ABSOLUTELY SUMMING MORPHISMS BETWEEN HILBERT C*-MODULES AND MODULAR PIETSCH FACTORIZATION PROBLEM

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Abstract: Motivated from the theory of Hilbert-Schmidt morphisms between Hilbert C*-modules over commutative C*-algebras by Stern and van Suijlekom *[J. Funct. Anal., 2021]*, we introduce the notion of p-absolutely summing morphisms between Hilbert C*-modules over commutative C*-algebras. We show that an adjointable morphism between Hilbert C*-modules over monotone closed commutative C*-algebra is 2-absolutely summing if and only if it is Hilbert-Schmidt. We formulate version of Pietsch factorization problem for p-absolutely summing morphisms and solve partially.
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1. Introduction

In his *Resume*, A. Grothendieck studied 1-absolutely and 2-absolutely summing operators between Banach spaces [22] (also see [8]). In 1967, for each $1 \leq p < \infty$, A. Pietsch introduced the notion of p-absolutely summing operators which became an area around the end of 20 century [7, 9, 18, 24, 26, 28–30, 38–41, 48, 50–52, 64, 65]. In 1979, Tomczak-Jaegermann studied p-summing operators by fixing by fixing number of points [63]. In 1970, Kwapien defined the notion of 0-summing operators [36]. In 2003, Farmer and Johnson introduced the notion of Lipschitz p-summing operators between metric spaces [11] (also see [4–6, 49]).

Definition 1.1. *[1, 50] Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, $\mathcal{X}^*$ be the dual of $\mathcal{X}$ and $1 \leq p < \infty$. A bounded linear operator $T : \mathcal{X} \to \mathcal{Y}$ is said to be **p-absolutely summing** if there is a real constant $C > 0$ satisfying following: for every $n \in \mathbb{N}$ and for all $x_1, \dots, x_n \in \mathcal{X}$,*

$$\left(\sum_{j=1}^{n}\|Tx_j\|^p\right)^{\frac{1}{p}} \leq C \sup_{f\in\mathcal{X}^*, \|f\|\leq 1}\left(\sum_{j=1}^{n}|f(x_j)|^p\right)^{\frac{1}{p}}. \tag{1}$$

*In this case, the **p-absolutely summing norm** of T is defined as*

$$\pi_p(T) \coloneqq \inf\{C : C \text{ satisfies Inequality } (1)\}.$$

The set of all p-absolutely summing operators from $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $\Pi_p(\mathcal{X}, \mathcal{Y})$.

Following are most important results in the theory of p-absolutely summing operators.

Theorem 1.2. *[50, 51] Let $1 \leq p < \infty$ and $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces. Then $(\Pi_p(\mathcal{X}, \mathcal{Y}), \pi_p(\cdot))$ is an operator ideal.*

Theorem 1.3. *[1, 50] Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Then a bounded linear operator $T : \mathcal{H} \to \mathcal{K}$ is 2-absolutely summing if and only if it is Hilbert-Schmidt. Moreover, $\|T\|_{HS} = \pi_2(T)$.*

Theorem 1.4. *[1, 50] (**Pietsch Factorization Theorem**) Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. A bounded linear operator $T : \mathcal{X} \to \mathcal{Y}$ is p-absolutely summing if and only if there is a real constant $C > 0$ and a regular Borel probability measure on $B_{\mathcal{X}^*} \coloneqq \{f : f \in \mathcal{X}^*, \|f\| \leq 1\}$ in weak*-topology such that*

$$\|Tx\| \leq C \left(\int\limits_{B_{\mathcal{X}^*}} |f(x)|^p \, d\mu_{B_{\mathcal{X}^*}}(f)\right)^{\frac{1}{p}}, \quad \forall x \in \mathcal{X}. \tag{2}$$

Moreover, $\pi_p(T) = \inf\{C : C$ satisfies Inequality (2)$\}$.

In this paper, we define the notion of p-absolutely summing morphisms between Hilbert C*-modules over commutative C*-algebras (Definition 2.1). We derive in Theorem 2.5 that an adjointable morphism between Hilbert C*-module over a monote closed C*-algebra is 2-summing if and only if modular Hilbert-Schmidt. We then formulate version of Pietsch factorization problem for p-absolutely summing morphisms and solve partially.

2. p-absolutely summing morphisms

We define modular version of Definition 1.1 as follows. For the theory of Hilbert C*-modules we refer [32, 47, 56].

Definition 2.1. *Let $1 \leq p < \infty$. Let $\mathcal{M}$ and $\mathcal{N}$ be Hilbert C*-modules over a commutative C*-algebra $\mathcal{A}$. An adjointable morphism $T : \mathcal{M} \to \mathcal{N}$ is said to be **modular p-absolutely summing** if there is a real constant $C > 0$ satisfying following: for every $n \in \mathbb{N}$ and for all $x_1, \dots, x_n \in \mathcal{M}$,*

$$\left\|\sum_{j=1}^{n} \langle Tx_j, Tx_j\rangle^{\frac{p}{2}}\right\|^{\frac{1}{p}} \leq C \sup_{x \in \mathcal{M}, \|x\| \leq 1} \left\|\sum_{j=1}^{n} (\langle x, x_j\rangle \langle x_j, x\rangle)^{\frac{p}{2}}\right\|^{\frac{1}{p}}. \tag{3}$$

In this case, the p-absolutely summing norm of T is defined as

$$\pi_p(T) \coloneqq \inf\{C : C \text{ satisfies Inequality } (3)\}.$$

The set of all p-absolutely summing morphisms from $\mathcal{M}$ to $\mathcal{N}$ is denoted by $\Pi_p(\mathcal{M}, \mathcal{N})$.

In 2021, Stern and van Suijlekom introduced the notion of modular Schatten class morphisms [59].

Definition 2.2. *[59] Let $1 \leq p < \infty$. Let $\mathcal{A}$ be a C*-algebra and $\widehat{\mathcal{A}}$ be its Gelfand spectrum. Let $\mathcal{M}$ and $\mathcal{N}$ be Hilbert C*-modules over $\mathcal{A}$. Let $T : \mathcal{M} \to \mathcal{N}$ be an adjointable morphism. We say that T is in the **modular p-Schatten class** if the function*

$$\operatorname{Tr}|T|^p : \widehat{\mathcal{A}} \ni \chi \mapsto \operatorname{Tr}|\chi_* T|^p \in \mathbb{R} \cup \{\infty\}$$

lies in $\mathcal{A}$. The modular p-Schatten norm of T is defined as

$$\|T\|_p \coloneqq \|\operatorname{Tr}|T|^p\|_{\mathcal{A}}^{\frac{1}{p}}.$$

*Modular 2-Schatten (resp. 1-Schatten) class morphism is called as **modular Hilbert-Schmidt** (resp. **modular trace class**). We denote $\|T\|_2$ by $\|T\|_{HS}$.*

Using the theory of modular frames for Hilbert C*-modules (see [16]) Stern and van Suijlekom were able to derive following result.

Theorem 2.3. *[59] Let $\mathcal{M}$ and $\mathcal{N}$ be Hilbert C*-modules over $\mathcal{A}$. Let $T : \mathcal{M} \to \mathcal{N}$ be an adjointable morphism. Then T is modular Hilbert-Schmidt if and only if for every modular Parseval frame $\{\tau_n\}_n$ for $\mathcal{M}$, the series $\sum_{n=1}^{\infty}\langle |T|^p\tau_n, \tau_n\rangle$ converges in norm in $\mathcal{A}$ and*

$$\operatorname{Tr}|T|^p = \sum_{n=1}^{\infty}\langle |T|^p\tau_n, \tau_n\rangle.$$

We now derive modular version of Theorem 1.3 with the following notion.

Definition 2.4. *[62] A C*-algebra $\mathcal{A}$ is said to be **monotone closed** if every bounded increasing net in $\mathcal{A}$ has the least upper bound in $\mathcal{A}$.*

Theorem 2.5. *Let $\mathcal{M}$ and $\mathcal{N}$ be Hilbert C*-modules over a commutative C*-algebra $\mathcal{A}$. Assume $\mathcal{A}$ is monotone closed. Let $T : \mathcal{M} \to \mathcal{N}$ be an adjointable morphism. Then $T \in \Pi_2(\mathcal{M}, \mathcal{N})$ if and only if T is modular Hilbert-Schmidt. Moreover, $\|T\|_{HS} = \pi_2(T)$.*

Proof. ($\Rightarrow$) Let $T \in \Pi_2(\mathcal{M}, \mathcal{N})$. Let $\{\tau_n\}_{n=1}^{\infty}$ be a modular Parseval frame for $\mathcal{M}$. Then

$$\langle x, x\rangle = \sum_{n=1}^{\infty}\langle x, \tau_n\rangle\langle \tau_n, x\rangle, \quad \forall x \in \mathcal{M}, \tag{4}$$

where the series converges in the norm of $\mathcal{A}$. To show T is modular Hilbert-Schmidt, using Theorem 2.3, it suffices to show that the series $\sum_{n=1}^{\infty}\langle T\tau_n, T\tau_n\rangle$ converges in norm in $\mathcal{A}$. Note that the series $\sum_{n=1}^{\infty}\langle T\tau_n, T\tau_n\rangle$ is monotonically increasing. Since the C*-algebra is monotone closed, we are done if we show the sequence $\{\sum_{j=1}^{n}\langle T\tau_j, T\tau_j\rangle\}_{n=1}^{\infty}$ is bounded. Let $n \in \mathbb{N}$. Since T is 2-summing, using Equation (4) we have

$$\begin{aligned}\left\|\sum_{j=1}^{n}\langle T\tau_j, T\tau_j\rangle\right\| &\leq \pi_2(T)^2 \sup_{x\in\mathcal{M}, \|x\|\leq 1}\left\|\sum_{j=1}^{n}\langle x, \tau_j\rangle\langle \tau_j, x\rangle\right\| \leq \pi_2(T)^2 \sup_{x\in\mathcal{M}, \|x\|\leq 1}\left\|\sum_{j=1}^{\infty}\langle x, \tau_j\rangle\langle \tau_j, x\rangle\right\| \\ &= \pi_2(T)^2 \sup_{x\in\mathcal{M}, \|x\|\leq 1}\|x\|^2 = \pi_2(T)^2.\end{aligned}$$

($\Leftarrow$) Let $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathcal{M}$. Let $\{\omega_n\}_{n=1}^{\infty}$ be an orthonormal basis for $\mathcal{M}$. Define

$$S : \mathcal{M} \ni x \mapsto \sum_{j=1}^{n}\langle x, \omega_j\rangle x_j \in \mathcal{M}.$$

Then

$$\begin{aligned}\|S\|^2 = \|S^*\|^2 &= \sup_{x\in\mathcal{M}, \|x\|\leq 1}\|S^*x\|^2 = \sup_{x\in\mathcal{M}, \|x\|\leq 1}\left\|\sum_{n=1}^{\infty}\langle S^*x, \omega_n\rangle\omega_n\right\|^2 = \sup_{x\in\mathcal{M}, \|x\|\leq 1}\left\|\sum_{n=1}^{\infty}\langle x, S\omega_n\rangle\omega_n\right\|^2 \\ &= \sup_{x\in\mathcal{M}, \|x\|\leq 1}\left\|\sum_{j=1}^{n}\langle x, x_j\rangle\omega_j\right\|^2 = \sup_{x\in\mathcal{M}, \|x\|\leq 1}\left\|\sum_{j=1}^{n}\langle x, x_j\rangle\langle x_j, x\rangle\right\|.\end{aligned}$$

Hence

$$\left\|\sum_{j=1}^{n}\langle Tx_j, Tx_j\rangle\right\| = \left\|\sum_{j=1}^{n}\langle TS\omega_j, TS\omega_j\rangle\right\| = \|TS\|_{\text{HS}}^2 \leq \|T\|_{\text{HS}}^2\|S\| = \|T\|_{\text{HS}}^2 \sup_{x\in\mathcal{M}, \|x\|\leq 1}\left\|\sum_{j=1}^{n}\langle x, x_j\rangle\langle x_j, x\rangle\right\|.$$

□

Note that we have not used monotonic closedness of C*-algebra in "if" part. In view of Theorem 1.4, we formulate following problem.

Question 2.6. ***Whether there exists a modular Pietsch factorization theorem?***

We solve Question (2.6) partially in the following theorem. Integrals in the following theorem is in the Kasparov sense [33].

Theorem 2.7. *Let $\mathcal{M}$ and $\mathcal{N}$ be Hilbert C*-modules over a commutative C*-algebra $\mathcal{A}$. Let $T : \mathcal{M} \to \mathcal{N}$ be an adjointable morphism. Assume that there exists a Lie group $G \subseteq B_{\mathcal{M}} := \{x : x \in \mathcal{M}, \|x\| \leq 1\}$ satisfying following.*

(i) $\mu_G(G) = 1$.

(ii) *For each $x \in \mathcal{M}$, the map $G \ni y \mapsto \langle x.y\rangle\langle y, x\rangle$ is continuous.*

(iii) *There exists a real $C > 0$ such that*

$$\langle Tx, Tx\rangle^{\frac{p}{2}} \leq C^p \int_G (\langle x.y\rangle\langle y, x\rangle)^{\frac{p}{2}}\, d\mu_G(y), \quad \forall x \in \mathcal{M}.$$

Then T modular p-absolutely summing and $\pi_p(T) = C$.

Proof. Let $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathcal{M}$. Then

$$\left\|\sum_{j=1}^n \langle Tx_j, Tx_j\rangle^{\frac{p}{2}}\right\| \leq C^p \left\|\sum_{j=1}^n \int_G (\langle x_j, y\rangle\langle y, x_j\rangle)^{\frac{p}{2}}\, d\mu_G(y)\right\| = C^p \left\|\int_G \sum_{j=1}^n (\langle x_j, y\rangle\langle y, x_j\rangle)^{\frac{p}{2}}\, d\mu_G(y)\right\|$$

$$\leq C^p \int_G \left\|\sum_{j=1}^n (\langle x_j, y\rangle\langle y, x_j\rangle)^{\frac{p}{2}}\right\| d\mu_G(y) \leq C^p \int_G \sup_{y\in\mathcal{M}, \|y\|\leq 1} \left\|\sum_{j=1}^n (\langle x_j, y\rangle\langle y, x_j\rangle)^{\frac{p}{2}}\right\| d\mu_G(y)$$

$$= C^p \sup_{y\in\mathcal{M}, \|y\|\leq 1} \left\|\sum_{j=1}^n (\langle x_j, y\rangle\langle y, x_j\rangle)^{\frac{p}{2}}\right\| \mu_G(G) = C^p \sup_{y\in\mathcal{M}, \|y\|\leq 1} \left\|\sum_{j=1}^n (\langle x_j, y\rangle\langle y, x_j\rangle)^{\frac{p}{2}}\right\|.$$

□

3. Appendix

In this appendix we formulate some problems for Banach modules over C*-algebras based on the results in Banach spaces which influenced a lot in the modern development of Functional Analysis. Our first kind of problems come from the Dvoretzky theorem [12–15, 20, 21, 42–46, 53, 54, 58, 60]. Let $\mathcal{X}$ and $\mathcal{Y}$ be finite dimensional Banach spaces such that $\dim(\mathcal{X}) = \dim(\mathcal{Y})$. Remenber that the Banach-Mazur distance between $\mathcal{X}$ and $\mathcal{Y}$ is defined as

$$d_{BM}(\mathcal{X}, \mathcal{Y}) := \inf\{\|T\|\|T^{-1}\| : T : \mathcal{X} \to \mathcal{Y} \text{ is invertible linear operator}\}.$$

For $n \in \mathbb{N}$, let $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ be the standard Euclidean Hilbert space.

Theorem 3.1. *[1, 27]* ***(John Theorem) If $\mathcal{X}$ is any n-dimensional real Banach space, then***

$$d_{BM}(\mathcal{Y}, (\mathbb{R}^n, \langle\cdot,\cdot\rangle)) \leq \sqrt{n}.$$

Theorem 3.2. *[1, 10]* ***(Dvoretzky Theorem) There is a universal constant $C > 0$ satisfying the following property: If $\mathcal{X}$ is any n-dimensional real Banach space and $0 < \varepsilon < \frac{1}{3}$, then***

for every natural number

$$k \leq C \log n \frac{\varepsilon^2}{|\log \varepsilon|},$$

there exists a k-dimensional Banach subspace $\mathcal{Y}$ of $\mathcal{X}$ such that

$$d_{BM}(\mathcal{Y}, (\mathbb{R}^k, \langle \cdot, \cdot \rangle)) < 1 + \varepsilon.$$

Let $\mathcal{A}$ be a unital C*-algebra with invariant basis number property (see [19] for a study on such C*-algebras) and $\mathcal{E}$, $\mathcal{F}$ be finite rank Banach modules over $\mathcal{A}$ such that $\text{rank}(\mathcal{E}) = \text{rank}(\mathcal{F})$. Modular Banach-Mazur distance between $\mathcal{E}$ and $\mathcal{F}$ is defined as

$$d_{MBM}(\mathcal{E}, \mathcal{F}) \coloneqq \inf\{\|T\|\|T^{-1}\| : T : \mathcal{E} \to \mathcal{F} \text{ is invertible module homomorphism}\}.$$

Given a unital C*-algebra $\mathcal{A}$ and $n \in \mathbb{N}$, by $\mathcal{A}^n$ we mean the standard (left) module over $\mathcal{A}$. We equip $\mathcal{A}^n$ with the C*-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{A}^n \times \mathcal{A}^n \to \mathcal{A}$ defined by

$$\langle (a_j)_{j=1}^n, (b_j)_{j=1}^n \rangle \coloneqq \sum_{j=1}^n a_j b_j^*, \quad \forall (a_j)_{j=1}^n, (b_j)_{j=1}^n \in \mathcal{A}^n.$$

Hence norm on $\mathcal{A}^n$ is given by

$$\|(a_j)_{j=1}^n\| \coloneqq \left\|\sum_{j=1}^n a_j a_j^*\right\|^{\frac{1}{2}}, \quad \forall (a_j)_{j=1}^n \in \mathcal{A}^n.$$

Then it is well-known that $\mathcal{A}^n$ is a Hilbert C*-module. We denote this Hilbert C*-module by $(\mathcal{A}^n, \langle \cdot, \cdot \rangle)$.

Problem 3.3. ***(Modular Dvoretzky Problem) Let $\mathbb{A}$ be the set of all unital C*-algebras with invariant basis number property. What is the best function $\Psi : \mathbb{A} \times \left(0, \frac{1}{3}\right) \times \mathbb{N} \to (0, \infty)$ satisfying the following property: If $\mathcal{E}$ is any n-rank Banach module over a unital C*-algebra $\mathcal{A}$ with IBN property and $0 < \varepsilon < \frac{1}{3}$, then for every natural number***

$$k \leq \Psi(\mathcal{A}, \varepsilon, n),$$

there exists a k-rank Banach submodule $\mathcal{F}$ of $\mathcal{E}$ such that

$$d_{MBM}(\mathcal{F}, (\mathcal{A}^k, \langle \cdot, \cdot \rangle)) < 1 + \varepsilon.$$

A particular case of Problem 3.3 is the following conjecture.

Conjecture 3.4. ***(Modular Dvoretzky Conjecture) Let $\mathcal{A}$ be a unital C*-algebra with IBN property. There is a universal constant $C > 0$ (which may depend upon $\mathcal{A}$) satisfying the following property: If $\mathcal{E}$ is any n-rank Banach module and $0 < \varepsilon < \frac{1}{3}$, then for every natural number***

$$k \leq C \log n \frac{\varepsilon^2}{|\log \varepsilon|},$$

there exists a k-rank Banach submodule $\mathcal{F}$ of $\mathcal{E}$ such that

$$d_{MBM}(\mathcal{F}, (\mathcal{A}^k, \langle \cdot, \cdot \rangle)) < 1 + \varepsilon.$$

Our second kind of problems come from the type-cotype theory of Banach spaces [1, 9, 28, 29, 37–39, 61]. Let $\mathcal{H}$ be a Hilbert space, $n \in \mathbb{N}$. Recall that for any n points $x_1, \dots, x_n \in \mathcal{H}$, we have

$$\frac{1}{2^n}\sum_{\varepsilon_1,\dots,\varepsilon_n\in\{-1,1\}}\left\|\sum_{j=1}^{n}\varepsilon_j x_j\right\|^2=\sum_{j=1}^{n}\|x_j\|^2. \tag{5}$$

It is Equality (5) which motivated the definition of type and cotype for Banach spaces.

Definition 3.5. *[1] Let $1 \leq p \leq 2$. A Banach space $\mathcal{X}$ is said to be of* ***(Rademacher) type*** *p if there exists $T_p(\mathcal{X}) > 0$ such that*

$$\left(\frac{1}{2^n}\sum_{\varepsilon_1,\dots,\varepsilon_n\in\{-1,1\}}\left\|\sum_{j=1}^{n}\varepsilon_j x_j\right\|^p\right)^{\frac{1}{p}}\leq T_p(\mathcal{X})\left(\sum_{j=1}^{n}\|x_j\|^p\right)^{\frac{1}{p}},\quad \forall x_1,\dots,x_n\in\mathcal{X},\ \forall n\in\mathbb{N}.$$

Definition 3.6. *[1] Let $2 \leq q < \infty$. A Banach space $\mathcal{X}$ is said to be of* ***(Rademacher) cotype*** *q if there exists $C_q(\mathcal{X}) > 0$ such that*

$$\left(\sum_{j=1}^{n}\|x_j\|^q\right)^{\frac{1}{q}}\leq C_q(\mathcal{X})\left(\frac{1}{2^n}\sum_{\varepsilon_1,\dots,\varepsilon_n\in\{-1,1\}}\left\|\sum_{j=1}^{n}\varepsilon_j x_j\right\|^q\right)^{\frac{1}{q}},\quad \forall x_1,\dots,x_n\in\mathcal{X},\ \forall n\in\mathbb{N}.$$

Let $\mathcal{E}$ be a (left) Hilbert C*-module over a unital C*-algebra $\mathcal{A}$, $n \in \mathbb{N}$. We see that for any n points $x_1, \dots, x_n \in \mathcal{E}$, we have

$$\frac{1}{2^n}\sum_{\varepsilon_1,\dots,\varepsilon_n\in\{-1,1\}}\left\langle\sum_{j=1}^{n}\varepsilon_j x_j,\sum_{k=1}^{n}\varepsilon_k x_k\right\rangle=\sum_{j=1}^{n}\langle x_j, x_j\rangle. \tag{6}$$

Problem 3.7. ***(Modular Type-Cotype Problems) Whether there is a way to define type (we call modular-type) and cotype (we call modular-cotype) for Banach modules over C*-algebras which reduces to Equality (6) for Hilbert C*-modules?***

Problem 3.8. ***Whether there is a notion of type and cotype for Banach modules over C*-algebras such that Kwapien theorems holds?, In other words, whether following statements hold?***

- **(i)** ***A Banach module $\mathcal{M}$ over a unital C*-algebra $\mathcal{A}$ has modular-type 2 and modular-cotype 2 if and only if $\mathcal{M}$ is isomorphic to a Hilbert C*-module over $\mathcal{A}$.***
- **(ii)** ***If $\mathcal{M}$ and $\mathcal{N}$ are Banach modules over a unital C*-algebra $\mathcal{A}$ of modular-type 2 and modular-cotype 2, respectively, then a bounded module morphism $T : \mathcal{M} \to \mathcal{N}$ factors through a Hilbert C*-module over $\mathcal{A}$.***

Problem 3.9. ***(Modular Khinchin-Kahane Inequalities Problems) Whether there is a Khinchin-Kahane inequalities for Banach modules over C*-algebras which reduce to Equality (6) for Hilbert C*-modules?***

Our third kind of problems come from Grothendieck inequality [1–3, 8, 17, 22, 23, 25, 31, 55, 57].

Theorem 3.10. *[1, 3, 8, 17, 22, 57]* ***(Grothendieck Inequality) There is a universal constant K_G satisfying the following: For any Hilbert space $\mathcal{H}$ and any $m, n \in \mathbb{N}$, if a scalar matrix***

$[a_{j,k}]_{1\leq j\leq m, 1\leq k\leq n}$ ***satisfy***

$$\left|\sum_{j=1}^{m}\sum_{k=1}^{n}a_{j,k}s_jt_k\right|\leq 1, \quad \forall s_j, t_k\in \mathbb{K}, |s_j|\leq 1, |t_k|\leq 1,$$

then

$$\left|\sum_{j=1}^{m}\sum_{k=1}^{n}a_{j,k}\langle u_j, v_k\rangle\right|\leq K_G, \quad \forall u_j, v_k\in \mathcal{H}, \|u_j\|\leq 1, \|v_k\|\leq 1.$$

Problem 3.11. ***(Modular Grothendieck Inequality Problem - 1) Let*** $\mathbb{A}$ ***be the set of all unital C*-algebras. Let*** $\mathcal{E}$ ***be a Hilbert C*-module over a unital C*-algebra*** $\mathcal{A}$***. Let*** $\mathcal{A}^+$ ***be the set of all positive elemnts in*** $\mathcal{A}$***. What is the best function*** $\Psi : \mathbb{A}\times\mathbb{N}\times\mathbb{N}\to\mathcal{A}^+$ ***satisfying the following property: If*** $[a_{j,k}]_{1\leq j\leq m, 1\leq k\leq n}\in \mathbb{M}_{m\times n}(\mathcal{A})$ ***satisfy***

$$\left\langle\sum_{j=1}^{m}\sum_{k=1}^{n}a_{j,k}s_jt_k, \sum_{p=1}^{m}\sum_{q=1}^{n}a_{p,q}s_pt_q\right\rangle\leq 1, \quad \forall s_j, t_k\in \mathcal{A},$$
$$s_js_j^*=s_j^*s_j=1, \forall 1\leq j\leq m, t_kt_k^*=t_k^*t_k=1, \forall 1\leq k\leq n,$$

then

$$\left\langle\sum_{j=1}^{m}\sum_{k=1}^{n}a_{j,k}\langle u_j, v_k\rangle, \sum_{p=1}^{m}\sum_{q=1}^{n}a_{p,q}\langle u_p, v_q\rangle\right\rangle\leq \Psi(\mathcal{A}, m, n), \quad \forall u_j, v_k\in \mathcal{E},$$
$$\langle u_j, u_j\rangle=1, \forall 1\leq j\leq m, \langle v_k, v_k\rangle=1, \forall 1\leq k\leq n.$$

In particular, whether Ψ ***depends on*** m ***and*** n***?***

Problem 3.12. ***(Modular Grothendieck Inequality Problem - 2) Let*** $\mathbb{A}$ ***be the set of all unital C*-algebras. Let*** $\mathcal{E}$ ***be a Hilbert C*-module over a unital C*-algebra*** $\mathcal{A}$***. Let*** $\mathcal{A}^+$ ***be the set of all positive elemnts in*** $\mathcal{A}$***. What is the best function*** $\Psi : \mathbb{A}\times\mathbb{N}\times\mathbb{N}\to\mathcal{A}^+$ ***satisfying the following property: If*** $[a_{j,k}]_{1\leq j\leq m, 1\leq k\leq n}\in \mathbb{M}_{m\times n}(\mathcal{A})$ ***satisfy***

$$\left\langle\sum_{j=1}^{m}\sum_{k=1}^{n}a_{j,k}s_jt_k, \sum_{p=1}^{m}\sum_{q=1}^{n}a_{p,q}s_pt_q\right\rangle\leq 1, \quad \forall s_j, t_k\in \mathcal{A}, \|s_j\|\leq 1, \forall 1\leq j\leq m, \|t_k\|\leq 1, \forall 1\leq k\leq n,$$

then

$$\left\langle\sum_{j=1}^{m}\sum_{k=1}^{n}a_{j,k}\langle u_j, v_k\rangle, \sum_{p=1}^{m}\sum_{q=1}^{n}a_{p,q}\langle u_p, v_q\rangle\right\rangle\leq \Psi(\mathcal{A}, m, n), \quad \forall u_j, v_k\in \mathcal{E},$$
$$\|u_j\|\leq 1, \forall 1\leq j\leq m, \|v_k\|\leq 1, \forall 1\leq k\leq n.$$

In particular, whether Ψ ***depends on*** m ***and*** n***?***

We believe strongly that Ψ depends on $\mathcal{A}$.

Remark 3.13. ***Modular Bourgain-Tzafriri restricted invertibility conjecture*** *and* ***Modular Johnson-Lindenstrauss flattening conjecture*** *have been stated in [34, 35].*

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