

# Riemann's "Functional" Equation is Not Valid and its Implication on the Riemann Hypothesis

By

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## ABSTRACT

**Riemann's "functional" equation** was formulated by Riemann that if valid, would mean that a real function that has zeros similar to the hypothesized zeros of the zeta "function" (which has something to do with the distribution of prime numbers) is obtained. This led him to obtain a formula for relating the hypothesized zeros of the zeta "function" to the number of primes given a certain magnitude [1]. The **Riemann Hypothesis** states that all the nontrivial zeros of the zeta "function" are at  $s = \frac{1}{2} + \omega i$ . Thus, the Riemann's functional equation is the foundation upon which the Riemann Hypothesis is based. But the "functional" equation, as shall be shown here, is not valid such that the Riemann Hypothesis crumbles on its claim.

## Introduction

The Riemann zeta “function” is shown below

$$(1) \quad \zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where  $n$  is the independent variable while  $s$  is a complex constant with real part  $\sigma$  and imaginary part  $\omega$ , and  $i$  is the imaginary unit  $i = \sqrt{-1}$ .  $\zeta(s)$  is a constant since we are taking the sum for all values of  $n$  and the value of  $s$  must be chosen such that  $|\zeta(s)| < \infty$ .

Consider the widely known Riemann’s “functional” equation

$$(2) \quad \zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s).$$

The series (1) absolutely converges if  $\sigma > 1$  while (2) converges if  $\sigma < 0$ . Thus, (1) and (2) don’t have common values that would make them equal so that

$$\zeta(s) \neq 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s),$$

which proves that (2) is **not** a valid equation.

Riemann obtained (2) from the contour integral

$$I(s) = \int_C \frac{(-z)^{s-1}}{e^z - 1} dz,$$

and also

$$(3) \quad \zeta(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_{\infty}^{\infty} \frac{(-x)^{s-1}}{e^x - 1} dx,$$

which is widely accepted as the “analytic continuation” of  $\zeta(s)$  for all  $s$  except at  $s=1$ . I will show next that

$$\zeta(s) \neq -\frac{\Gamma(1-s)}{2\pi i} \int_{\infty}^{\infty} \frac{(-x)^{s-1}}{e^x - 1} dx$$

or (3) is **not** a valid equation.

## Riemann's "Functional" Equation Is Not Valid

If  $x > 0$ ,

$$\frac{1}{e^z - 1} = \sum_{n=1}^{\infty} e^{-nz}$$

and

$$I(s) = (-1)^{s-1} \int_C \sum_{n=1}^{\infty} e^{-nz} (z)^{s-1} dz = (-1)^{s-1} \int_C \sum_{n=1}^{\infty} \frac{1}{n^s} z^{s-1} e^{-z} dz$$

$$I(s) = (-1)^{s-1} \zeta(s) \int_C z^{s-1} e^{-z} dz \quad \sigma > 1.$$

The contour integral of  $I(s)$  if  $z = x$  and  $x$  going from  $+\infty$  and back again to  $+\infty$  is

$$I_1(s) = (-1)^{s-1} \zeta(s) \int_{\infty}^{\infty} x^{s-1} e^{-x} dx = (-1)^{s-1} \zeta(s) \left\{ \int_{\infty}^0 e^{-x} x^{s-1} dx + \int_0^{\infty} e^{-x} x^{s-1} dx \right\}$$

$$I_1(s) = (-1)^{s-1} \zeta(s) [-\Gamma(s) + \Gamma(s)] = 0 \quad \sigma > 1$$

Because  $I_1(s) = 0$ ,

$$I_1(s) = 0 \neq -2i \sin(\pi s) \zeta(s) \Gamma(s)$$

and so

$$\zeta(s) \neq -\frac{\Gamma(1-s)}{2\pi i} \int_{\infty}^{\infty} \frac{(-x)^{s-1}}{e^x - 1} dx \neq -\left\{ \frac{\Gamma(1-s)}{2\pi i} (0) = 0 \right\}.$$

Therefore, (3) is **not** a valid equation.

## The Misapplication of the Residue Theorem

Riemann applied the Residue Theorem on (4) by assuming the quantities  $2\pi ni$  for  $n = \pm 1, \pm 2, \pm 3$ , and so on, as the *poles* of (4). The function  $q(z) = 1/(e^z - 1)$  in (4) is a periodic function of  $z$ ,

$$q(z + 2\pi i) = \frac{1}{e^{(z+2\pi i)} - 1} = \frac{1}{e^z - 1} \quad z \neq 0,$$

since the exponential function  $e^z$  is a periodic function with period  $2\pi i$ . Hence the term  $2\pi ni$  are the multiple of the *fundamental period*  $2\pi i$ . It is, therefore, not valid to treat them as the poles of (6). In fact, the only pole of  $q(z)$  is at  $z = 0$  with residue 1,

$$\oint \frac{dz}{e^z - 1} = 2\pi i.$$

Unfortunately, one can not use  $z = 0$  on  $I(s)$  because it will be undefined

$$\begin{aligned} \int_C \frac{(-z)^{s-1}}{e^z - 1} dz &= (0)^{s-1} - 2\pi i \left( \sum_{n=1}^{\infty} (-2\pi n i)^{s-1} + (2\pi n i)^{s-1} \right) = \text{undefined} \\ &= \text{undefined} \neq -2\pi i \sin\left(\frac{\pi s}{2}\right) 2^s \pi^{s-1} \zeta(1-s). \end{aligned}$$

Thus, the integral on this contour is **not** valid.

Or consider the contour integral

$$I_2(s) = (-1)^{s-1} \int_{a-i\infty}^{a+i\infty} \frac{z^{s-1}}{e^z - 1} dz = (-1)^{s-1} \zeta(s) \int_{a-i\infty}^{a+i\infty} z^{s-1} e^{-z} dz$$

$$M = |(-1)^{s-1} \zeta(s)| |(a+yi)^{s-1} e^{-(a+yi)}| = |\zeta(s)| R^{\sigma-1} e^{-a} \quad \text{and} \quad L = \int_a^b |z'(\tau)| d\tau = 2R$$

$$ML = \lim_{R \rightarrow \infty} 2 |\zeta(s)| R^{\sigma} e^{-a} = \begin{cases} \infty & \sigma > 1 \\ \text{indeterminate} & \sigma < 0 \end{cases}$$

As you can see that if  $-\infty < y < +\infty$ , the resulting contour integral is undefined.

Also, if  $x < 0$

$$I(s) = -(-1)^{s-1} \int_C \sum_{n=0}^{\infty} e^{nz} (z)^{s-1} dz = (-1)^s \int_C \sum_{n=0}^{\infty} \frac{1}{n^s} z^{s-1} e^z dz = (-1)^s \int_C \sum_{n=0}^{\infty} \frac{1}{n^s} z^{s-1} e^z dz = \text{undefined}$$

Thus, if we want a  $I(s)$  to be finite,  $x$  must be greater than zero ( $x > 0$ ) or  $0 < |z| < 2\pi$  (see Appendix).

## CONCLUSION

Therefore, since Riemann's functional equation is not valid, it follows that the Riemann hypothesis is also not valid.

## Appendix

### Some Contour Integrals of $I(s)$

Let us now look closely at  $I(s)$ ,

$$(4) \quad I(s) = \int_C \frac{(-z)^{s-1}}{e^z - 1} dz = \int_{z_1}^{z_2} \frac{(-z)^{s-1}}{e^z - 1} dz$$

The integral  $I(s)$  depends on  $s$ , the endpoints  $z_1$  and  $z_2$ , and the chosen path itself. Where  $z (= x + yi)$  is the independent variable and  $s (= \sigma + ti)$  is a complex constant.

Since  $s$  is constant

$$I'(s) = \frac{dI}{ds} = \frac{0}{0} = \text{indeterminate} \quad \text{and} \quad \int_s^s I(s) ds = 0,$$

its role is to ensure that  $I(s)$  is finite for a given value of  $s$  ( $|I(s)| < \infty$ ),

$$|I(s)| = \left| \int_C \frac{(-z)^{s-1}}{e^z - 1} dz \right| < M L < \infty$$

where

$$M = \left| \frac{(-z)^{s-1}}{e^z - 1} \right| \quad \text{and} \quad L = \int_a^b |z'(\tau)| d\tau.$$

**Note:**  $(-1)^{s-1}$  is simply a multi-valued complex quantity with principal values

$$(-1)^{-1}(-1)^s = -e^{\pm(\pi is)}.$$

**If  $x > 0$ ,**

$$I(s) = (-1)^{s-1} \int_C \sum_{n=1}^{\infty} e^{-nz} (z)^{s-1} dz = (-1)^{s-1} \int_C \sum_{n=1}^{\infty} \frac{1}{n^s} z^{s-1} e^{-z} dz$$

$$I(s) = (-1)^{s-1} \zeta(s) \int_C z^{s-1} e^{-z} dz \quad \sigma > 1$$

- Path  $C_1$ :  $z = x$ ,  $x$  from  $+\infty$  to 0 and from 0 to  $+\infty$

$$I_1(s) = (-1)^{s-1} \zeta(s) \int_{\infty}^0 x^{s-1} e^{-x} dx = (-1)^{s-1} \zeta(s) \left\{ \int_{\infty}^0 e^{-x} x^{s-1} dx + \int_0^{\infty} e^{-x} x^{s-1} dx \right\}$$

$$I_1(s) = (-1)^{s-1} l \zeta(s) \{-\Gamma(s) + \Gamma(s)\} = 0 \quad \sigma > 1$$

- Path  $C_2$ :  $z = x, 0 < x < \infty$

$$I_2(s) = (-1)^{s-1} \zeta(s) \int_0^{\infty} e^{-x} x^{s-1} dx = (-1)^{s-1} \zeta(s) \Gamma(s) \quad \sigma > 1$$

- Path  $C_3$ :  $z = a + yi$ , for constant  $a > 0$  and  $-\infty < y < +\infty$

$$I_3(s) = \int_C \frac{(-z)^{s-1}}{e^z - 1} dz = (-1)^{s-1} \int_{a-\infty i}^{a+\infty i} \frac{z^{s-1}}{e^z - 1} dz = (-1)^{s-1} \zeta(s) \int_{a-\infty i}^{a+\infty i} z^{s-1} e^{-z} dz$$

$$M = |(-1)^{s-1} \zeta(s)| |(a+yi)^{s-1} e^{-(a+yi)}| = |\zeta(s)| R^{\sigma-1} e^{-a} \quad \text{and} \quad L = \int_a^b |z'(\tau)| d\tau = 2R$$

$$ML = \lim_{R \rightarrow \infty} 2 |\zeta(s)| R^{\sigma} e^{-a} = \begin{cases} \infty & \sigma > 1 \\ \text{indeterminate} & \sigma < 0 \end{cases}$$

**If  $x < 0$ ,**

$$I(s) = -(-1)^{s-1} \int_C \sum_{n=0}^{\infty} e^{nz} (z)^{s-1} dz = (-1)^s \int_C \sum_{n=0}^{\infty} \frac{1}{n^s} z^{s-1} e^z dz = (-1)^s \int_C \sum_{n=0}^{\infty} \frac{1}{n^s} z^{s-1} e^z dz = \text{undefined}$$

- Path  $C_4$ :  $z = x, -\infty < x < 0$

$$I_4(s) = \int_{-\infty}^0 \frac{(-x)^{s-1}}{e^x - 1} dx = \int_0^{\infty} \frac{x^{s-1}}{e^{-x} - 1} dx = - \int_0^{\infty} \sum_{n=0}^{\infty} e^{-nx} x^{s-1} dx = - \left\{ \zeta(s) \Gamma(s) + \left( \int_0^{\infty} x^{s-1} dx = \infty \right) \right\} = \infty$$

**Circular path:**  $z = r e^{i\theta} \quad 0 < r < 2\pi$

$$\frac{z}{e^z - 1} = \sum_{m=0}^{\infty} \frac{B_m z^m}{m!}$$

where  $B_m$  are the Bernoulli numbers,

$B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, B_5 = 0, B_6 = 1/42, B_7 = 0, B_8 = -1/30, B_9 = 0$ , etc.

$$I(s) = \int_r^{re^{2\pi i}} \frac{(-z)^{s-1}}{e^z - 1} dz = \int_r^{re^{2\pi i}} \sum_{m=0}^{\infty} \frac{B_m z^m}{m!} (-1)^{s-1} z^{s-1} \frac{dz}{z} = \sum_{m=0}^{\infty} \frac{B_m}{m!} (-1)^{s-1} \int_r^{re^{2\pi i}} z^{m+s-1} \frac{dz}{z}$$

For complex  $s$

$$I(s) = \sum_{m=0}^{\infty} \frac{B_m}{m!} (-1)^{s-1} \int_1^{e^{2\pi i}} z^{m+s-1} \frac{dz}{z} = \sum_{m=0}^{\infty} \frac{B_m}{m!} (-1)^{s-1} r^{(m+s-1)} \int_0^{2\pi} e^{(m+s-1)i\theta} i d\theta$$

$$I(s) = -2i \sin(\pi s) \sum_{m=0}^{\infty} \frac{B_m r^{m+s-1}}{m! (m+s-1)}$$

When  $s = -n$  ( $n = 0, 1, 2, \dots$ ),

$$I(-n) = \int_r^{re^{2\pi i}} \sum_{m=0}^{\infty} \frac{B_m}{m!} (-1)^{-n-1} z^{m-n-1} \frac{dz}{z} = \sum_{m=0}^{\infty} \frac{B_m}{m!} (-1)^{-n-1} \int_0^{2\pi} z^{m-n-1} i d\theta$$

$$I(-n) = 2\pi i \frac{B_{n+1}}{(n+1)!} (-1)^{-n-1}$$

When  $s = 1$

$$I(1) = \sum_{m=0}^{\infty} \frac{B_m}{m!} \int_0^{2\pi} z^m i d\theta = \sum_{m=0}^{\infty} \frac{B_m}{m!} r^m \int_0^{2\pi} e^{mi\theta} i d\theta = 2\pi i B_0$$

When  $s = n$  ( $n = 2, 3, 4, \dots$ ),

$$I(n) = \sum_{m=0}^{\infty} \frac{B_m}{m!} (-1)^{n-1} \int_r^{re^{2\pi i}} z^{m+n-2} dz = \sum_{m=0}^{\infty} \frac{B_m}{m!} (-1)^{n-1} r^{(m+n-1)} \int_0^{2\pi} e^{(m+n-1)i\theta} i d\theta = 0$$

## REFERENCES

- [1] Riemann, Bernhard (1859). *On the Number of Prime Numbers less than a Given Magnitude*.
- [2] Brown, James Ward; Churchill, Ruel V. (1996). *Complex Variables and Applications* (6<sup>th</sup> ed.). Singapore: McGraw Hill International Editions.

## LINKS

- [https://en.wikipedia.org/wiki/Riemann\\_zeta\\_function](https://en.wikipedia.org/wiki/Riemann_zeta_function)
- [https://en.wikipedia.org/wiki/Riemann\\_hypothesis](https://en.wikipedia.org/wiki/Riemann_hypothesis)