, in Theorem 1 of [4], a celebrated fixed point theorem which we state as follows:

Theorem 2.1. *([4]) Let T be a self map of a complete metric space (X, d). If there is a constant $\alpha \in (0, 1)$ such that*

$$d(Tx, Ty) \leq \alpha \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \qquad (C)$$

for all $x, y \in X$, then T has a unique point $z \in X$ and $d(z, T^n x_0) \to 0$ as $n \to \infty$, for all $x_0 \in X$.

In the old articles [8, 9, 19] it was explored the problem of extending Ćirić's theorem to quasi-metric spaces. The absence of symmetry in the quasi-metric framework makes it difficult to obtain a suitable contraction condition that coincides with condition (C) when we have a metric space. Thus, in an initial attempt it seems reasonable to repeat verbatim condition (C). However, the following easy example shows that the value $d(x, y)$ is not an appropriate candidate to appear in a suitable contraction condition of type (C).

Example 2.2. Let d be the quasi-metric on $\mathbb{N}$ given by $d(n, n) = 0$ for all $n \in \mathbb{N}$ and $d(n, m) = 1/m$ otherwise. Clearly $(\mathbb{N}, d)$ is a complete T_1 quasi-metric space (observe that the non-eventually constant Cauchy sequences are τ_d-convergent to any $n \in \mathbb{N}$).

Now let T be the self map of $\mathbb{N}$ defined as $Tn = 2n$ for all $n \in \mathbb{N}$. Then, for $n \neq m$, we get $d(Tn, Tm) = 1/2m = d(n, m)/2$. However T is free of fixed points.

In addition, it was presented in [19] an example of a self map T of a compact Hausdorff quasi-metric space (X,d) satisfying $d(Tx,Ty) \leq \alpha \max\{d(x,y), d(x,Tx)\}$, for all $x, y \in X$, with $\alpha = 9/10$, and such that T has no fixed points.

Related to the preceding examples, Jachymski proved in [9] that a self map T of a complete T_1 quasi-metric space (X,d) has a unique fixed point provided there is a constant $\alpha \in (0,1)$ satisfying the following contraction condition for all $x, y \in X$:

$$d(Tx,Ty) \leq \alpha \max\{d(y,x), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}. \qquad (C')$$

On the other hand, Kumam, Dung and Sitytithakerngkiet obtained in [13, Theorem 3.1] the following nice improvement of Ćirić's theorem.

Theorem 2.3. *([13]) Let T be a self map of a complete metric space (X,d). If there is a constant $\alpha \in (0,1)$ such that*

$$\begin{aligned} d(Tx,Ty) \leq\ & \alpha \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx), \\ & d(x,T^2x), d(Tx,T^2x), d(x,T^2y), d(T^2x,Ty)\}, \end{aligned} \qquad (KDS)$$

for all $x, y \in X$, then T has a unique point $z \in X$ and $d(z, T^n x_0) \to 0$ as $n \to \infty$, for all $x_0 \in X$.

In order to obtain a suitable quasi-metric generalization of Theorem 2.3 we change the values $d(x,y)$ and $d(Ty,T^2x)$ with $d(y,x)$ and $d(T^2x,Ty)$, respectively, in condition (KDS). In fact, we shall consider the next more general contraction condition, where $\alpha \in (0,1)$:

$$\begin{aligned} d(Tx,Ty) \leq\ & \alpha \max\{d(y,x), d^s(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx), \\ & d^s(x,T^2x), d^s(Tx,T^2x), d(x,T^2y), d(T^2x,Ty)\}. \end{aligned} \qquad (KDS')$$

In the sequel, by a (KDS')-contraction on a quasi-metric space (X,d) we mean a self map T of X satisfying condition (KDS') for all $x, y \in X$.

If T satisfies

$$\begin{aligned} d(Tx,Ty) \leq\ & \alpha \max\{d(y,x), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx), \\ & d(x,T^2x), d(Tx,T^2x), d(x,T^2y), d(T^2x,Ty)\}, \end{aligned} \qquad (KDS'')$$

for all $x, y \in X$, we say that T is a (KDS'')-contraction on (X,d).

It is evident that every (KDS'')-contraction is a (KDS')-contraction. In Example 2.7 we present a (KDS')-contraction which is not a (KDS'')-contraction.

We now get the following results.

Lemma 2.4. *Let T be a (KDS')-contraction on quasi-metric space (X,d). Then, the sequence $(T^n x_0)_{n\in\omega}$ is a Cauchy sequence for all $x_0 \in X$.*

Proof. It is almost obvious that if T satisfies condition (KDS'), then it satisfies condition (KDS) on the metric space (X,d^s). Fix $x_0 \in X$. The first part of the proof of [13, Theorem 3.1] shows that $(T^n x_0)_{n\in\omega}$ is a Cauchy sequence in (X,d^s), so it is a Cauchy sequence in (X,d). □

Theorem 2.5. *Let T be a (KDS')-contraction on a complete quasi-metric space (X,d). Then, there is $z \in X$ such that $d(z,Tz) = 0$, and $d(z, T^n x_0) \to 0$ as $n \to \infty$, for all $x_0 \in X$.*

Proof. Let $\alpha \in (0,1)$ for which T condition (KDS') holds. Fix $x_0 \in X$. For each $n \in \omega$ put $x_n := T^n x_0$. By Lemma 2.4, $(x_n)_{n\in\omega}$ is a Cauchy sequence, so there is $z \in X$ such that $d(z, x_n) \to 0$ as $n \to \infty$.

We shall show that $d(x_n, Tz) \to 0$ as $n \to \infty$.

Choose an arbitrary $\varepsilon > 0$. Then, there is $n_\varepsilon \in \mathbb{N}$ such that $d(z, x_n) < r$ and $d^s(x_n, x_m) < r$ for all $n, m \geq n_\varepsilon$, where $r = (1-\alpha)\varepsilon/2\alpha$.

By mathematical induction we shall prove that

$$d(x_{n+k}, Tz) < r\alpha(\sum_{j=1}^{k} \alpha^{j-1}) + \alpha^{k-1} d(x_n, Tz), \tag{2.1}$$

for all $n \geq n_\varepsilon$ and $k \in \mathbb{N}$.

Indeed, by the contraction condition (KDS'), for each $n \geq n_\varepsilon$ we get

$$d(x_{n+1}, Tz) < \alpha \max\{r, d(z, Tz), d(x_n, Tz), d(x_{n+2}, Tz)\}. \tag{2.2}$$

From the triangle inequality it follows

$$d(z, Tz) < r + d(x_n, Tz) \quad \textit{and} \quad d(x_{n+2}, Tz) < r + d(x_n, Tz), \tag{2.3}$$

for all $n \geq n_\varepsilon$.

Combining (2.2) and (2.3) we obtain

$$d(x_{n+1}, Tz) < \alpha(r + d(x_n, Tz)), \tag{2.4}$$

for all $n \geq n_\varepsilon$.

Since, by (2.2), $d(x_{n+2}, Tz) < \alpha(r + d(x_{n+1}, Tz))$ we deduce

$$d(x_{n+2}, Tz) < r\alpha(1+\alpha) + \alpha d(x_n, Tz).$$

Now assume that $d(x_{n+k}, Tz) < r\alpha(\sum_{j=1}^{k} \alpha^{j-1}) + \alpha^{k-1} d(x_n, Tz)$, with $n \geq n_\varepsilon$ and $k \in \mathbb{N}$.

Then, by applying (2.4), we obtain

$$d(x_{n+k+1}, Tz) < \alpha(r + d(x_{n+k}, Tz)) < r\alpha(\sum_{j=1}^{k+1} \alpha^{j-1}) + \alpha^{k} d(x_n, Tz),$$

and thus the inequality (2.1) remains proved.

Choose k_0 such that $\alpha^{k_0-1} d(x_{n_\varepsilon}, Tz) < \varepsilon/2$. For each $n > n_\varepsilon + k_0$ there is $k \in \mathbb{N}$ such that $n = n_\varepsilon + k$, so $k > k_0$. Hence, by applying (2.1), we deduce

$$d(x_n, Tz) < r\alpha \frac{1}{1-\alpha} + \alpha^{k-1} d(x_{n_\varepsilon}, Tz) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2},$$

for all $n > n_\varepsilon + k_0$.

Therefore $d(T^n x_0, z) \to 0$ as $n \to \infty$. Since $d(z, T^n x_0) \to 0$ as $n \to \infty$, we conclude that $d(z, Tz) = 0$. □

Corollary 2.6. *Let T be a (KDS'')-contraction on a complete quasi-metric space (X, d). Then, there is $z \in X$ such that $d(z, Tz) = 0$, and $d(z, T^n x_0) \to 0$ as $n \to \infty$, for all $x_0 \in X$.*

The next modification of [13, Example 2.5] provides an instance where we can apply Theorem 2.5 but not Corollary 2.6.

Example 2.7. Let $X = \{0,1,2,3,4\}$ and let $d : X \times X \to \mathbb{R}^+$ defined as:

$d(x,x) = 0$ for all $x \in X$;

$d(0,1) = d(0,2) = 0$, $d(0,3) = d(0,4) = 2$;

$d(1,0) = d(1,2) = 1$, $d(1,3) = d(1,4) = 2$;

$d(2,0) = d(2,1) = 1$, $d(2,3) = d(2,4) = 2$;

and

$d(x,y) = 1$ in the rest of cases.

It is routine to check that d is a quasi-metric on X. Notice also that the Cauchy sequences in (X,d^s) are those that are eventually constant, so (X,d) is complete.

Now let $T : X \to X$ defined as $T0 = T1 = T2 = 0$, $T3 = 1$ and $T4 = 2$.

We have

- $d(Tx,Ty) = d(0,0) = 0$ if $x,y \in \{0,1,2\}$.
- $d(Tx,T3) = d(0,1) = 0$ if $x \in \{0,1,2\}$.
- $d(Tx,T4) = d(0,2) = 0$ if $x \in \{0,1,2\}$.
- $d(T3,Ty) = d(1,0) = 1$ if $y \in \{0,1,2\}$.

Since $d^s(3,T3) = d^s(3,1) = d(1,3) = 2$, we get

$$d(T3,Ty) = \frac{1}{2}d^s(3,T3),$$

whenever $y \in \{1,2,3\}$.

- $d(T4,Ty) = d(2,0) = 1$ if $y \in \{0,1,2\}$.

Since $d^s(4,T4) = d^s(4,2) = d(2,4) = 2$, we get

$$d(T4,Ty) = \frac{1}{2}d^s(4,T4),$$

whenever $y \in \{1,2,3\}$.

- $d(T3,T4) = d(1,2) = 1 = d^s(3,T3)/2$.
- $d(T4,T3) = d(2,1) = 1 = d^s(4,T4)/2$.

We deduce that, for $\alpha = 1/2$ and $x,y \in X$,

$$\begin{aligned} d(Tx,Ty) \quad \leq \quad & \alpha \max\{d(y,x), d^s(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx), \\ & d^s(x,T^2x), d^s(Tx,T^2x), d(x,T^2y), d(T^2x,Ty)\}. \end{aligned}$$

This implies that all conditions of Theorem 2.5 are satisfied, and hence, there is $z \in X$ such that $d(z,Tz) = 0$. In this case $z = 0$, and, in addition, it is the unique fixed point of T.

Finally, we shall see that T is not a (KDS'')-contraction on (X,d). To reach it is suffices to observe that $d(T3,T4) = d(1,2) = 1$, and

$$\max\{d(4,3), d(3,T3), d(4,T4), d(3,T4), d(4,T3), d(3,T^23), d(T3,T^23), d(4,T^23), d(T^23,T4)\} = 1.$$

As a consequence of the preceding corollary we have that deleting the values $d(y,x)$ and $d(T^2x,Ty)$, it is possible to show the existence and uniqueness of fixed point. To this end, we introduce the notion of a (KDS''')-contraction as follows:

A self map T of a quasi-metric space (X,d) is called a (KDS''')-contraction on (X,d) if there is a constant $\alpha \in (0,1)$ such that

$$d(Tx,Ty) \leq \alpha \max\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx), d(x,T^2x), d(Tx,T^2x), d(y,T^2x)\}, \tag{2.5}$$

for all $x,y \in X$.

Corollary 2.8. *Let T be a (KDS''')-contraction on a complete quasi-metric space (X,d). Then, there is $z \in X$ such that $d(z,Tz) = 0$ and Tz is the unique fixed point of T.*

Proof. As every (KDS''')-contraction is a (KDS'')-contraction, we deduce from Corollary 2.6 that there is $z \in X$ such that $d(z,Tz) = 0$.

We first show that $d(Tz,T^2z) = 0$.

Indeed, by the contraction condition (2.5) we obtain

$$d(Tz,T^2z) \leq \alpha \max\{d(z,Tz), d(Tz,T^2z), d(z,T^2z), d(Tz,Tz)\}.$$

Since $d(z,T^2z) \leq d(z,Tz) + d(Tz,T^2z) = d(Tz,T^2z)$, we deduce that

$$d(Tz,T^2z) \leq \alpha d(Tz,T^2z).$$

Hence $d(Tz,T^2z) = 0$.

Next we show that $d(T^2z,Tz) = 0$.

Indeed, as $d(z,Tz) = d(Tz,T^2z) = 0$, we have $d(z,T^2z) = 0$, and, by (2.5),

$$d(T^2z,Tz) \leq \alpha \max\{d(T^2z,T^3z), d(Tz,T^3z)\}. \tag{2.6}$$

Since, by (2.5),

$$d(T^2z,T^3z) \leq \alpha \max\{d(T^2z,T^3z), d(Tz,T^3z)\},$$

and, on the other hand,

$$d(Tz,T^3z) \leq d(Tz,T^2z) + d(T^2z,T^3z) = d(T^2z,T^3z),$$

we get $d(T^2z,T^3z) \leq \alpha d(T^2z,T^3z)$, which implies $d(T^2z,T^3z) = 0$. From the triangle inequality it follows that $d(Tz,T^3z) = 0$. By (2.6), we deduce that $d(T^2z,Tz) = 0$. Hence $Tz = T^2z$.

Finally, suppose that $u \in X$ is another fixed point of T. Then

$$d(u,z) = d(Tu,Tz) \leq \alpha \max\{d(u,z), d(z,u)\} \leq \alpha\ d(z,u).$$

Similarly $d(z,u) \leq \alpha\ d(u,z)$. Hence $d(u,z) \leq \alpha\ d(u,z)$, so $d(u,z) = 0$.

Analogously we prove that $d(z,u) = 0$. Consequently $u = z$. □

Remark 2.9. Regarding Corollary 2.8, note that its proof shows that for each $z \in X$ such that $d(z,Tz) = 0$, one has that Tz is the unique fixed point of T.

In the next corollary we show that under the hypotheses of Theorem 2.5, if (X,d) is a T_1 quasi-metric space, then T has a unique fixed point.

Corollary 2.10. *Let T be a self map of a complete T_1 quasi-metric space (X,d). If there is a constant $\alpha \in (0,1)$ such that the contraction condition (KDS') is satisfied by all $x,y \in X$, then T has a unique fixed point $z \in X$ and $d(z,T^nx_0) \to 0$ as $n \to \infty$, for all $x_0 \in X$.*

Proof. By Theorem 2.5 there is $z \in X$ such that $d(z, Tz) = 0$ and $d(z, T^n x_0) \to 0$ as $n \to \infty$, for all $x_0 \in X$. As (X, d) is T_1, we have $z = Tz$.

Suppose that $u \in X$ is another fixed point of T. By (DKS') we deduce

$$d(z, u) = d(Tz, Tu) \leq \alpha \max\{d(u, z), d(z, u)\} = \alpha\ d^s(u, z),$$

and

$$d(u, z) = d(Tu, Tz) \leq \alpha \max\{d(z, u), d(u, z)\} = \alpha\ d^s(u, z).$$

Hence $d^s(u, z) \leq \alpha d^s(u, z)$, so $u = z$. □

The following is an easy example of a self map T of a complete (non-T_1) quasi-metric space (X, d) fulfilling for all $x, y \in X$ and any $\alpha \in (0, 1)$, $d(Tx, Ty) \leq \alpha D(x, y)$, where $D(x, y)$ is any of the seven values that appear on the right side of the contraction condition (KDS'''). Hence T has a unique fixed point. Furthermore, there is $z \in X$ such that $d(z, Tz) = 0$ but z is not a fixed point of T.

Example 2.11. Let $X = \{0, 1\}$ and let d be the quasi-metric on X given by $d(0, 0) = d(1, 1) = d(0, 1) = 0$, and $d(1, 0) = 1$. Clearly (X, d) is complete. Define $T : X \to X$ as $T0 = T1 = 1$. Then $d(T0, T1) = d(T1, T0) = d(1, 1) = 0$. By Corollary 2.8, T has a unique fixed point. In fact, 1 is the unique fixed point of T. Note also that $d(0, T0) = 0$ but 0 is not a fixed point of T (compare Theorem 2.5).

The next example illustrates Corollary 2.8.

Example 2.12. Let $(\mathbb{R}, d)$ be the complete quasi-metric space of Example 1.1.

Let $T : \mathbb{R} \to \mathbb{R}$ defined as $Tx = 0$ for all $x \leq 1$, and $Tx = x/3$ otherwise.

For each $x, y \in \mathbb{R}$ we have

- If $x \leq 1$, $d(Tx, Ty) = d(0, 0) = 0$.
- If $y \leq 1 < x$, $d(Tx, Ty) = d(x/3, 0) = x/3$. Hence
$$d(Tx, Ty) = \frac{1}{3} d(x, Ty) = \frac{3}{8} d(x, T^2 x) = \frac{1}{2} d(x, Tx).$$
- If $x > y > 1$, $d(Tx, Ty) = (x - y)/3$. Hence
$$d(Tx, Ty) \leq \frac{1}{3} d(x, Ty) \quad \text{and} \quad d(Tx, Ty) < \frac{1}{2} d(x, Tx) < \frac{1}{2} d(x, T^2 x).$$
- If $y > x > 1$, $d(Tx, Ty) = 0$.

Therefore for each $x, y \in \mathbb{R}$ we get

$$d(Tx, Ty) \leq \frac{1}{3} d(x, Ty) \quad \text{and} \quad d(Tx, Ty) \leq \frac{1}{2} d(x, Tx) \leq \frac{1}{2} d(x, T^2 x).$$

So T more than meets the conditions of Corollary 2.8 (see also Remark 2.13 below).

We conclude the paper with a characterization of complete quasi-metric spaces that involves Theorem 2.5 and Corollary 2.8.

Let us recall (see e.g. [18]) that a self map T of a quasi-metric space (X, d) is said to be

(a) a Kannan contraction on (X, d) if there is a constant $\alpha \in (0, 1/2)$ such that

$$d(Tx, Ty) \leq \alpha(d(x, Tx) + d(y, Ty))$$

for all $x, y \in X$.

(b) a Chatterjea contraction on (X, d) if there is a constant $\alpha \in (0, 1/2)$ such that

$$d(Tx, Ty) \leq \alpha(d(x, Ty) + d(y, Tx))$$

for all $x, y \in X$.

Remark 2.13. Although the self map T of Example 2.12 is loosely a (KDS''')-contraction, we shall show that it is not a Kannan contraction. Indeed, let $x > 1$ and $y = 0$. Then

$$d(Tx, Ty) = \frac{x}{3} = \frac{1}{2}(d(x, Tx) + d(y, Ty)).$$

Theorem 2.14. *For a quasi-metric space (X, d) the following assertions are equivalent:*

(A) (X, d) is complete.

(B) For each (KDS')-contraction T on (X, d) there is $z \in X$ such that $d(z, Tz) = 0$.

(C) Each (KDS''')-contraction on (X, d) has a fixed point.

(D) Each Kannan contraction on (X, d) has a fixed point.

(E) Each Chatterjea contraction on (X, d) has a fixed point.

Proof. $(A) \implies (B)$ Theorem 2.5.

$(B) \implies (C)$ Let T be a (KDS''')-contraction on (X, d). Then T is a (KDS')-contraction on (X, d), so, by our assumption, there is $z \in X$ such that $d(z, Tz) = 0$. It follows from Corollary 2.8 and Remark 2.9 that Tz is the unique fixed point of T.

$(C) \implies (D)$ and $(C) \implies (E)$ are obvious because both a Kannan contraction and a Chatterjea contraction is a (KDS''')-contraction.

$(D) \implies (A)$ [18, Theorem 3.5].

$(E) \implies (A)$ [18, Theorem 3.5]. □

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