

Quantum Free Particle and Internal Force

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In a previous note (1) we argued that a quantum free particle, represented by $\exp(ipx)$, carries the information of the force that created it and which it may also impart when striking a target in the distributions $\cos(px)$, $\sin(px)$ of its wavefunction $\exp(ipx)$. In this note, we try to quantify this argument by first examining a second collective force in a quantum bound state other than $-dV(x)/dx$, namely $KE(x)$ ($-dW/dx / W$) where W is the wavefunction. We argue this force should be present in a particle in a box with infinite walls and ultimately in a quantum free particle. Thus $-id/dx$, related to the translational operator, is linked to momentum and its conservation, but also ultimately to impulse carried by such a momentum, namely p . The force is found from $pp/2m$ where $pp/2m$ is the frequency. Thus p is enough information for both momentum and force.. A high resolution (i.e. small wavelength) of a high momentum particle is directly linked to the high force it may impart when striking a target. Alternatively, the notion that an impulse $\text{Integral } F dt$ occurs at a point x , as in two-body scattering in a Maxwell-Boltzmann gas, is only achieved in the high p limit, but an MB case is considered to be a high energy, high p limit i.e. a classical situation.

Two “Collective” Forces in a Quantum Bound State

In a classical Maxwell-Boltzmann gas one calculates pressure by finding the average of:

Constant $p v$ ((1))

where $2p$ is the impulse imparted by a particle with momentum p that bounces back with $-p$ and v is related to the flux. The $1/\text{time}$ from the flux cancels the time in $\text{Integral } F dt$. The constant is related to the dimensions etc.

((1)) leads to the ideal gas law: $PV = nRT$ or $\text{Pressure} = \text{density}(x) RT$ ((2)).

There exists a balance of pressure on two sides of a little box with $-dV/dx$ density(x) where $-dV/dx$ is the Newtonian force and $V(x)$ the potential in the MB gas.

For a classical bound state problem which satisfies: $KE(x) + V(x) = \text{Energy}$, one does not usually think of pressure although a force $-dV/dx$ is present.

What is the situation in a quantum bound state? In (1) we described two collective force terms which “balance” the average microscopic force. In a quantum bound state, one considers an ensemble of free particle states represented by:

$W(x) = \text{Sum over } p a(p)\exp(ipx)$ ((3))

$\exp(ipx)$ is probabilistic because it contains the two x -probability distributions $\cos(px)$ and $\sin(px)$ which are at odds with $x=p/m t$ within a wavelength even though on average the particle does follow $x=p/m t$. We have argued that the x -distributions are reminiscent of a force being present i.e. in classical statistical mechanics a spatial distribution is associated with a force:

$$\text{density}(x) = \exp(-V(x)/T) \text{ where } T = \text{temperature} \quad ((4))$$

How does one show this explicitly? We suggest that each $\exp(ipx)$ represents a force given by:

$$p^2/2m \quad ((5))$$

Here p is the impulse, but only for the particle to change from p to $p=0$ (not bounce back with $-p$), but unlike an MB gas there is no sense of time and hence no flux. $1/\text{time}$ is given by the frequency $p^2/2m = \text{energy}$. With ((5)) one may now consider the time-independent Schrodinger equation for a bound state as in (1):

$$d/dx \{ \{-1/2m d/dx dW/dx / W + V(x)\} = d/dx E_n \quad ((6a)) \text{ or:}$$

$$\{ \text{Sum over } p \ 1/2m p^2 a(p) \exp(ipx) \} / W(x) = -dV/dx + KE(x) dW/dx / W \quad ((6b))$$

The term on the LHS of ((6b)) represents the average force due to the various $\exp(ipx)$ s in the ensemble $W(x) = \text{Sum over } p \ a(p) \exp(ipx)$. This is like the “pressure” in a classical Maxwell-Boltzmann gas. On the RHS of ((6b)) one has two collective forces, namely the usual Newtonian force $-dV/dx$ and $KE(x) dW/dx / W$ where $KE(x)$ is like a frequency associated with average kinetic energy and $dW/dx / W$ is the magnitude of the average impulse in the bound state system.

Thus the presence of $dW/dx / W$ multiplied by an average kinetic energy $KE(x)$ must represent a collective force because $-dV/dx$ represents one. This may be applied to the case of a particle in a box with infinite walls.

Particle in a Box with Infinite Walls

As is well-known, a particle in a box with infinite walls at $x=0$ and $x=L$ has the solution:

$$W_n(x) = C \sin(p_n x) \quad \text{where } \hbar = 1 \text{ and } p_n = n \pi / L \quad ((7))$$

In this case, Newtonian force $-dV/dx = 0$ within the one dimensional box. This is consistent with the existence of an average momentum p_n . $\sin(p_n x)$ is like one distribution of the free particle wavefunction $\exp(ipx)$.

Nevertheless, ((6b)) implies that if $KE(x) dW/dx / W$ is not zero, there should be a collective force in the system at any x . For this case:

$$\text{Collective force} = -p_n \cos(p_n x) / \sin(p_n x) \quad ((8))$$

Thus a constant momentum system (away from the walls) is still associated with an x dependent collective force.

Finally, one may consider a quantum free particle.

Quantum Free Particle

We suggest that one may apply the collective force ((6b)) $KE(x) dW/dx / W$ to a free particle for which:

$$KE(x) = p^2/2m \quad \text{and} \quad |dW/dx / W| = p \quad ((9))$$

Thus a single particle is associated with a force: $p^2/2m$, but this is already the assumption made in ((5)) The notion of a collective force is consistent with this result. Actually $p^2/2m$ representing the force associated with a single particle is not altogether an assumption because the average of $p^2/2m$ appears on the RHS side of ((6b)) and collective forces on the left. Thus $p^2/2m$ must be a force associated with $\exp(ipx)$.

A free particle moving at constant p is still associated with a force $p^2/2m$ at every x point because a force was needed to create it and it may impart this same force on a target.

Force and Resolution

The quantum free particle wavefunction $\exp(ipx)$ is an eigenfunction of both the kinetic energy and momentum operators. What is the momentum operator? It is $-i\hbar d/dx$ where d/dx is the translation operator. We have argued previously that this operator appears in Fermat's time extremization procedure and is linked with momentum conservation. Momentum conservation is also a notion of Newtonian mechanics. Here we stress that d/dx is associated with momentum which represents the impulse carried by a particle with momentum p . Thus we suggest that the small wavelength associated with a high momentum p may be ascribed to the high impulse that p delivers.

In Newtonian mechanics, an impulse is represented by $\int F dt$ and it is usually assumed that this occurs at a point x , e.g. collisions in a Maxwell-Boltzmann gas. As p increases a natural interval of x , namely the wavelength $= \hbar/p$, becomes smaller and smaller. Thus, only in the very high p limit does an impulse actually become associated with an x interval approaching a point.

Entropy Considerations

Consider a particle in a one dimensional box with infinite walls. A high p average implies many peaks within L , i.e. a small wavelength. This means that one has a $1/L$ number of peaks probability of "hitting" a peak. This suggests high entropy. In other words, a high impulse (and not disorder) is associated with a high entropy of knowing the peak point of a particle.

Thermal De Broglie Wavelength

As noted above, $\exp(ipx)$ is associated with an impulse p , but this impulse is linked to the length \hbar/p . For a Maxwell-Boltzmann gas with temperature T , the de Broglie wavelength is given by: $\text{wavelength} = \sqrt{\hbar^2 / 2m kT}$. This is the length associated with an impulse and should be much smaller than interparticle spacing, if impulses are really taking place at a point x as assumed by the form: $\exp(- (p^2/2m + V(x))/T)$.

Conclusion

In conclusion, we try to show explicitly that $\cos(px)$, $\sin(px)$ i.e. two x -probability distributions in $\exp(ipx)$, which statistically indicate the presence of a force are actually associated with an impulse p and hence a force through $p^2/2m$. We do this by taking d/dx of the time-independent Schrodinger equation and associate $p^2/2m$ with the force associated with $\exp(ipx)$ and identify two collective forces $-dV/dx$ and $\hbar^2/(2m) d^2W/dx^2 / W$. If $-dV/dx=0$ as in the case of a particle in a well with infinite potential walls, the collective force is $\hbar^2/(2m) d^2W/dx^2 / W$. Applying this to $\exp(ipx)$ yields $p^2/2m$ which was already considered to be force. Thus we argue that a particle with momentum p is associated with an impulse p and a force $p^2/2m$ and this in turn is associated with $\cos(px)$ and $\sin(px)$. Thus, the higher the p , the smaller the wavelength, associated with the impulse p . As p becomes very large, the wavelength goes to 0 suggesting that an impulse occurs at a point x which is usually the assumption associated with the Newtonian Integral $\int F dt$. For example, in two body collisions in a Maxwell-Boltzmann gas, the impulses occur at a point x .

References

1. Ruggeri, Francesco R. Quantum Average Momentum Compared with Classical Statistical Pressure Part II (preprint, zenodo, 2023)
2. https://en.wikipedia.org/wiki/Thermal_de_Broglie_wavelength