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A VARIATION ON PERFECT NUMBERS

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Abstract

For $k \in \mathbb{N}$ we define a new divisor function s_k called the k^{th} prime symmetric function. By analogy with the sum of divisors function σ, we use the functions s_k to consider variations on perfect numbers, namely k-symmetric-perfect numbers as well as k-cycles. We find all k-symmetric-perfect numbers for $k = 1, 2, 3$. We also consider the problem of whether a natural number n can be expressed in the form $s_k(n)$, and show that for n large enough, it always can be for $k = 1, 2$.

1. Introduction

Definition 1: Let k be a nonnegative integer. We define $s_k : \mathbb{N} \to \mathbb{N} \cup \{0\}$ as follows: If $k = 0$, $s_k(n) \equiv 1$. If $k > 0$, and $n = p_1 \cdots p_r$, where $r = \Omega(n)$ is the number of prime factors (with multiplicity) of n, then

$$s_k(n) = \sum p_{i_1} \cdots p_{i_k},$$

where the sum is taken over all products of k prime factors from the set $\{p_1, \ldots, p_r\}$. We say s_k is the k^{th} prime symmetric function.

Note that if $\Omega(n) < k$, we have $s_k(n) = 0$.

There is an alternate way of defining the functions s_k. Given $n = p_1 \cdots p_r \in \mathbb{N}$, set

$$S_n(x) = \prod_{i=1}^{r} (x + p_i).$$

Then $s_k(n)$ is the coefficient of x^{r-k} in $S_n(x)$. The empty product is taken to be 1.

Example 1: $s_0(12) = 1$, $s_1(12) = 2 + 2 + 3 = 7$, $s_2(12) = 2 \cdot 2 + 2 \cdot 3 + 2 \cdot 3 = 16$, $s_3(12) = 12$, and $s_4(12) = 0$.

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Several good texts detailing the basic theory of perfect numbers exist, see for instance [1], [4], [7], and [13]. In addition, many variations on perfect numbers have been defined and studied. For examples, see the remaining references. We now define a new variation of perfect, defective, and excessive numbers using the divisor functions s_k.

Definition 2: Let $n \in \mathbb{N}$. If $s_k(n) < n$, we say n is k-symmetric-defective. If $s_k(n) > n$, we say n is k-symmetric-excessive. If $s_k(n) = n$, and $\Omega(n) = k$, we say n is trivially k-symmetric-perfect. If $s_k(n) = n$, and $\Omega(n) > k$, we say n is k-symmetric-perfect. If n is k-symmetric-perfect or k-symmetric-excessive, we say n is k-symmetric-special.

Notation: For the sake of brevity we write k-SD for k-symmetric-defective, k-SP for k-symmetric-perfect, k-SE for k-symmetric-excessive, and k-SS for k-symmetric-special.

Example 2: If p is prime, then p^p is a $(p-1)$-SP number, since

$$s_{p-1}(p^p) = \binom{p}{p-1} p^{p-1} = p^p.$$

In fact, this example has a form of converse:

Theorem 1: The prime power p^α is k-SP if and only if $\alpha = p$ and $k = p-1$.

Proof. We have seen that this is sufficient, now suppose $k < \alpha$, and $s_k(p^\alpha) = p^\alpha$. Then

$$\binom{\alpha}{k} = p^{\alpha-k}. \tag{1}$$

For $1 < k < \alpha - 1$, $\binom{\alpha}{k}$ is divisible by two distinct prime factors, hence we must have $k = 1$ or $\alpha - 1$. Now $4 = 2^2$, is the only 1-SP number, and corresponds to the case where $k = 1 = \alpha - 1$. Hence we may assume $k = \alpha - 1$, which from (1) implies that $\alpha = p$ and $k = p - 1$. This proves the theorem. □

Definition 3: A finite sequence $\{n_0, \dots, n_\ell\}$ is called a k-cycle if the following conditions are satisfied:

1. $\ell > 1$,
2. $n_0, \dots, n_{\ell-1}$ are distinct and $n_\ell = n_0$, and
3. $s_k(n_i) = n_{i+1}$, for $i = 0, 1, \dots, \ell - 1$.

2. Basic Properties of Prime Symmetric Functions

The following proposition is an immediate consequence of the definition.

Proposition 2: If $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, then

$$s_k(n) = \sum_{\substack{i_1 + \cdots + i_r = k \\ i_1, \ldots, i_r \geq 0}} \binom{\alpha_1}{i_1} \cdots \binom{\alpha_r}{i_r} p_1^{i_1} \cdots p_r^{i_r}.$$

Proposition 3:

$$s_k(mn) = \sum_{i=0}^{k} s_{k-i}(m) s_i(n).$$

Proof. If $m = 1$, or $n = 1$, the result is immediate, as it is if $k = 0$. If $k > 0$, $m = p_1 \cdots p_r$, and $n = q_1 \cdots q_s$, let

$$S = \{p_1, \ldots, p_r, q_1, \ldots, q_s\}.$$

Then

$$s_k(mn) = \sum_{\{r_1, \ldots, r_k\} \subset S} r_1 \cdots r_k.$$

We collect the terms of this sum having $k - i$ factors from m, and i factors from n. The sum of these is equal to $s_{k-i}(m) s_i(n)$. Summing as i ranges from 0 to k gives the desired result. □

Corollary 4: Let n, $k \in \mathbb{N}$, and let p and q be primes, with $p < q$, and suppose $\Omega(pn) > k$. If $pn - s_k(pn) > 0$, then $pn - s_k(pn) < qn - s_k(qn)$.

Proof. The following inequality

$$\begin{aligned} qn - s_k(qn) &= qn - q s_{k-1}(n) - s_k(n) \\ &> pn - p s_{k-1}(n) - s_k(n) \\ &= pn - s_k(pn) \end{aligned}$$

is true if $n > s_{k-1}(n)$. But

$$pn - s_k(pn) = pn - p s_{k-1}(n) - s_k(n) > 0$$

by assumption, so

$$n > s_{k-1}(n) + s_k(n)/p > s_{k-1}(n).$$

□

In searching for k-cycles and k-SP numbers, it is essential to know when $s_k(n) \geq n$. We search by fixing $\Omega(n)$, and systematically checking all products of $\Omega(n)$ primes. The corollary tells us that if $s_k(pn) < pn$, then for any $q > p$, qn is also k-SD.

Lemma 5: Let k, $n \in \mathbb{N}$. Then there exists an $r > k$ such that if $\Omega(n) \geq r$, then n is k-SD. Let $r(k)$ denote the least such r. Then

$$r(k) = \min\{\, r \;:\; \binom{r}{k} < 2^{r-k} \,\}\,.$$

Proof. There is an $r > k$ such that the function

$$f(t) = \binom{t}{k}$$

satisfies $f(t) < g(t)$ for all $t \geq r$, where

$$g(t) = 2^{t-k},$$

since f is a polynomial, and g is an exponential function. Now suppose $t \geq r$, and let $p_1, \dots, p_t$ be t primes. Then

$$\binom{t}{k} = \binom{t}{t-k} < 2^{t-k}, \text{ which implies } \sum \frac{1}{p_{i_1} \cdots p_{i_{t-k}}} \leq \binom{t}{t-k} \frac{1}{2^{t-k}} < 1,$$

where the sum is taken over all $i_1, \dots, i_{t-k}$ such that $1 \leq i_1 < \cdots < i_{t-k} \leq t$. This implies

$$\sum p_{i_1} \cdots p_{i_k} < p_1 \cdots p_t,$$

where the sum is taken over all $i_1, \dots, i_k$ such that $1 \leq i_1 < \cdots < i_k \leq t$. This in turn implies that

$$s_k(p_1 \cdots p_t) < p_1 \cdots p_t.$$

Now we prove the second statement. The inequality

$$\binom{2k}{k} \geq 2^k$$

holds for all $k \geq 1$, and so $r(k) > 2k$. This in mind, let $r(k)$ be as claimed in the statement of the theorem. We argue inductively. Let $t > r$, and suppose that

$$\binom{t-1}{k} < 2^{t-1-k}.$$

Then

$$2\binom{t-1}{k} < 2^{t-k}.$$

Since $t > 2k$, we have $t < 2(t-k)$, and so

$$\binom{t}{k} = \frac{t(t-1)\cdots(t-k+1)}{k!} < \frac{2(t-1)(t-2)\cdots(t-k)}{k!} = 2\binom{t-1}{k}.$$

Hence

$$\binom{t}{k} < 2\binom{t-1}{k} < 2^{t-k},$$

and the proof is complete by induction. □

The first few values of $r(k)$ are given in the following table:

k	$r(k)$
1	3
2	6
3	10
4	14
5	19
6	23
7	27
8	31
9	36
10	40

The properties of 1-symmetric-perfection etc. corresponding to the first prime symmetric function s_1 are easily characterized. The primes are the trivial 1-SP numbers, 4 is the only 1-SP number, and all other numbers are 1-SD. Clearly there are no 1-cycles. We now investigate these properties in the second prime symmetric function.

3. The Second Prime Symmetric Function

Let n be an integer greater than 1. By a family $E_k(n,r)$ of k-SS numbers, we mean a set

$$E_k(n,r) = \{np_1 \cdots p_r | p_1, \ldots, p_r \text{ are primes}\}$$

such that if $m \in E_k(n,r)$, then m is k-SS. The family $E_2(4,1)$ of numbers of the form $4p$, where p is prime, is one such set, since the elements satisfy $s_2(4p) = 4p + 4 > 4p$. $E_k(n,0)$ merely denotes the singleton set of a k-SS number. To find all 2-SP numbers and all 2-cycles we need to find all numbers n such that $2 < \Omega(n) < 6$, with $s_2(n) \geq n$, since $r(2) = 6$. To do this we use the algorithm mentioned after Corollary 4.

3.1. $\Omega(n) = 3$

$$s_2(2 \cdot 2 \cdot p) = 4p + 4 > 4p,$$
$$s_2(2 \cdot 3 \cdot p) = 5p + 6 < 6p, \text{ when } p > 6.$$

This shows that there are no other infinite families of 2-SS numbers satisfying $\Omega(n) = 3$. Below we find all 2-SS numbers not belonging to this family.

$$s_2(2 \cdot 3 \cdot 3) = 21 > 18,$$
$$s_2(2 \cdot 3 \cdot 5) = 31 > 30,$$
$$s_2(2 \cdot 3 \cdot 7) = 41 < 42,$$
$$s_2(3 \cdot 3 \cdot 3) = 27,$$
$$s_2(3 \cdot 3 \cdot 5) = 39 < 45.$$

Thus 27 is the only 2-SP number satisfying $\Omega(n) = 3$. Iterating on the above 2-SE numbers shows none belong to 2-cycles. For example

$$18 \xrightarrow{s_2} 21 \xrightarrow{s_2} 10 \xrightarrow{s_2} 10 \xrightarrow{s_2} \cdots .$$

3.2. $\Omega(n) = 4$

$$s_2(2 \cdot 2 \cdot 2 \cdot p) = 6p + 12 < 8p, \text{ when } p > 6.$$

Thus there are no infinite families of 2-SS numbers with $\Omega(n) = 4$. Iterating on $8p$ for $p = 2$, 3, 5, shows that none belong to a 2-cycle. Checking other cases:

$$s_2(2 \cdot 2 \cdot 3 \cdot 3) = 37 > 36,$$
$$s_2(2 \cdot 2 \cdot 3 \cdot 5) = 51 < 60,$$
$$s_2(2 \cdot 3 \cdot 3 \cdot 3) = 45 < 54.$$

Hence there are no 2-SP numbers satisfying $\Omega(n) = 4$. Iterating on the above 2-SE numbers shows that none belong to a 2-cycle.

3.3. $\Omega(n) = 5$

$$s_2(2 \cdot 2 \cdot 2 \cdot 2 \cdot p) = 8p + 24 < 16p, \text{ when } p > 3.$$

Thus there are no infinite families of 2-SS numbers with $\Omega(n) = 5$. Iterating on $16p$ for $p = 2$, 3, shows that 48 is in fact 2-SP, and 32, which is 2-SE, does not belong to a 2-cycle. Checking other cases:

$$s_2(2 \cdot 2 \cdot 2 \cdot 3 \cdot 3) = 57 < 72.$$

Hence 48 is the only 2-SP number satisfying $\Omega(n) = 5$. We have proved the following theorem.

Theorem 6: 27 and 48 are the only 2-SP numbers.

Theorem 7: There are no 2-cycles.

Proof. A 2-cycle must have a least element that is 2-SE. We have shown that any such element must belong to the family of numbers of the form $4p$. We will show that in all but a few trivial cases $s_2(s_2(4p)) < 4p$, giving a contradiction. Now, $s_2(4p) = 8((p+1)/2)$. We may assume that p is odd, and set $m = (p+1)/2$. Thus we will have a contradiction if the following holds:

$$s_2(8m) < 8m - 4.$$

This is equivalent to

$$12 + 6s_1(m) + s_2(m) < 8m - 4, \tag{2}$$

which is equivalent to

$$\frac{16}{p_1 \cdots p_s} + 6\sum_{i=1}^{s} \frac{1}{p_1 \cdots \hat{p_i} \cdots p_s} + \sum_{1 \le i < j \le s} \frac{1}{p_1 \cdots \hat{p_i} \cdots \hat{p_j} \cdots p_s} < 8,$$

where $m = p_1 \cdots p_s$. Here $p_1 \cdots \hat{p_i} \cdots p_s$ is defined to be $p_1 \cdots p_s/p_i$, and $p_1 \cdots \hat{p_i} \cdots \hat{p_j} \cdots p_s$ is defined to be $p_1 \cdots p_s/p_ip_j$.

Since $p_i \ge 2$, this expression is implied by:

$$\frac{16}{2^s} + \frac{6s}{2^{s-1}} + \frac{s(s-1)}{2}\frac{1}{2^{s-2}} < 8,$$

which holds for all $s \ge 4$. If $s = 1$, then $m = p$ is prime, and so condition (2) becomes:

$$12 + 6p < 8p - 4,$$

which holds for all $p > 8$. It is easily verified for $p = 2$, 3, 5 and 7, that $8p$ does not belong to a 2-cycle.

For $s = 2$, we can write $m = pq$. The only values of m for which (2) fails are determined by the prime pairs $(p, q) = (2, 2)$, $(2, 3)$. In both cases, $8m$ does not belong to a 2-cycle.

Finally for $s = 3$, if $m = pqr$, only for the triple $(p, q, r) = (2, 2, 2)$ does m fail to satisfy (2). Again, in this case, $8m$ does not belong to a 2-cycle. $\square$

Definition 4: A sequence $\{n_i\}$ (finite or infinite) is called a k-ascending sequence if $n_i < s_k(n_i) = n_{i+1}$. If $\{n_i\} = \{n_i\}_{i=0}^{t}$, then $\{n_i\}$ is said to have length t.

Remark: The longest 2-ascending sequence is

$$8 \xrightarrow{s_2} 12 \xrightarrow{s_2} 16 \xrightarrow{s_2} 24 \xrightarrow{s_2} 30 \xrightarrow{s_2} 31.$$

Definition 5: Let $k \in \mathbb{N} \cup \{0\}$. We define $r_k : \mathbb{N} \to \mathbb{N} \cup \{0\}$ by

$$r_k(n) = |\{s_k^{-1}[\{n\}]|$$

Example 3: $r_1(1) = 0$, but for all $n \geq 2$, $r_1(n) \geq 1$. In fact, $\lim_{n\to\infty} r_1(n) = \infty$. To see this, simply set

$$n = s_1(2^a 3^b) = 2a + 3b,$$

and observe that the number of pairs (a, b) satisfying this equation can be made arbitrarily large for all n sufficiently large.

We prove a weaker result for r_2.

Theorem 8: There exists an $N \in \mathbb{N}$ such that for all $m \geq N$, $r_2(m) \geq 1$.

Proof. It suffices to show that for m sufficiently large, $m = s_2(2^a 3^b 5^c 7^d)$, for some a, b, c, and $d \geq 0$. In general,

$$\begin{aligned} s_2(2^a 3^b 5^c 7^d) &= 4\binom{a}{2} + 9\binom{b}{2} + 25\binom{c}{2} + 49\binom{d}{2} \\ &+ 6\binom{a}{1}\binom{b}{1} + 10\binom{a}{1}\binom{c}{1} + 14\binom{a}{1}\binom{d}{1} \\ &+ 15\binom{b}{1}\binom{c}{1} + 21\binom{b}{1}\binom{d}{1} + 35\binom{c}{1}\binom{d}{1} \\ &= \frac{1}{2}\left[(2a+3b+5c+7d)^2 - (4a+9b+25c+49d)\right] \end{aligned}$$

So, given m, we need only find solutions to the equations:

$$\begin{aligned} 2a + 3b + 5c + 7d &= R, \\ 4a + 9b + 25c + 49d &= R^2 - 2m, \end{aligned}$$

with nonnegative integers a, b, c, d, and $R \in \mathbb{N}$. These equations are equivalent to:

$$2a - 10c - 28d = 3R - R^2 + 2m, \tag{3}$$

$$3b + 15c + 35d = R^2 - 2R - 2m, \tag{4}$$

Since a and b must be nonnegative integers, we have the following necessary and sufficient conditions for a solution to (3) and (4):

1. $2m \equiv R^2 + R + d \pmod 3$,

2. $R^2 - 3R - 10c - 28d \leq 2m$,
3. $2m \leq R^2 - 2R - 15c - 35d$.

Note that equation (3) is always satisfied modulo 2. Condition 1 results from taking equation (4) modulo 3, and conditions 2 and 3 are derived from the fact that $a, b \geq 0$.

Consider the interval

$$I_R(c,d) = [R^2 - 3R - 10c - 28d, R^2 - 2R - 15c - 35d].$$

For fixed d, let $c_R(d)$ be the least c such that $\ell(I_R(c,d)) < 15$, where $\ell(I)$ denotes the length of an interval I. We use the notation $L(I)$ and $R(I)$ to denote the left and right endpoints of an interval I, respectively. Since $R(I_R(c,d)) = R(I_R(c+1,d)) + 15$, when they exist, we have that

$$\bigcup_{c=0}^{c_R(d)} I_R(c,d) = [R^2 - 3R - 10c_R(d) - 28d, R^2 - 2R - 35d].$$

Denote the above interval by $\mathcal{I}_R(d)$. By definition of $c_R(d)$,

$$\ell(I_R(c_R(d),d)) = R - 5c_R(d) - 7d < 15, \text{ so } \quad -10c_R(d) < -2R + 30 + 14d,$$

and $c_R(d)$ is the least such c. Consider the interval $\bigcap_{d=0}^{2} \mathcal{I}_R(d)$. Clearly $R(\bigcap_{d=0}^{2} \mathcal{I}_R(d)) = R^2 - 2R - 70$. We now wish to find an upper bound for $L(\bigcap_{d=0}^{2} \mathcal{I}_R(d))$. From the above inequality, we have that

$$L(\mathcal{I}_R(d)) = R^2 - 3R - 10c_R(d) - 28d < R^2 - 5R - 14d + 30.$$

Thus

$$\begin{aligned} L(\bigcap_{d=0}^{2} \mathcal{I}_R(d)) &= \max\{R^2 - 3R - 10c_R(d) - 28d | d = 0,1,2\} \\ &< \max\{R^2 - 5R - 14d + 30 | d = 0,1,2\} \\ &= R^2 - 5R + 30. \end{aligned}$$

Let $J_R = [R^2 - 5R + 30, R^2 - 2R - 70]$. Then $J_R \subset \bigcap_{d=0}^{2} \mathcal{I}_R(d)$. Now

$$L(J_{R+1}) \leq R(J_R), \text{ if and only if } R^2 - 3R + 26 \leq R^2 - 2R - 70,$$

which holds for all $R \geq 96$. So if $2m \geq L(J_{96}) = 8766$, then there is an $R \geq 96$ such that $2m \in J_R \subset \bigcap_{d=0}^{2} \mathcal{I}_R(d)$. Choose $d \in \{0,1,2\}$ such that condition 1 is satisfied. Since $2m \in \mathcal{I}_R(d)$, there is a $c \geq 0$ such that $2m \in I_R(c,d)$. For these values of R, c, and d, conditions 2 and 3 are satisfied. In other words, there exists an n such that $m = s_2(n)$. This completes the proof. □

We end this section with a conjecture.

Conjecture 1: For every $k \in \mathbb{N}$, $\lim_{n\to\infty} r_k(n) = \infty$.

4. Higher Prime Symmetric Functions

Theorem 9: (1) Let $n \in \mathbb{N}$. If n is k-SS then pn is $(k+1)$-SE for every prime p.

(2) If pn is $(k+1)$-SS for every prime p, then n is k-SS, and hence by (1), pn is $(k+1)$-SE for every prime p.

Proof. (1) Suppose n is k-SS. Then since $\Omega(n) > k$, we have $s_{k+1}(n) > 0$. So

$$\begin{aligned} s_{k+1}(pn) &= ps_k(n) + s_{k+1}(n) \\ &\geq pn + s_{k+1}(n) \\ &> pn. \end{aligned}$$

(2) If $s_{k+1}(pn) = ps_k(n) + s_{k+1}(n) \geq pn$, for every prime p, then $s_k(n) \geq n - s_{k+1}(n)/p$. Letting $p \to \infty$, we have $s_k(n) \geq n$. □

Corollary 10: For $k \in \mathbb{N}$, there are only finitely many k-SP numbers.

Proof. By the previous theorem, any family $E_{k+1}(n, r+1)$ is of the form $pE_k(n, r)$, where p ranges over the primes. Furthermore, this family contains only $(k+1)$-SE numbers. There are only finitely many $(k+1)$-SS numbers not belonging to any such family. □

Thus the infinite families of 3-SE numbers are: $E_3(4, 2)$, $E_3(16, 1)$, $E_3(18, 1)$, $E_3(24, 1)$, $E_3(27, 1)$, $E_3(30, 1)$, $E_3(32, 1)$, $E_3(36, 1)$, $E_3(40, 1)$, $E_3(48, 1)$.

By exhaustive search (as was done with $k = 2$), all other 3-SS numbers can be found. They constitute the following set:

$$\{42p|p = 7, 11, \ldots, 41\} \cup \{56p|p = 7, 11, \ldots, 43\} \cup \{64p|p = 2, 3, \ldots, 37\} \cup$$
$$\{726, 858, 250, 350, 225, 315, 968, 1144, 300, 420, 162, 270, 378, 243, 400, 560,$$
$$216, 360, 504, 324, 288, 480, 672, 432, 256, 384, 640, 576, 512, 768\}.$$

None of the elements in the above sets are 3-SP, hence there are no 3-SP numbers. The diversity of possible 3-ascending sequences makes it difficult to rule out the existence of 3-cycles as we did 2-cycles. This is illustrated in the following example.

Example 4: If p_1, q_1 are odd primes, then $s_3(4p_1q_1) = 4(p_1q_1 + p_1 + q_1)$. It is possible that $p_1q_1 + p_1 + q_1 = p_2q_2$, where p_2, q_2 are again odd primes, and so on. Several such sequences exist the longest one with $p_1q_1 < 50000$, and $p_i, q_i > 3$ is:

$$184892 = 4 \cdot 17 \cdot 2719 \xrightarrow{s_3} 195836 = 4 \cdot 173 \cdot 283 \xrightarrow{s_3} 197660 = 4 \cdot 5 \cdot 9883$$
$$\xrightarrow{s_3} 237212 = 4 \cdot 31 \cdot 1913 \xrightarrow{s_3} 244988 = 4 \cdot 73 \cdot 839 \xrightarrow{s_3} 248636 = 4 \cdot 61 \cdot 1019$$
$$\xrightarrow{s_3} 252956 = 4 \cdot 11 \cdot 5749 \xrightarrow{s_3} 275996 = 4 \cdot 7 \cdot 9857 \xrightarrow{s_3} 315452 = 4 \cdot 17 \cdot 4639$$
$$\xrightarrow{s_3} 334076 = 4 \cdot 47 \cdot 1777 \xrightarrow{s_3} 341372 = 4 \cdot 31 \cdot 2753 \xrightarrow{s_3} 352508 = 4 \cdot 13 \cdot 6779$$
$$\xrightarrow{s_3} 379676 = 4 \cdot 11 \cdot 8629 \xrightarrow{s_3} 414236 = 4 \cdot 29 \cdot 3571 \xrightarrow{s_3} 428636 = 4 \cdot 13 \cdot 8243$$
$$\xrightarrow{s_3} 461660 = 4 \cdot 5 \cdot 41 \cdot 563.$$

It seems highly unlikely, however, that a 3-ascending sequence be infinite. This is part of our final conjecture:

Conjecture 2: Any k-ascending sequence is finite.

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