ON THE SMALL PRIME FACTORS OF A NON-DEFICIENT NUMBER

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Abstract

Let $\sigma(n)$ be the sum of the positive divisors of n. A number n is non-deficient if $\sigma(n) \geq 2n$. We establish new lower bounds for the number of distinct prime factors of an odd non-deficient number in terms of its second smallest, third smallest and fourth smallest prime factors. Tighter bounds are obtained for odd perfect numbers. We also discuss the behavior of $\sigma(n!+1)$, $\sigma(2^n+1)$, and related sequences.

1. Introduction

We will write $\sigma(n)$ to be the sum of the positive divisors of n. Recall, that $\sigma(n)$ is a multiplicative function. Recall further that numbers where $\sigma(n) = 2n$ are said to be perfect, numbers where $\sigma(n) < 2n$ are said to be deficient, and numbers where $\sigma(n) > 2n$ are said to be abundant. A large amount of prior work has been done on estimating the density of abundant numbers. See for example [7] and [8].

We will throughout this paper write $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ where $p_1, p_2 \cdots p_k$ are primes satisfying $p_1 < p_2 \cdots < p_k$. Bounds giving a lower bound for n in terms of p_i for either small i or i close to k exist already in the literature, under the assumption that n is non-deficient or under the stronger assumption that n is an odd perfect number.

Acquaah and Konyagin showed [1] that if N is an odd perfect number, then

$$p_k < (3N)^{\frac{1}{3}}, \tag{1}$$

and

$$p_{k-1} < (2N)^{\frac{1}{4}}. \tag{2}$$

Building on their work, the author [15] showed that

$$p_{k-1} < (2N)^{\frac{1}{5}}, \tag{3}$$

and that

$$p_{k-1}p_k < 6^{\frac{1}{4}} N^{\frac{1}{2}}. \tag{4}$$

However, Inequalities 1 through 4 give a lower bound for N in terms of p_k and p_{k-1}. They do not give a lower bound for k, except in so far as we have upper bounds for N in terms of k, and the inequalities can then be chained together. In contrast, for p_i, when i is small, the natural route seems to go mainly in the other direction, using p_i to bound k from below, and then use those results to obtain a lower bound on N.

The main results of this paper are improved lower bounds on k in terms of p_2 and p_3 when n is odd and not deficient. We prove similar bounds with p_4 when n is odd and perfect, and a slightly weaker result when n is a primitive non-deficient number. Using the same techniques we will also produce estimates for $\sigma(m)$ when m is near but not equal to $n!$.

Kishore [6] proved that if n is an odd perfect number with k distinct prime factors, and $2 \le i \le 6$, then one must have

$$p_i < 2^{2^{i-1}} (k - i + 1).$$

Note that these bounds are all linear in p_i and k. In this paper, for i in the range $2 \le i \le 4$, we improve on Kishore's results and give better than linear bounds. Moreover, for $i = 2$ and $i = 3$, our results apply not just to odd perfect numbers but also to odd abundant numbers.

For p_1, better than linear bounds are known, due to [10] and the author [14]. Most of the results in this paper rely on the same fundamental techniques as in [10] and [14]. The first section of this paper extends those techniques to these small prime divisors. The second section uses the same techniques to look at a related problem involving the sum of the divisors of numbers near $n!$.

We will write $h(n) = \frac{\sigma(n)}{n}$. Note that $h(n)$ is frequently called the *abundancy* of n, but it has also been referred to in the literature as the *abundancy index* of n or the *index* of n. A number is perfect if and only if $h(n) = 2$, is abundant if and only $h(n) > 2$, and is deficient if and only $h(n) < 2$.

One has that

$$h(n) = \frac{1}{n}\sum_{d|n} d = \sum_{d|n} \frac{1}{d}$$

from which it follows that

$$h(mn) > h(n) \tag{5}$$

whenever $m > 1$. A consequence of Inequality 5 is that any multiple of an abundant number is abundant, and any multiple of a perfect number other than the number itself is abundant. Motivated by this consequence, there is a notion of a *primitive abundant number* which is a number n which is abundant but where all proper divisors of n are not abundant. There is some inconsistency in the literature

here; some authors define primitive abundant numbers as including perfect numbers where all proper divisors are deficient. In some respects, this is a more natural set to investigate. Others call this set to be the set of *primitive non-deficient numbers*, which is the term we will use. The first substantial work on this topic was done by Dickson who proved that given a fixed k, there are only a finite number of primitive non-deficient numbers with k distinct prime factors[3]. Prior work on these sets looked at their density. See especially [2] as well as [9] and [13].

We will write $H(n) = \Pi_{i=1}^{k} \frac{p_i}{p_i - 1}$. It is well-known and not hard to show that $h(n) \leq H(n)$ (and this inequality is strict as long as $n > 1$). In fact,

$$\lim_{m \to \infty} h(n^m) = H(n).$$

Thus, $H(n)$ is the best possible upper bound on $h(n)$ when we know only the distinct prime divisors of n, but have no other information about n.

We will write P_j as the jth prime number (note the capital letter distinguishing this from the p_i). Given a prime P_j and a real number $\alpha > 1$, we will write $b_\alpha(P_j)$ to be the smallest positive integer b such that

$$\alpha < \left(\frac{P_j}{P_j - 1}\right)\left(\frac{P_{j+1}}{P_{j+1} - 1}\right) \cdots \left(\frac{P_{j+b-1}}{P_{j+b-1} - 1}\right).$$

We will similarly define $a_\alpha(j)$ as $b_\alpha(P_j)$.

These two functions, $a_\alpha(j)$ and $b_\alpha(P_j)$, are generalizations of the functions labeled as $a(j)$ and $b(P_j)$ in [10] and [14]. In particular, $a(j) = a_2(j)$ and $b(j) = b_2(j)$. The use of $\alpha = 2$ arises naturally in the context when one is interested in bounding k in terms of the smallest prime divisor of n, but examining other prime divisors requires looking at other values of α. Note that if P_j is the smallest prime divisor of an odd perfect or odd abundant number n then $k \geq a(j)$.

Norton proved the following theorem.

Theorem 1. *Let t be a positive integer. Then:*

$$a(t) > t^2 - 2t - \frac{t+1}{\log t} - \frac{5}{4} - \frac{1}{2t} - \frac{1}{4t \log t}. \tag{6}$$

In fact, $a(t)$ is asymptotic to $\frac{1}{2}t^2 \log t$. When Norton was writing, bounds on the prime number theorem and related functions were not strong enough to easily give an explicit lower bound on $a(t)$ which had the correct lead term. Subsequently, the author proved an explicit version with the correct asymptotic [14].

Theorem 2. *For all $t > 1$ we have*

$$a(t) \geq \frac{t^2}{2}\left(\log t - \frac{3}{2}\log\log t + \frac{1}{20} + \frac{\log\log t}{\log t}\right) - t + 1. \tag{7}$$

Notice that this theorem has the correct first lead term.

Norton also proved that

$$a(n) = \frac{1}{2}n^2 \log n + \frac{1}{2}n^2 \log \log n - \frac{3}{4}n^2 + \frac{n^2 \log \log n}{2 \log n} + O\left(\frac{n^2}{\log n}\right).$$

Thus, Inequality 7 is not as strong as it could be. Although Inequality 7 has the correct lead term, the later terms do not match the asymptotic.

In the final section of this paper we use the same techniques to give a new proof of a theorem about the abundancy of numbers of the form $m! + z$ where z is a fixed non-zero integer. We will prove the following.

Theorem 3. *Let z be a non-zero integer. Then*

$$\lim_{m \to \infty} h(m! + z) = h(|z|).$$

Theorem 3 appears to be a known folklore result; we are uncertain if it has been previously published.

2. Small Prime Factors of Non-deficient Numbers

We require the following explicit version of Mertens' Theorem.

Lemma 4. *We have*

$$e^\gamma (\log x)\left(1 - \frac{1}{2 \log^2 x}\right) < \prod_{p \le x} \frac{p}{p - 1} < e^\gamma (\log x)\left(1 + \frac{1}{2 \log^2 x}\right). \tag{8}$$

Here γ is Euler's constant. The lower bound is valid for $x > 3$ and the upper bound is valid for $x \ge 286$.

Proof. This is proven in [11]. □

The following asymptotic is a straightforward consequence of this explicit version of Mertens' Theorem and the Prime Number Theorem.

Theorem 5. *Let $\alpha > 1$ be fixed. Then*

$$b_\alpha(p) \sim \frac{p^\alpha}{\alpha \log p},$$

and

$$a_\alpha(n) \sim \frac{n^\alpha \log n}{\alpha}.$$

Much of our work will be trying to make Theorem 5 explicit for certain specific values of α. We will for our purposes only need to construct explicit lower bounds, but it may be interesting to construct similarly explicit upper bounds.

We will write $S(x) = \prod_{p \le x} \frac{p}{p-1}$, and will write

$$Q(x,y) = \prod_{y < p \le x} \frac{p}{p-1} = \frac{S(x)}{S(y)}.$$

Whenever we use this notation we will assume that $x > y > 3$.

Lemma 6. *Let A, B and α be real numbers. Assume that $A > B > 6$, and that $3 \ge \alpha > 1$. Assume further that*

$$A + \frac{1}{2A} \ge \alpha \left(B - \frac{1}{2B} \right).$$

Then $A \ge \alpha B - \frac{2}{B}$.

Proof. We note that $A + \frac{1}{A}$ is an increasing function in A for all $A > 1$. So we just need to verify that $\left(\alpha B - \frac{2}{B} \right) + \frac{1}{2(\alpha B - \frac{2}{B})} \le \alpha(B - \frac{1}{2B})$. This is the same as verifying that

$$\alpha B - \frac{\alpha}{2B} - \left(\left(\alpha B - \frac{2}{B} \right) + \frac{1}{2(\alpha B - \frac{2}{B})} \right) \ge 0.$$

We can rewrite the expression on the right-hand side above as

$$\frac{(-\alpha + 4)\left(\alpha B + \frac{2}{B}\right) - B}{2B\left(\alpha B - \frac{2}{B}\right)}$$

which has a positive denominator (since $B > \sqrt{2}$), so we just need to look at

$$(-\alpha + 4)(\alpha B + \frac{2}{B} - B).$$

Since $\alpha \le 3$, we have

$$(-\alpha + 4)(\alpha B + \frac{2}{B} - B) \ge (\alpha B + \frac{2}{B}) - B > \frac{2}{B} > 0.$$

$\square$

We will need a few explicit estimates of certain functions of primes. We have from [4] that

$$\frac{x}{\log x}\left(1 + \frac{0.992}{\log x}\right) \le \pi(x) \le \frac{x}{\log x}\left(1 + \frac{1.2762}{\log x}\right), \tag{9}$$

with the lower bound valid if $x \ge 599$ and the upper bound valid for all $x > 1$.

We also need from [14] that for $n \geq 2$,

$$P_n \geq n \left(\log n + \log \log n - 1 + \frac{32}{31(\log n)^2} \right). \tag{10}$$

Theorem 7. *Let α be a real number such $1 < \alpha \leq 3$ and let p be a prime with $p \geq 853$. Then*

$$b_\alpha(p) \geq \frac{p^\alpha}{\alpha \log p} - \frac{1.669 p^\alpha}{\alpha \log^2 p} - \frac{0.662 p^\alpha}{\alpha \log^3 p} - \frac{p}{\log p} - \frac{1.2762 p}{\log^2 p} + 1.$$

Proof. Let $p \geq 601$ be a prime. Let q be the smallest prime such that $Q(p, q) \geq \alpha$.
We have from Lemma 4 that

$$\alpha \leq Q(q, p) \leq \frac{e^\gamma (\log q) \left(1 + \frac{1}{2\log^2 q} \right)}{e^\gamma (\log p) \left(1 - \frac{1}{2\log^2 p} \right)} = \frac{(\log q) \left(1 + \frac{1}{2\log^2 q} \right)}{(\log p) \left(1 - \frac{1}{2\log^2 p} \right)}. \tag{11}$$

If we set $A = \log q$, and $p = \log B$, we have

$$A + \frac{1}{2A} \geq \alpha B (1 - \frac{2}{B^2}),$$

so we may apply Lemma 6 to get that $\log q \geq \alpha \log p - \frac{2}{\log p}$. Then we have $q \geq p e^{\frac{2}{\log p}}$.

We have then
$$b_{\alpha(p)} \geq \pi(p^\alpha e^{\frac{-2}{\log p}}) - \pi(p) + 1.$$

We note that $e^{\frac{-2}{\log p}} \geq 1 - \frac{2}{\log p}$ and so

$$b_{\alpha(p)} \geq \pi \left(p^\alpha (1 - \frac{2}{\log p}) \right) - \pi(p) + 1.$$

Since $p \geq 853$, we have $p^\alpha \left(1 - \frac{2}{\log p} \right) \geq 599$, and may apply Lemma 9 to get that

$$b_{\alpha(p)} \geq \frac{p^\alpha (1 - \frac{2}{\log p})}{\log p^\alpha (1 - \frac{2}{\log p})} \left(1 + \frac{0.992}{\log p^\alpha (1 - \frac{2}{\log p})} \right) - \frac{p}{\log p} \left(1 + \frac{1.2762}{\log p} \right) + 1.$$

So
$$b_{\alpha(p)} \geq \frac{p^\alpha (1 - \frac{2}{\log p})}{\alpha \log p} \left(1 + \frac{0.992}{\alpha \log p} \right) - \frac{p}{\log p} \left(1 + \frac{1.2762}{\log p} \right) + 1.$$

Since $\alpha \leq 3$, we have

$$b_{\alpha(p)} \geq \frac{p^\alpha (1 - \frac{2}{\log p})}{\alpha \log p} \left(1 + \frac{0.331}{\log p} \right) - \frac{p}{\log p} \left(1 + \frac{1.2762}{\log p} \right) + 1.$$

Multiplying out the terms gives us the desired inequality. $\qquad\square$

We will need some specific versions of the above theorem for certain values of α.

Theorem 8. *For all primes $p \geq 3$, one has*

$$b_{\frac{4}{3}}(p) \geq \frac{p^{\frac{4}{3}}}{\frac{4}{3}\log p} - \frac{1.25175 p^{\frac{4}{3}}}{\log^2 p} - \frac{0.4965 p^{\frac{4}{3}}}{\log^3 p} - \frac{p}{\log p} - \frac{1.2762 p}{\log^2 p} + 1, \qquad (12)$$

$$b_{\frac{16}{15}}(p) \geq \frac{p^{\frac{16}{15}}}{\frac{16}{15}\log p} - \frac{1.5646875 p^{\frac{16}{15}}}{\log^2 p} - \frac{0.620625 p^{\frac{16}{15}}}{\log^3 p} - \frac{p}{\log p} - \frac{1.2762 p}{\log^2 p} + 1, \qquad (13)$$

$$b_{\frac{256}{255}}(p) \geq \frac{p^{\frac{256}{255}}}{\frac{256}{255}\log p} - \frac{1.662481 p^{\frac{256}{255}}}{\log^2 p} - \frac{0.65942 p^{\frac{256}{255}}}{\log^3 p} - \frac{p}{\log p} - \frac{1.2762 p}{\log^2 p} + 1, \qquad (14)$$

and

$$b_{\frac{26325}{26257}}(p) \geq \frac{p^{\frac{26325}{26257}}}{\frac{26325}{26257}\log p} - \frac{1.6647 p^{\frac{26325}{26257}}}{\log^2 p} - \frac{0.6603 p^{\frac{26325}{26257}}}{\log^3 p} - \frac{p}{\log p} - \frac{1.2762 p}{\log^2 p} + 1. \qquad (15)$$

Proof. We will only discuss how to prove Equation 12. The other equations have very similar proofs. Theorem 7 implies that Equation 12 is true for all primes $p \geq 853$. We then use a computer program to check that it is true for all primes p with $3 \leq p \leq 853$. The others are done similarly. $\qquad \square$

Proposition 9. *Suppose that N is an odd number which is either perfect or abundant. Let p_2 be second smallest prime divisor of N. Let k be the number of distinct prime divisors of N. Then $k \geq b_{\frac{4}{3}}(p_2) + 1$.*

Proof. We have $h(N) \geq 2$. We assume that N has the factorization $p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ with $p_1 < p_2 < p_3 \cdots < p_k$. We have then

$$2 \leq h(N) < H(n) = \frac{p_1}{p_1 - 1}\frac{p_2}{p_2 - 1} \cdots \frac{p_k}{p_k - 1}.$$

We have that in the most extreme case, $p_1 = 3$, in which case $\frac{p_1}{p_1-1} = \frac{3}{2}$. We thus have

$$\frac{3}{2}\frac{p_2}{p_2 - 1} \cdots \frac{p_k}{p_k - 1} > 2,$$

and hence

$$\frac{p_2}{p_2 - 1} \cdots \frac{p_k}{p_k - 1} > \frac{4}{3}.$$

We thus have $k - 1 \geq b_{\frac{4}{3}}(p_2)$. $\qquad \square$

Applying Proposition 9 with Inequality 12 we obtain the next result.

Corollary 10. *Suppose that N is an odd number which is either perfect or abundant. Let p_2 be the second smallest prime divisor of N. Let k be the number of distinct prime divisors of N. Then*

$$k \geq \frac{p_2^{\frac{4}{3}}}{\frac{4}{3}\log p_2} - \frac{1.25175 p_2^{\frac{4}{3}}}{\log^2 p_2} - \frac{0.4965 p_2^{\frac{4}{3}}}{\log^3 p_2} - \frac{p_2}{\log p_2} - \frac{1.2762 p_2}{\log^2 p_2} + 2.$$

The bound from Corollary 10 exceeds Kishore's bound when $p_2 \geq 139$. The bound also applies to all odd non-deficient numbers, whereas Kishore's bound applies only to odd perfect numbers. With a little computation we can use Corollary 10 to obtain the following bound which is stronger than Kishore's bound for any value of p_2.

Corollary 11. *If N is an odd abundant or odd perfect number with k distinct prime factors and second smallest prime factor p_2, then $p_2 \leq 3k - 1$.*

Proposition 12. *Suppose that N is an odd number which is either perfect or abundant. Let p_3 be the third smallest prime divisor of N. Let k be the number of distinct prime divisors of N. Then $k \geq b_{\frac{16}{15}}(p_3) + 3$.*

Proof. This proof is essentially the same as the proof of Proposition 9. However, now the worst case scenario is that the smallest prime is 3 and the second smallest prime is 5. □

Corollary 13. *Suppose that N is an odd number which is either perfect or abundant. Let p_2 be the smallest prime divisor of N. Let k be the number of distinct prime divisors of N. Then*

$$b_{\frac{16}{15}}(p_3) \geq \frac{p_3^{\frac{16}{15}}}{\frac{16}{15}\log p_3} - \frac{1.5646875 p_3^{\frac{16}{15}}}{\log^2 p_3} - \frac{0.620625 p_3^{\frac{16}{15}}}{\log^3 p_3} - \frac{p_3}{\log p_3} - \frac{1.2762 p_3}{\log^2 p_3} + 3.$$

Here the inequality is better than Kishore's inequality when $p \geq 282767$. Due to the large value where the inequality in Corollary 13 becomes better than Kishore's inequality, it is not feasible to construct a nice linear inequality like Corollary 11 for p_3.

We now turn to smallest bounds on the fourth prime factor. Here, we must restrict ourselves finally to odd perfect numbers. This is because there actually are odd abundant numbers with three distinct prime factors; $N = 945$ is an example.

We recall one of the oldest results about odd perfect numbers which is due to Euler.

Lemma 14. *(Euler). If N is an odd perfect number, then we have $N = p^e m^2$ for some prime p where $(p, m) = 1$ and $p \equiv e \equiv 1 \pmod 4$.*

In fact, Euler's result applies more generally to any odd number n satisfying $\sigma(n) \equiv 2 \pmod 4$.

We also note the following well known lemma where we include a proof for completeness.

Lemma 15. *Let N be an odd perfect number. Then $105 \nmid N$.*

Proof. Let N be an odd perfect number, and assume that $105|N$. Since $3 \equiv 3$ (mod 4), and $7 \equiv 3$ (mod 4), by Euler's result, we must have $3^2|N$ and $7^2|N$. But $(3^2)(5)(7^2)$ is an abundant number. $\square$

Using the same approach we can further restrict the prime divisors of an odd perfect number.

Lemma 16. *Let N be an odd perfect number. Assume that $3|N$, $5|N$ and $11|N$. Then one must have $5||N$.*

Proof. Suppose that N is an odd perfect number where $(3)(5)(11)|N$. Note that since $3 \equiv 11 \equiv 3$ (mod 4), we must from Euler's result, have $3^2|N$ and $11^2|N$. First, we will show that $3^4|N$. Assume that $3^4 \nmid N$. Since $3 \equiv 3$ (mod 4), we must have $3^2|N$. Thus, $\sigma(3^2) = 13|N$. But $(3^2)5(11^2)(13)$ is abundant. We thus may assume that $3^4|N$.

Note that $\sigma(11^2) = (7)(19)$, so if $11^2||N$, we would have $105|N$ which by the prior Lemma is impossible. Thus, we conclude that $11^4|N$. However, $(3^4)(5^2)(11^4)$ is an abundant number. So we must have that $5|N$. $\square$

Lemma 17. *Let N be an odd perfect number. Suppose that $3|N$, $5|N$, and $13|N$. Then we must have either $5||N$ or $3^2||N$.*

Proof. Assume that N is an odd perfect number and that $3|N$, $5|n$ and $13|N$. We note that if 13 were the special prime, then since $7|(13+1)$, we would have $7|N$, which is impossible since no odd perfect number is divisible by 105. Assume that $5^2|N$ and that $3^3|N$; the second implies that we must have $3^4|N$. But $(5^2)(3^4)(13^2)|N$ and then $(5^2)(3^4)(13^2)$ is abundant, which is a contradiction. $\square$

Lemma 18. *Let N be an odd perfect number, with $N = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_k^{a_k}$ with $p_1 < p_2 < p_3 \cdots < p_k$. Then $h(p_1^{a_1} p_2^{a_2} p_3^{a_3}) < \frac{255}{128}$.*

Proof. Note that if $p_1 \geq 5$, then $H(p_1^{a_1} p_2^{a_2} p_3^{a_3}) \leq \frac{5}{4} \frac{7}{6} \frac{11}{10} < \frac{255}{128}$. Since $h(m) < H(m)$ for all $m > 1$, this means that we would have $h(p_1^{a_1} p_2^{a_2} p_3^{a_3}) < \frac{255}{128}$. So we may assume that $p_1 = 3$.

Similarly, since $\frac{3}{2} \frac{7}{6} \frac{11}{10} < \frac{255}{128}$ we may assume that $p_2 = 5$. Since $\frac{3}{2} \frac{5}{4} \frac{17}{16} = \frac{255}{128}$, we may assume that $p_3 = 11$ or that $p_3 = 13$.

Let us first consider the situation where $p_3 = 11$. Then from Lemma 16, we must have $a_2 = 1$. Then

$$h(p_1^{a_1} p_2^{a_2} p_3^{a_3}) = h(3^{a_1})h(5)h(11^{a_3}) < H(3)h(5)H(11) = \frac{3}{2}\frac{6}{5}\frac{11}{10} < \frac{255}{128}.$$

The situation where $p_3 = 13$ is nearly identical, but we use Lemma 17 instead of Lemma 16. $\qquad\square$

Proposition 19. *Suppose that N is an odd perfect number, and let p_4 be the fourth smallest prime factor of N. Assume that N has k distinct prime factors. Then*

$$k \geq b_{\frac{256}{255}}(p_4) + 3.$$

Proof. Let N be an odd perfect number. Assume that N has the factorization $p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ with $p_1 < p_2 < p_3 \cdots < p_k$. We must have that

$$h(N) = 2 = h(p_1^{a_1} p_2^{a_2} p_3^{a_3})h(p_4^{a_4} \cdots p_k^{a_k}) < \frac{255}{128}H(p_4^{a_4} \cdots p_k^{a_k}).$$

We have then that $H(p_4^{a_4} \cdots p_k^{a_k}) \geq \frac{256}{255}$ The result then follows from the same sort of argument as before. $\qquad\square$

We then obtain a corresponding corollary as before.

Corollary 20. *Suppose that N is an odd perfect number, and let p_4 be the fourth smallest prime factor of N. Assume that N has k distinct prime factors. Then*

$$k \geq \frac{p^{\frac{256}{255}}}{\frac{256}{255}\log p} - \frac{1.662481p^{\frac{256}{255}}}{\log^2 p} - \frac{0.65942p^{\frac{256}{255}}}{\log^3 p} - \frac{p}{\log p} - \frac{1.2762p}{\log^2 p} + 4.$$

We can make a slightly weaker version of Proposition 19 which does apply to some non-deficient numbers. Recall a number M is a primitive non-deficient number if M is not deficient and all divisors of M are deficient.

Lemma 21. *Let M be a primitive non-deficient number with at least four distinct prime factors. Suppose that $3|M$, $5|M$, and $7|M$. Then one has either $3||M$ or $3^2||M$. Moreover, if $3^2||M$ then $5||M$ and $7||M$.*

Proof. First, note that $(3^3)(5)(7)$ is abundant. So one must have $3||M$ or $3^2||M$. Now assume that $3^2||M$. Since $(3^2)(5^2)7$ and $(3^2)5(7^2)$ are both abundant, we must have $5||M$ and $7||M$. $\qquad\square$

Lemma 22. *Let M be a primitive non-deficient number with at least four distinct prime factors. Suppose that $3|M$, $5^2|M$, and $11|M$. Then either $3||N$ or $3^2||N$.*

Proof. This follows from noting that $(3^3)(5^2)(11)$ is abundant. $\square$

Lemma 23. *Let M be a primitive non-deficient number with at least four distinct prime factors. Suppose that $3|M$, $5^2|M$, and $13|M$. Then $3^5 \nmid M$. If $3^4|M$, then $5^2||M$ and $13^2 \nmid M$. If $3^3|M$, then either $5^3 \nmid M$ or $13^2 \nmid M$.*

Proof. This follows from noting that $(3^5)(5^2)(13)$, $(3^4)(5^3)(13)$, $(3^4)(5^2)(13^2)$, and $(3^3)(5^3)(13^2)$ are all abundant numbers. $\square$

Theorem 24. *Let M be a primitive non-deficient number with at least four distinct prime factors. Assume further that $M = p_1^{a_1} p_2^{a_2} p_3^{a_3} p_4^{a_4} \cdots p_k^{a_k}$ where p_1, p_2, $p_3 \cdots p_k$ are primes and $p_1 < p_2 < p_3 < p_4 < \cdots p_k$. Then we have $h(p_1^{a_1} p_2^{a_2} p_3^{a_3}) \le \frac{52514}{26325} = 1.9948\ldots$.*

Proof. The method is essentially the same as our earlier approach. Set $M = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_a}$. If $p_1 > 3$, then

$$h(p_1^{a_1} p_2^{a_2} p_3^{a_3}) < H((5)(7)(11)) = \frac{5}{4}\frac{7}{6}\frac{11}{10} = \frac{77}{48} < \frac{52514}{26325}.$$

We thus may assume that $p_1 = 3$. By similar logic, we may assume that $p_2 = 5$, and that p_3 is one 7, 11 or 13.

Consider the case where $p_3 = 7$. Then by Lemma 21 we have

$$h(p_1^{a_1} p_2^{a_2} p_3^{a_3}) \le \max(h((3^2)(5)(7), h(3)H(5,7)) = \max(\frac{208}{105}, \frac{35}{18}) < \frac{52514}{26325}.$$

Consider now the case where $p_3 = 11$. Consider the possible values of a_2. If $a_2 = 1$ (that is $5||N$), then we have

$$h(p_1^{a_1} p_2^{a_2} p_3^{a_3}) \le H(3)h(5)H(11) = \frac{99}{50} < \frac{52514}{26325}.$$

Now assume $a_2 \ge 2$. This is equivalent to $5^2|N$. Then by Lemma 22, we have $a_1 \le 2$. Thus

$$h(p_1^{a_1} p_2^{a_2} p_3^{a_3}) \le h(13)H(5)H(11) = \frac{143}{72} < \frac{52514}{26325}.$$

Thus, in all situations where $p_3 = 11$ we have the desired inequality.

Finally, consider the case where $p_3 = 13$. If $a_2 = 1$, then we have

$$h(p_1^{a_1} p_2^{a_2} p_3^{a_3}) \le H(3)h(5)H(13) = \frac{3}{2}\frac{6}{5}\frac{14}{13} = \frac{39}{20} < \frac{52514}{26325}.$$

Assume $a_2 \ge 2$. Then by Lemma 23, $a_1 \le 4$. We will consider four cases depending on the value of a_1.

`Case I`: Assume that $a_1 = 1$. Then

$$h(p_1^{a_1} p_2^{a_2} p_3^{a_3}) \leq h(3)H(5)H(13) = \frac{4}{3}\frac{5}{4}\frac{13}{12} = \frac{65}{36} < \frac{52514}{26325}.$$

`Case II`: Assume that $a_1 = 2$. Then

$$h(p_1^{a_1} p_2^{a_2} p_3^{a_3}) \leq h(9)H(5)H(13) = \frac{13}{9}\frac{5}{4}\frac{13}{12} = \frac{845}{432} < \frac{52514}{26325}.$$

`Case III`: Assume that $a_1 = 3$. If $a_3 = 1$, then

$$h(p_1^{a_1} p_2^{a_2} p_3^{a_3}) \leq h(27)H(5)h(13) = \frac{40}{27}\frac{5}{4}\frac{14}{13} = \frac{700}{351} < \frac{52514}{26325}.$$

Therefore, we may assume that $a_3 \geq 2$. Thus, by Lemma 23, we have that $a_2 \leq 2$. Thus,

$$h(p_1^{a_1} p_2^{a_2} p_3^{a_3}) \leq h(27)h(25)H(13) = \frac{40}{27}\frac{31}{25}\frac{13}{12} = \frac{806}{405} < \frac{52514}{26325}.$$

`Case IV`: $a_1 = 4$. Thus, by Lemma 23, we must have $a_2 \leq 2$, and $a_3 = 1$. Then

$$h(p_1^{a_1} p_2^{a_2} p_3^{a_3}) \leq h(81)h(25)H(13) = \frac{121}{81}\frac{31}{25}\frac{14}{13} = \frac{52514}{26325}.$$

Note that it is only in Case IV above that we hit the worst case scenario. $\square$

Using Theorem 24 along with Equation 15, we may use the exact same method of proof to prove the next result.

Proposition 25. *Let M be a primitive non-deficient number. (That is, M is not deficient and it has no non-deficient divisors.) And assume that M has at least four distinct prime divisors. Let k be the number of distinct prime divisors of M, and let p_4 be the fourth smallest prime factor of M. Then we have*

$$k \geq b_{\frac{26325}{26257}}(p_4) + 4.$$

As before we have a corresponding corollary.

Corollary 26. *Let M be a primitive non-deficient number. (That is, M is not deficient and it has no non-deficient divisors.) And assume that M has at least four distinct prime divisors. Let k be the number of distinct prime divisors of M, and let p_4 be the fourth smallest prime factor of M. Then we have*

$$b_{\frac{26325}{26257}}(p_4) \geq \frac{p_4^{\frac{26325}{26257}}}{\frac{26325}{26257}\log p_4} - \frac{1.6647 p_4^{\frac{26325}{26257}}}{\log^2 p_4} - \frac{0.6603 p_4^{\frac{26325}{26257}}}{\log^3 p_4} - \frac{p_4}{\log p_4} - \frac{1.2762 p_4}{\log^2 p_4} + 5.$$

Note that the bounds in this section are weak in the following sense. Let $\omega(n)$ be the number of distinct prime divisors of n. Given a function $f(n)$ with $\lim_{n \to \infty} f(n)$, then for any positive integer i, the statement "the ith smallest prime divisor of n is at most $f(n)$" will apply to almost all positive integers n (in the sense that the exceptional set has density zero). Similarly, any such restriction for odd perfect numbers is a weak restriction in the sense discussed in section seven of [14].

It is not too hard to see that all the results in this section which apply to non-deficient numbers in general or apply to primitive non-deficient numbers are asymptotically best possible. Of course, for the results which only concern odd perfect numbers, it is likely that the results are far from best possible, since very likely no odd perfect numbers exist. This is connected to a question raised in [14]. In particular, the author defined $b_o(p)$ to be the minimum of the number of distinct prime divisors of any odd perfect number with smallest prime divisor p and set $b_o(p) = \infty$ when there are no odd perfect numbers with smallest prime divisor p. The author asked if it was possible to prove that $b_o(p) > b(p)$, or in the notation of our paper, if $b_o(p) > b_2(p)$. The notation here is less than ideal; note that the small circle in $b_o(p)$ is a lower case O, not a zero. Since we now have a numeric subscript of $b(p)$ in this paper, a slightly better notation for our purposes is as follows: We will write $b_{(O,i)}(p)$ to be the to be the minimum of the number of distinct prime divisors of any odd perfect number with ith smallest prime divisor p and set $b_{(O,i)}(p) = \infty$ when no such odd perfect number exists. The above mentioned question then becomes whether it is possible to prove that $b_{(O,1)}(p) > b_2(p)$. Following this, we can ask similar questions for other values of i. In particular, can we prove any of the following:

1. $b_{(O,2)}(p) > b_{\frac{4}{3}}(p)$,

2. $b_{(O,3)}(p) > b_{\frac{16}{15}}(p)$,

3. $b_{(O,4)}(p) > b_{\frac{256}{255}}(p)$.

Two other directions which may make sense for investigation are to extend these results for larger prime factors and to extend them to multiperfect numbers. Recall, a number n is said to be c-multiperfect if $\sigma(n) = cN$ for some integer c. One expects analogous results for multiperfect numbers in terms of c.

We can also extend various other results about $a(n) = a_2(n)$ to $a_\alpha(n)$ for $\alpha > 1$. In [14], it is proven that $a_2(n)$ is a strictly increasing function. In general, this is false for $a_{\alpha(n)}$. However, we strongly suspect that for any $\alpha > 1$, $a_\alpha(n)$ is strictly increasing for sufficiently large n. In fact, we suspect that the following conjecture is true.

Conjecture 27. For any $\alpha > 1$, and $C > 0$, for sufficiently large n one has $a_\alpha(n + 1) > a_\alpha(n) + C$.

In [14], the second difference of $a(n)$ was studied. The author defined $f(n) = a(n+1) + a(n-1) - 2a(n)$. Extending this notation, let us write $f_\alpha(n) = a_\alpha(n+1) + a_\alpha(n-1) - 2a_\alpha(n)$. Note that $f_2(n) = f(n)$.

In that paper, it was noted that $f_2(n)$ is generally positive and that negative values occurred generally at a prime gap. For example, $f(31) = -5$. This corresponds to the fact that 30th prime is 113, right before a record setting gap; the next prime is 127. Motivated by this observation, we ask the following questions for all $\alpha > 1$.

1. Are there infinitely many values of n where $f_\alpha(n)$ is negative?

2. Is the set of n where $f_\alpha(n) < 0$ a subset of those n where P_n is right after a record setting gap?

3. Are there infinitely many n where P_n occurs at a record setting gap and $f_\alpha(n)$ is positive?

4. Does $f_\alpha(n)$ take on any integer value? In particular, is $f(n)$ ever zero?

5. Does $a_\alpha(n+1) - a_\alpha(n)$ take on every positive integer value?

We suspect that there are values of α where questions 4 and 5 have an answer of no. We are substantially more uncertain about the other questions.

3. Sums of Divisors of Numbers Near $n!$

It is well known that for any $n > 1$, any number of the form $n! \pm 1$ is deficient. We reproduce a proof here.

Lemma 28. *For any prime $p > 2$, we have $b_2(p) \geq p$.*

Proof. Assume that $b_2(p) = m$. Then since $\frac{x}{x-1}$ is a decreasing function for positive x and the ith prime after p is at least $p + i$, we must have

$$2 < \left(\frac{p}{p-1}\right)\left(\frac{p+1}{p}\right)\left(\frac{p+2}{p+1}\right)\cdots\left(\frac{p+b(p)-1}{p+b(p)-2}\right) = \frac{p+b(p)-1}{p-1}.$$

Thus

$$2p - 2 < p + b(p) - 1,$$

and so $b_2(p) > p - 1$, and so $b_2(p) \geq p$. $\square$

Variants of Lemma 28 are very old and to date back to Servais [12] who used a variant to prove that an odd perfect number with smallest prime divisor p must have at least p distinct prime factors. It immediately follows from this Lemma that

any non-deficient number with smallest prime factor p must have at least p distinct prime divisors.

To prove that any number of the form $n!\pm 1$ must be deficient, note that any such number must have smallest prime factor at least $n+1$. If $n!\pm 1$ were not-deficient it would need to have at least $n+1$ distinct prime factors, but then we would have $(n+1)^{n+1} \le n!\pm 1$, which is absurd.

In this section, we will use the same methods as earlier in this paper to prove the related result that in fact

$$\lim_{n\to\infty} h(n!\pm 1) = 1.$$

In fact, we will prove a stronger claim.

Theorem 29. *Let a, b and c be positive integers. Then*

$$\lim_{n\to\infty} h(\frac{a}{b}n!\pm c) = h(c).$$

Note that simply using the method of Servais's Lemma is insufficient to prove Theorem 29. Using only that method, one would just get that a number n with $h(n) \ge \alpha$ with smallest prime factor p has at least about $(\alpha - 1)p$ distinct prime factors. So we really do need to use Mertens' theorem.

To prove Theorem 29, we will prove a slightly stronger claim.

Theorem 30.
$$\lim_{n\to\infty} H\left(\frac{a}{b}n!\pm 1\right) = 1.$$

Proof. We will prove the result with $\frac{a}{b}n!-1$. The proof is nearly identical for $\frac{a}{b}n!+1$.

Fix positive integers a and b. Fix an $\alpha > 1$, and assume that there is an increasing sequence $n_1, n_2 \cdots$ such that for all k, $\frac{a}{b}n!$ is an integer and $H(\frac{a}{b}n_k!\pm 1) \ge \alpha$. For sufficiently n_k, the smallest prime divisor of $\frac{a}{b}n_k! - 1$ is at least $n+1$. According to Theorem 5 for any $\epsilon > 0$, for sufficiently large n_k we have that $\frac{a}{b}n_k!\pm 1$ has at least $(1-\epsilon)\frac{(n_k+1)^{\alpha}}{\alpha\log(n_k+1)}$ distinct prime factors. Choose $\epsilon = 1/2$. Then for sufficiently large n_k, we must have

$$(n_k+1)^{\frac{1}{2}\frac{(n_k+1)^{\alpha}}{\alpha\log(n_k+1)}} < \frac{a}{b}n_k! - 1 < an_k{}^{n_k}.$$

This implies that

$$(n_k)^{\frac{1}{2}\frac{(n_k)^{\alpha}}{\alpha\log(n_k+1)}} < \frac{a}{b}n_k! - 1 < an_k{}^{n_k},$$

and hence

$$\frac{1}{2}\frac{(n_k)^{\alpha}}{\alpha\log(n_k+1)} < n_k.$$

However, this inequality cannot be true for any sufficiently large n_k, since $\alpha > 1$. Thus, there are only finitely many possible values for n_k and the result is proved. $\square$

With a little work one can use essentially the exact same method as above to prove the following slightly more general result.

Theorem 31. *Let $g(n)$ be an increasing function satisfying the following conditions:*

1. *If n is sufficiently large, then for any prime p with $p \leq n$, $(p, g(n)) = 1$.*

2. *There is a constant C such that $g(n) < n^{Cn}$ for all sufficiently large n.*

Then

$$\lim_{n \to \infty} H(g(n)) = 1.$$

Four functions which seem natural to investigate in the context of Theorem 30, are $A(n) = h(n! + 1)$, $B(n) = h(n! - 1)$, $C(n) = H(n! + 1)$, and $D(n) = H(n! + 1)$. We will focus on $A(n)$, although the same remarks apply essentially to the other three.

One might hope that $A(n)$ would be a decreasing function given our earlier limit and the fact that $A(1) = \frac{3}{2}$, and that $A(n) > 1$ for all $n > 1$. However, this is not the case. $A(3) = h(7) = \frac{8}{7} = 1.1428\cdots$, and $A(4) = h(25) = \frac{31}{25} = 1.24$. It is easy to find similar values for $B(n)$, $C(n)$, and $D(n)$.

We suspect that this non-decreasing behavior is not simply an artifact of small numbers. In particular we have the following conjecture.

Conjecture 32. There are arbitrarily large n such that $A(n) < A(n + 1)$.

We have similar conjectures for the other three functions.

Conjecture 33. There are arbitrarily large n such that $B(n) < B(n + 1)$.

Conjecture 34. There are arbitrarily large n such that $C(n) < C(n + 1)$.

Conjecture 35. There are arbitrarily large n such that $D(n) < D(n + 1)$.

Note that standard conjectures about primes imply these conjectures. For example, whenever $n! + 1$ is prime and $(n + 1)! + 1$ is composite, one will have that $A(n) < A(n + 1)$ and $C(n) < C(n + 1)$. Since it is believed that there are infinitely many prime values of the form $n! + 1$, and since there are infinitely many values of $n! + 1$ which are composite (by Wilson's Theorem), we should strongly believe Conjecture 32. One hopes that these conjectures may be substantially easier to prove than statements like there being infinitely many primes of the form $n! + 1$.

One obvious question is whether there are infinitely many n where $n! + 1$ is composite but $A(n) < A(n + 1)$. This question seems difficult but we tentatively conjecture that its answer is also yes. To see why this seems likely, note that if $n! + 1$ is a semiprime with both prime factors near the square root, and $n! + 1$ has a small prime divisor substantially smaller than its square root, we will have $A(n) < A(n + 1)$.

Let S_A be the set of positive integers such that $A(n) < A(n+1)$. Define S_B, S_C and S_D similarly for $B(n)$, $C(n)$, and $D(n)$. What are the densities of these sets? The obvious conjectural natural density for S_A is $1/2$ given the apparent slow rate at which $A(n)$ tends towards 1. We do not see any way of even proving that this density exists. Curiously, we do not at this time see a way of even ruling out that S_A has density 1. If that were the case, it would mean that $A(n) < A(n+1)$ on almost all n, but that $A(n)$ takes large drops at isolated n which drive it downwards overall. This seems like it should be obviously false but we do not see any easy proof. The following conjecture seems likely.

Conjecture 36. We have $\lim_{n \to \infty} H\left((n!)^n + 1\right) = 1$.

Unfortunately, Mertens' Theorem by itself does not seem to be strong enough to prove Conjecture 36. It would likely require some result of the sort that the number of distinct prime factors of $(n!)^n + 1$ must be much smaller than $\log_n(n!)$.

We use this paper as an opportunity to ask similar questions about $h(f(n))$ for other somewhat well-behaved fast growing functions $f(n)$. Let us consider the functions $J(n) = h(2^n - 1)$ and $K(n) = h(2^n + 1)$, $L(n) = H(2^n - 1)$, and $M(n) = H(2^n + 1)$. Unlike with our earlier quadruplet, the behavior in the -1 case here is substantially different than the $+1$ case. It is a straightforward consequence from Fermat's Little Theorem, and the fact that the sum of the reciprocals of the primes diverges, that $J(n)$ can be arbitrarily large.

In contrast, it is less obvious that $K(n)$ can be arbitrarily large.

Theorem 37. *The function $K(n)$ can be arbitrarily large.*

Proof. Consider a prime $p \equiv 3 \pmod 8$. Note that 2 is a quadratic non-residue modulo p. Thus, by Gauss's Lemma, $2^{\frac{p-1}{2}} \equiv -1 \pmod p$. So, whenever n is odd, and $\frac{p-1}{2} | n$, we have $p | 2^n + 1$. By the strong form of Dirichlet's theorem, the sum of the reciprocals of the primes which are 3 mod 8 diverges. Thus, to make $h(2^n + 1)$ arbitrarily large, we just need to set n to be the odd part of $k!$ for large k. $\qquad \square$

One natural problem in this context is to look for record setting values of $K(n)$. This becomes an analog of the superabundant numbers restricted to just looking at numbers which are one more than a power of 2. More generally, given an increasing function $f(n)$, we will define a number to be an f-*superabundant* if $h(f(n)) > h(f(i))$ for any $i < n$. Note that the usual superabundant numbers are just the f-superabundant numbers where $f(x)$ is the identity function.

In our context then, we are interested in the K-superabundant numbers. We will write the nth K-superabundant number as a_n. The first few are 1, 3, 5, 9, 15, 45, 135, 315, 945 $\cdots$. Note that the first few a_i are odd. It seems likely that all the a_i are odd because if n is even, then $2^n + 1$ is not divisible by 3, and that makes making $2^n + 1$ large substantially more difficult. Proving that they are all

odd seems difficult. A related open question is whether for any odd prime p we have $p|a_n$ for all arbitrarily large n. The answer seems very likely to be yes.

Note that the J-superabundant numbers behave very differently. They appear to be always even except the first term, and seem to grow much slower. The first few are 1, 2, 4, 6, 8, 12, 24, 36, 60, 120, and 180. (Here we are using the convention that $\sigma(0) = 0$.) If we set t_n as the nth J-superabundant number, we can similarly ask if it is true that $p|t_n$ for all arbitrarily large n. Again, the answer seems likely to be yes.

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