THE AVERAGE BEHAVIOR OF A HYBRID ARITHMETIC FUNCTION INVOLVING HECKE EIGENVALUES

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Abstract

Let $\lambda_f(n)$, $\sigma(n)$ and $\varphi(n)$ be the nth Hecke eigenvalue of normalized cuspidal Hecke eigenform, the sum-of-divisors function and the Euler totient function, respectively. In this paper, we investigate the asymptotic behavior of the summatory function

$$\sum_{n\leq x}\lambda_f^j(n)\sigma^b(n)\varphi^c(n)$$

as $x\to\infty$, where $j\geq 9$ is any fixed integer. This generalizes the previous work in this direction.

1. Introduction

The Fourier coefficients of automorphic forms are interesting and important research objects in modern number theory. Let H_k be the set of normalized primitive holomorphic cusp forms of even integral weight k for the full modular group

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$\Gamma = SL(2,\mathbb{Z})$, which consists of the eigenfunctions for all the Hecke operators T_n. The Fourier coefficients of $f \in H_k$ at the cusp ∞ admits the Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e(nz), \qquad e(z) := e^{2\pi i z},$$

where $\lambda_f(n) \in \mathbb{R}$ are the normalized Fourier coefficients (Hecke eigenvalues) of f, and $\lambda_f(1) = 1$. It is well-known that the Hecke eigenvalues $\lambda_f(n)$ satisfy the Hecke relation

$$\lambda_f(n)\lambda_f(m) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right),$$

where m and n are positive integers. In 1974, P. Deligne [7] proved the celebrated Ramanujan−Petersson conjecture which asserts that

$$|\lambda_f(n)| \leq d(n), \tag{1}$$

where $d(n)$ denotes the classical divisor function.

The Inequality (1) implies that for any prime number p, there exist two complex numbers $\alpha_f(p), \beta_f(p)$ such that

$$\lambda_f(p) = \alpha_f(p) + \beta_f(p), \qquad |\alpha_f(p)| = |\beta_f(p)| = \alpha_f(p)\beta_f(p) = 1. \tag{2}$$

The average behavior of Hecke eigenvalues of normalized cuspidal Hecke eigenforms is an important topic in modern number theory. In 1927, Hecke [11] proved that

$$\sum_{n \leq x} \lambda_f(n) \ll x^{\frac{1}{2}}. \tag{3}$$

Later, the upper bound in Inequality (3) was improved by several authors (see, for example, [7, 12, 25]). In particular, Wu [27] has shown that

$$\sum_{n \leq x} \lambda_f(n) \ll x^{\frac{1}{3}} \log^{\rho} x,$$

where

$$\rho = \frac{102 + 7\sqrt{21}}{210}\left(\frac{6-\sqrt{21}}{5}\right)^{\frac{1}{2}} + \frac{102 - 7\sqrt{21}}{210}\left(\frac{6+\sqrt{21}}{5}\right)^{\frac{1}{2}} - \frac{33}{35} = -0.118\cdots.$$

In the 1930s, Rankin [24] and Selberg [26] independently proved the asymptotic formula

$$\sum_{n \leq x} \lambda_f^2(n) = c_f x + O(x^{3/5}) \tag{4}$$

for any $\varepsilon > 0$, where $c_f > 0$ is a positive constant depending on f. Very recently, the exponent in Equation (4) was improved to $\frac{3}{5} - \delta$ in place of $\frac{3}{5}$ by Huang [13], where $\delta \leq 1/560$. This remains the best possible result to date.

In 2015, Manski, Mayle and Zbacnik [20] considered the average behavior of a hybrid arithmetic function and proved that

$$\sum_{n\leq x} d^a(n)\sigma^b(n)\varphi^c(n) = x^{b+c+1}P_{2^a-1}(\log x) + O\big(x^{b+c+r_a+\varepsilon}\big)$$

where $a, b, c \in \mathbb{R}$ and $\frac{1}{2} \leq r_a < 1$, here $P_l(t)$ denotes the polynomial of t with degree l. Later, Li [19] and Cui [6] investigated the average behavior of the sum

$$\sum_{n\leq x} \lambda_f^j(n)\sigma^b(n)\varphi^c(n) \tag{5}$$

for $1 \leq j \leq 6$. Very recently, Wei and Lao [28] refined the results for $j = 2, 4, 6$ and gave the asymptotic behavior of Equation (5) for $j = 7, 8$.

Inspired by the above results, in this paper we consider the asymptotic behavior of Equation (5) for $j \geq 9$ by invoking the recent work of Newton and Thorne [21, 22] which asserts that $\mathrm{sym}^r f$ is an automorphic cuspidal representation of $GL(r + 1)$ for all $r \geq 1$. The well-known analytic properties, such as the individual and averaged convexity or subconvexity bounds for the associated L-functions, also play an important role in the proof of the main result. More precisely, we prove the following theorem.

Theorem 1.1. *Let $b, c \in \mathbb{R}$, and let $f \in H_k$ be a Hecke eigenform. Let $j \geq 9$ be any given positive integer. Then:*
(i) For $j = 2l$ an even integer, we have

$$\sum_{n\leq x} \lambda_f^j(n)\sigma^b(n)\varphi^c(n) = x^{b+c+1}P_{A_j-1}(\log x) + O_f\Big(x^{b+c+1-\frac{420}{105\cdot 2^{2l+1}-80A_l-63C_l+63}+\varepsilon}\Big)$$

for any $\varepsilon > 0$, where $P_l(t)$ denotes a polynomial in t with degree l, and A_l and C_l are given by

$$A_l = \frac{(2l)!}{l!(l+1)!}, \qquad C_l = \frac{3\cdot(2l)!}{(l-1)!(l+2)!}.$$

(ii) For $j = 2l + 1$ an odd integer, we have

$$\sum_{n\leq x} \lambda_f^j(n)\sigma^b(n)\varphi^c(n) = O_f\Big(x^{b+c+1-\frac{3}{3\cdot 2^{2l}-B_l}+\varepsilon}\Big)$$

for any $\varepsilon > 0$, where B_l is given by

$$B_l = 2\frac{(2l+1)!}{l!(l+2)!}.$$

Throughout the paper, we always assume that $f \in H_k$ is a Hecke eigenform. Also, let $\varepsilon > 0$ be an arbitrarily small positive constant that may vary in different contexts. The symbol p always denotes a prime number.

2. Preliminaries

In this section, we review some relevant facts about the symmetric power L-functions and collect some important lemmas which play an important role in the proof of the main result in this paper.

Let $f \in H_k$ be a Hecke eigenform. The Hecke L-function associated with $f(z)$ is defined by

$$L(f,s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1}, \ \Re(s) > 1,$$

where the local parameters $\alpha_f(p)$ and $\beta_f(p)$ are defined as in Equation (2).

We can also define the jth symmetric power L-function attached to f by

$$L(\mathrm{sym}^j f, s) := \prod_p \prod_{m=0}^{j} \left(1 - \frac{\alpha_f(p)^{j-m}\beta_f(p)^m}{p^s}\right)^{-1} \tag{6}$$

for $\Re(s) > 1$. We can rewrite it as a Dirichlet series

$$\begin{aligned} L(\mathrm{sym}^j f, s) \quad &= \quad \prod_p \left(1 + \frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s} + \ldots + \frac{\lambda_{\mathrm{sym}^j f}(p^k)}{p^{ks}} + \ldots\right) \\ &:= \quad \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s}, \ \Re(s) > 1. \end{aligned} \tag{7}$$

It is well-known that $\lambda_{\mathrm{sym}^j f}(n)$ is a real multiplicative function. And from Equations (2), (6), (7) and Hecke operator theory, we infer that

$$\lambda_f(p^j) = \sum_{m=0}^{j} \alpha_f(p)^{j-2m} = \lambda_{\mathrm{sym}^j f}(p), \ j \geq 1. \tag{8}$$

Let π_f be an automorphic cuspidal automorphic representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$. It is well-known that an automorphic cuspidal representation π of $GL_2(\mathbb{A}_{\mathbb{Q}})$ is associated with a primitive form f, and hence an automorphic function $L(\pi_f, s)$ coincides with $L(f, s)$. Denote by $\mathrm{sym}^j \pi_f$ the jth symmetric power lift of π_f. For $2 \leq$

$j \leq 8$, the automorphy of $\text{sym}^j \pi_f$ were proved by a series of important work of Gelbart and Jacquet [9], Kim and Shahidi [15, 16, 17], Dieulefait [8], and Clozel and Thorne [3, 4, 5]. Very recently, Newton and Thorne [21, 22] showed that there exists a cuspidal automorphy representation of $GL_{j+1}(\mathbb{A}_{\mathbb{Q}})$ whose L-function equals $L(\text{sym}^j f, s)$ for all $j \geq 1$. Hence for $j \geq 1$, the L-function $L(\text{sym}^j f, s)$ is an entire function and satisfies a functional equation of certain Riemann-type with degree $j+1$.

In the next lemma, we introduce the truncated Perron's formula, which is given in Karatsuba and Voronin by [14, pp. 334-336].

Lemma 2.1. *Suppose that the series $f(s) = \sum_{n \geq 1} a_n n^{-s}$ converges absolutely in $\Re(s) > 1$, and $|a(n)| \leq A(n)$, where $A(n)$ is a positive monotonously increasing function and*

$$\sum_{n \geq 1} |a_n| n^{-\sigma} = O\big((\sigma - 1)^{-\alpha}\big)$$

for some $\alpha > 1$ as $\sigma \to 1^+$. Then

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s) \frac{x^s}{s} ds + O\bigg(\frac{x^b}{T(b-1)^\alpha}\bigg) + O\bigg(\frac{xA(2x)\log x}{T}\bigg)$$

holds for any $1 < b \leq b_0, T \geq 2, x = N + \frac{1}{2}$ (the constants in O-terms depend on b_0).

Next, we introduce a series of helpful lemmas which are important in the proof of the main results in this paper. First of all, we give the decompositions of the generating L-functions into some lower degree L-functions, which is significant in the computations via Perron's formula. Next, we invoke the subconvexity bounds for some associated L-functions, which gives better results.

Lemma 2.2. *Let $b, c \in \mathbb{R}$ and $j \geq 9$ be a fixed integer, and let $f \in H_k$ be a Hecke eigenform. Let $j = 2l$ be an even integer, and define*

$$L_{j,b,c}(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^j(n) \sigma^b(n) \varphi^c(n)}{n^s}.$$

Then

$$\begin{aligned} L_{j,b,c}(s) &= \zeta(s-b-c)^{A_l} L(sym^{2l} f, s-b-c) \\ &\quad \times \prod_{1 \leq r \leq l-1} L(sym^{2r} f, s-b-c)^{C_l(r)} U_{j,b,c}(s), \end{aligned}$$

where A_l and $C_l(r)$ are given by

$$A_l = \frac{(2l)!}{l!(l+1)!}, \qquad C_l(r) = \frac{(2l)!(2r+1)}{(l-r)!(l+r+1)!}.$$

The L-function $L_{j,b,c}(s)$ is of degree 2^j and all coefficients are nonnegative, and $U_{j,b,c}(s)$ is a Dirichlet series which converges absolutely and uniformly in the half-plane $\Re(s) > b + c + \frac{1}{2}$.

Proof. Since $\lambda_f^j(n)\sigma^b(n)\varphi^c(n)$ is a multiplicative function, then for $\Re(s) \gg 1$ we have the Euler product

$$\begin{aligned} L_{j,b,c}(s) &= \prod_p \left(1 + \sum_{k\geq 1} \frac{\lambda_f^j(p^k)\sigma^b(p^k)\varphi^c(p^k)}{p^{ks}}\right) \\ &= \prod_p \left(1 + \frac{\lambda_f^j(p)\sigma^b(p)\varphi^c(p)}{p^s} + \frac{\lambda_f^j(p^2)\sigma^b(p^2)\varphi^c(p^2)}{p^{2s}} + \dots\right). \end{aligned}$$

In the half-plane $\Re(s) > b + c + \frac{1}{2}$, the p-th coefficient of the L-function determines the analytic properties of $L_{j,b,c}(s)$.

By Equation (8) and [18, Lemma 7.1], we get

$$\begin{aligned} \lambda_f^j(p)\sigma^b(p)\varphi^c(p) &= \lambda_f(p)^j(p+1)^b(p-1)^c \\ &= \left(A_l + \sum_{1\leq r\leq l-1} C_l(r)\lambda_{\mathrm{sym}^{2r}f}(p) + \lambda_{\mathrm{sym}^{2l}f}(p)\right)(p+1)^b(p-1)^c \end{aligned}$$

since in this case $j = 2l$ is an even integer.

Let $s = \sigma + it$. By following an argument similar to that of Wei and Lao [28, Lemma 2.4], we get

$$\begin{aligned} L_{j,b,c}(s) &= \prod_p \left(1 + \frac{\lambda_f^j(p)(1+\chi(p))}{p^{s-b-c}} + O\big(p^{2(b+c-\sigma)} + p^{(b+c-1-\sigma)}\big)\right) \\ &:= \zeta(s-b-c)^{A_l} L(\mathrm{sym}^{2l} f, s-b-c) \\ &\quad \times \prod_{1\leq r\leq l-1} L(\mathrm{sym}^{2r} f, s-b-c)^{C_l(r)} U_{j,b,c}(s), \end{aligned}$$

where the Dirichlet series $U_{j,b,c}(s)$ converges absolutely and uniformly in the half-plane $\Re(s) \geq b + c + \frac{1}{2} + \varepsilon$ and $U_{j,b,c}(s) \neq 0$ with $\Re(s) = b + c + 1$. □

Lemma 2.3. *Let $b, c \in \mathbb{R}$ and $j \geq 9$ be a fixed integer, and let $f \in H_k$ be a Hecke eigenform. Let $j = 2l + 1$ be an odd integer, and define*

$$L^*_{j,b,c}(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^j(n)\sigma^b(n)\varphi^c(n)}{n^s}.$$

Then

$$\begin{aligned} L^*_{j,b,c}(s) &= L(f, s-b-c)^{B_l} L(sym^{2l+1} f, s-b-c) \\ &\quad \times \prod_{1\leq r\leq l-1} L(sym^{2r+1} f, s-b-c)^{D_l(r)} U^*_{j,b,c}(s), \end{aligned}$$

where B_l and $D_l(r)$ are given by

$$B_l = 2\frac{(2l+1)!}{l!(l+2)!}, \qquad D_l(r) = \frac{(2l+1)!(2r+2)}{(l-r)!(l+r+2)!}.$$

The L-function $L_{j,b,c}(s)$ is of degree 2^j, and $U^*_{j,b,c}(s)$ is a Dirichlet series which converges absolutely and uniformly in the half-plane $\Re(s) > b+c+\frac{1}{2}$.

Proof. This follows essentially the same argument as in Wei et al. [28] and note Lau and Lü [18, Lemma 7.1]. □

Lemma 2.4. *For any $\varepsilon > 0$, we have*

$$\begin{aligned} \zeta(\sigma+it) &\ll \big(1+|t|\big)^{\max\{\frac{13}{42}(1-\sigma),0\}+\varepsilon}, \\ L(f,\sigma+it) &\ll \big(1+|t|\big)^{\max\{\frac{2}{3}(1-\sigma),0\}+\varepsilon}, \\ L(sym^2 f,\sigma+it) &\ll \big(1+|t|\big)^{\max\{\frac{27}{20}(1-\sigma),0\}+\varepsilon}. \end{aligned}$$

uniformly for $\frac{1}{2} \le \sigma \le 2$ and $|t| \ge 1$.

Proof. The first result is the new breakthrough of Bourgain [2], the second is due to Good [10], and the third result follows from the work of Aggarwal [1]. □

Now, we state some basic definitions and analytic properties about general L-functions. A general L-function $L(\phi,s)$ is a Dirichlet series (associated with the object ϕ) that admits an Euler product of degree $m \ge 1$, namely

$$L(\phi,s) = \sum_{n=1}^{\infty} \frac{\lambda_\phi(n)}{n^s} = \prod_{p<\infty} \prod_{j=1}^{m} \left(1 - \frac{\alpha_\phi(p,j)}{p^s}\right)^{-1},$$

where $\alpha_\phi(p,j), j = 1,2,\cdots,m$ are the local parameters of $L(\phi,s)$ at a finite prime p. Suppose that this series and its Euler product are absolutely convergent for $\Re(s) > 1$. We denote the gamma factor by

$$L_\infty(\phi,s) = \prod_{j=1}^{m} \pi^{-\frac{s+\mu_\phi(j)}{2}} \Gamma\left(\frac{s+\mu_\phi(j)}{2}\right)$$

with local parameters $\mu_\phi(j), j = 1,2,\cdots,m$ of $L(\phi,s)$ at ∞. The complete L-function $\Lambda(\phi,s)$ is defined by

$$\Lambda(\phi,s) = q(\phi)^{\frac{s}{2}} L_\infty(\phi,s) L(\phi,s),$$

where $q(\phi)$ is the conductor of $L(\phi,s)$. We assume that $\Lambda(\phi,s)$ admits an analytic continuation to the the whole complex plane $\mathbb{C}$ and is holomorphic everywhere except for possible poles of finite order at $s = 0, 1$. Furthermore, it satisfies a functional equation of the Riemann-type

$$\Lambda(\phi, s) = \epsilon_\phi \Lambda(\tilde{\phi}, 1 - s)$$

where ϵ_ϕ is the root number with $|\epsilon_\phi| = 1$ and $\tilde{\phi}$ is dual of ϕ such that $\lambda_{\tilde{\phi}}(n) = \overline{\lambda_\phi(n)}, L_\infty(\tilde{\phi}, s) = L_\infty(\phi, s)$ and $q(\tilde{\phi}) = q(\phi)$. We say that L-function $L(\phi, s)$ satisfies the Ramanujan conjecture if $\lambda_\phi(n) \ll n^\varepsilon$ for any ε.

From above we observe that the L-functions $L(\mathrm{sym}^j f, s), j \geq 1$ are general L-functions in the sense of Perelli [7]. For the general L-functions, we have the following averaged or individual convexity bounds.

Lemma 2.5. *Assume that $\mathfrak{L}(s)$ is a general L-function of degree m. Then*

$$\int_T^{2T} \left|\mathfrak{L}(\sigma + it)\right|^2 dt \ll T^{m(1-\sigma)+\varepsilon},$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1$ and $T \geq 1$, and

$$\mathfrak{L}(\sigma + it) \ll \big(1 + |t|\big)^{\max\{\frac{m}{2}(1-\sigma), 0\}+\varepsilon},$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1 + \varepsilon$ and $|t| \geq 1$.

Proof. This follows from the results of Perelli's mean value theorem and convexity bounds for general L-functions [7]. □

3. Proof of Theorem 1.1

We firstly consider the case that the $j = 2\ell$ is an even integer. By applying Lemma 2.1, we obtain

$$\sum_{n \leq x} \lambda_f^j(n) \sigma^b(n) \varphi^c(n) = \frac{1}{2\pi i} \int_{b+c+1+\varepsilon-iT}^{b+c+1+\varepsilon+iT} L_{j,b,c}(s) \frac{x^s}{s} ds + O\left(\frac{x^{b+c+1+\varepsilon}}{T}\right),$$

where $s = \sigma + it$ and $1 \leq T \leq x$ is some parameter to be chosen later. For the sake of simplicity, by Lemma 2.2 we will write

$$L_{j,b,c}(s) = H_{j,b,c}(s - b - c) U_{j,b,c}(s),$$

where

$$H_{j,b,c}(s) := \zeta(s)^{A_l} L(\mathrm{sym}^{2l} f, s) \prod_{1 \leq r \leq l-1} L(\mathrm{sym}^{2r} f, s)^{C_l(r)}.$$

By shifting the line of integration to the parallel segment with $\Re(s) = b + c + \frac{1}{2} + \varepsilon$ and invoking Cauchy's residue theorem, by Lemma 2.2 we have

$$\begin{aligned}
&\sum_{n\le x} \lambda_f^j(n)\sigma^b(n)\varphi^c(n) = \mathrm{Res}_{s=b+c+1}\Big\{L_{j,b,c}(s)\frac{x^s}{s}\Big\}\\
&+\frac{1}{2\pi i}\Big\{\int_{b+c+\frac{1}{2}+\varepsilon-iT}^{b+c+\frac{1}{2}+\varepsilon+iT}+\int_{b+c+1+\varepsilon-iT}^{b+c+\frac{1}{2}+\varepsilon-iT}+\int_{b+c+\frac{1}{2}+\varepsilon+iT}^{b+c+1+\varepsilon+iT}\Big\}L_{j,b,c}(s)\frac{x^s}{s}ds\\
&+O\Big(\frac{x^{b+c+1+\varepsilon}}{T}\Big)\\
&:= x^{b+c+1}P_{A_l-1}(\log x) + J_1 + J_2 + J_3 + O\Big(\frac{x^{b+c+1+\varepsilon}}{T}\Big), \qquad (9)
\end{aligned}$$

where the term $x^{b+c+1}P_{A_l-1}(\log x)$ comes from the residue of the function

$$L_{j,b,c}(s)\frac{x^s}{s}$$

at $s = 1$, and $P_l(t)$ denotes a polynomial of t with degree l.

Let

$$H_{j,b,c}(s) = \zeta(s)^{A_l}G_{j,b,c}(s)L(\mathrm{sym}^2 f, s)^{C_l(1)},$$

where

$$G_{j,b,c}(s) := L(\mathrm{sym}^{2l} f, s)\prod_{2\le r\le l-1} L(\mathrm{sym}^{2r} f, s)^{C_l(r)}.$$

It is not hard to find that $G_{j,b,c}(s)$ is an L-function of degree $2^{2l} - A_l - 3C_1(1)$.

Now we need to handle the three terms J_1, J_2 and J_3. For the integrals over the horizontal segments J_2 and J_3, we have

$$\begin{aligned}
J_2 + J_3 &\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} \big|H_{j,b,c}(\sigma+iT)\big|x^{b+c+\sigma}T^{-1}d\sigma\\
&\ll x^{b+c}\int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} \big|H_{j,b,c}(\sigma+iT)\big|x^{\sigma}T^{-1}d\sigma.
\end{aligned}$$

By Lemmas 2.4-2.5, we have

$$\begin{aligned}
J_2 + J_3 &\ll x^{b+c}\max_{\frac{1}{2}+\varepsilon\le\sigma\le 1+\varepsilon} x^{\sigma}T^{(\frac{13}{42}A_l+\frac{27}{20}C_l(1)+\frac{1}{2}(2^{2l}-A_l-3C_l(1)))(1-\sigma)+\varepsilon}T^{-1}\\
&\ll \frac{x^{b+c+1+\varepsilon}}{T} + x^{b+c+\frac{1}{2}+\varepsilon}T^{2^{2l-2}-\frac{2}{21}A_l-\frac{3}{40}C_l(1)-1+\varepsilon}. \qquad (10)
\end{aligned}$$

For J_1, by Lemmas 2.4 and 2.5 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} J_1 &\ll x^{b+c+\frac{1}{2}+\varepsilon}\int_1^T \left|H_{j,b,c}\left(\frac{1}{2}+it\right)\right| t^{-1}dt + x^{b+c+\frac{1}{2}+\varepsilon} \\ &\ll x^{b+c+\frac{1}{2}+\varepsilon}\log T \max_{1\le T_1\le T/2}\left(\max_{T_1\le t\le 2T_1} T_1^{-1}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{A_l}\right. \\ &\quad \times\left|L\left(\mathrm{sym}^2 f,\frac{1}{2}+it\right)\right|^{C_l(1)-1}\left(\int_{T_1}^{2T_1}\left|L\left(\mathrm{sym}^2 f,\frac{1}{2}+it\right)\right|^2 dt\right)^{\frac{1}{2}} \\ &\quad \times\left.\left(\int_{T_1}^{2T_1}\left|G_{j,b,c}\left(\frac{1}{2}+it\right)\right|^2 dt\right)^{\frac{1}{2}}\right) + x^{b+c+\frac{1}{2}+\varepsilon} \\ &\ll x^{b+c+\frac{1}{2}+\varepsilon}T^{\frac{13}{84}A_l+\frac{27}{40}(C_l(1)-1)+\frac{3}{4}+(2^{2l}-A_l-3C_l(1))\times\frac{1}{4}-1+\varepsilon} + x^{b+c+\frac{1}{2}+\varepsilon} \\ &\ll x^{b+c+\frac{1}{2}+\varepsilon}T^{2^{2l-2}-\frac{2}{21}A_l-\frac{3}{40}C_l(1)-\frac{37}{40}+\varepsilon}. \end{aligned} \tag{11}$$

Therefore, from Equations (9)-(11), we have

$$\begin{aligned} \sum_{n\le x}\lambda_f^j(n)\sigma^b(n)\varphi^c(n) &= x^{b+c+1}P_{A_j-1}(\log x) + O\left(\frac{x^{b+c+1+\varepsilon}}{T}\right) \\ &\quad + O\left(x^{b+c+\frac{1}{2}+\varepsilon}T^{2^{2l-2}-\frac{2}{21}A_l-\frac{3}{40}C_l(1)-\frac{37}{40}+\varepsilon}\right). \end{aligned} \tag{12}$$

On taking $T = x^{\frac{420}{105\cdot 2^{2l+1}-80A_l-63C_l(1)+63}}$ in Equation (12), we get

$$\sum_{n\le x}\lambda_f^j(n)\sigma^b(n)\varphi^c(n) = x^{b+c+1}P_{A_j-1}(\log x) + O\big(x^{b+c+1-\frac{420}{105\cdot 2^{2l+1}-80A_l-63C_l(1)+63}+\varepsilon}\big).$$

This completes the case that $j = 2l$.

Now we turn to the case that $j = 2l+1$. By applying Lemmas 2.1 and 2.3 and shifting the line of integration to the parallel segment with $\Re(s) = b+c+\frac{1}{2}+\varepsilon$, we obtain

$$\begin{aligned} &\sum_{n\le x}\lambda_f^j(n)\sigma^b(n)\varphi^c(n) = \\ &\qquad \frac{1}{2\pi i}\left\{\int_{b+c+\frac{1}{2}+\varepsilon-iT}^{b+c+\frac{1}{2}+\varepsilon+iT} + \int_{b+c+1+\varepsilon-iT}^{b+c+\frac{1}{2}+\varepsilon-iT} + \int_{b+c+\frac{1}{2}+\varepsilon+iT}^{b+c+1+\varepsilon+iT}\right\} L^*_{j,b,c}(s)\frac{x^s}{s}ds \\ &\qquad + O\left(\frac{x^{b+c+1+\varepsilon}}{T}\right) \\ :=&\quad I_1+I_2+I_3+O\left(\frac{x^{b+c+1+\varepsilon}}{T}\right), \end{aligned} \tag{13}$$

where $s = \sigma+it$ and $1 \le T \le x$ is a parameter to be chosen later.

For the sake of simplicity, by Lemma 2.3 we write

$$L^*_{j,b,c}(s) = H^*_{j,b,c}(s-b-c)U^*_{j,b,c}(s),$$

where

$$H^*_{j,b,c}(s) := L(f,s)^{B_l} L(\mathrm{sym}^{2l+1} f, s) \prod_{1\le r\le l-1} L(\mathrm{sym}^{2r+1} f, s)^{D_l(r)}.$$

And let

$$H^*_{j,b,c}(s) = L(f,s)^{B_l} G^*_{j,b,c}(s) L(\mathrm{sym}^3 f, s)^{D_l(1)},$$

where

$$G^*_{j,b,c}(s) := L(\mathrm{sym}^{2l+1} f, s) \prod_{2\le r\le l-1} L(\mathrm{sym}^{2r+1} f, s)^{D_l(r)}.$$

It is obvious that $G^*_{j,b,c}(s)$ is an L-function of degree $2^{2l+1} - 2B_l - 4D_l(1)$.

Next we begin to handle the three terms I_1, I_2 and I_3. For I_1, using Lemmas 2.4 and 2.5 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
I_1 &\ll x^{b+c+\frac{1}{2}+\varepsilon} \int_1^T \left| H^*_{j,b,c}\left(\frac{1}{2}+it\right)\right| t^{-1} dt + x^{b+c+\frac{1}{2}+\varepsilon} \\
&\ll x^{b+c+\frac{1}{2}+\varepsilon} \log T \max_{1\le T_1\le T/2} \Bigg(\max_{T_1\le t\le 2T_1} T_1^{-1} \left| L\left(f, \frac{1}{2}+it\right)\right|^{B_l} \\
&\quad \times \left| L\left(\mathrm{sym}^3 f, \frac{1}{2}+it\right)\right|^{D_l(1)-1} \left(\int_{T_1}^{2T_1} \left| L\left(\mathrm{sym}^3 f, \frac{1}{2}+it\right)\right|^2 dt \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{T_1}^{2T_1} \left| G^*_{j,b,c}\left(\frac{1}{2}+it\right)\right|^2 dt \right)^{\frac{1}{2}} \Bigg) + x^{b+c+\frac{1}{2}+\varepsilon} \\
&\ll x^{b+c+\frac{1}{2}+\varepsilon} T^{\frac{1}{3}B_l + \frac{1}{2}\times\frac{1}{2}\times 4\times(D_l(1)-1) + \frac{1}{2}\times\frac{1}{2}\times 4 + (2^{2l+1}-2B_l-4D_l(1))\times\frac{1}{4} - 1 + \varepsilon} \\
&\quad + x^{b+c+\frac{1}{2}+\varepsilon} \\
&\ll x^{b+c+\frac{1}{2}+\varepsilon} T^{2^{2l-1} - \frac{1}{6}B_l - 1 + \varepsilon}.
\end{aligned} \tag{14}$$

For the integrals over the horizontal segments I_2 and I_3, by Lemmas 2.4 and 2.5 we have

$$\begin{aligned}
I_2 + I_3 &\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} \left| H^*_{j,b,c}(\sigma+iT)\right| x^{b+c+\sigma} T^{-1} d\sigma \\
&\ll x^{b+c} \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} \left| H^*_{j,b,c}(\sigma+iT)\right| x^{\sigma} T^{-1} d\sigma \\
&\ll x^{b+c} \max_{\frac{1}{2}+\varepsilon\le\sigma\le b} x^{\sigma} T^{(\frac{2}{3}B_l + \frac{1}{2}(2^{2l+1}-2B_l))(1-\sigma)+\varepsilon} T^{-1} \\
&\ll \frac{x^{b+c+1+\varepsilon}}{T} + x^{b+c+\frac{1}{2}+\varepsilon} T^{2^{2l-1} - \frac{1}{6}B_l - 1 + \varepsilon}.
\end{aligned} \tag{15}$$

Combining Equations (13)-(15), we obtain

$$\sum_{n\leq x}\lambda_f^j(n)\sigma^b(n)\varphi^c(n) \ll \frac{x^{b+c+1+\varepsilon}}{T} + x^{b+c+\frac{1}{2}+\varepsilon}T^{2^{2l-1}-\frac{1}{6}B_l-1+\varepsilon}. \tag{16}$$

On taking $T = x^{\frac{3}{3\cdot 2^{2l}-B_l}}$ in Equation (16), we have

$$\sum_{n\leq x}\lambda_f^j(n)\sigma^b(n)\varphi^c(n) \ll x^{b+c+1-\frac{3}{3\cdot 2^{2l}-B_l}+\varepsilon}.$$

This completes the proof of Theorem 1.1.

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