

Speculation on Internal (Momentum) and External ($-dV/dx$) Forces in Quantum Mechanics

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Newton's second law $dp/dt = \text{Force}$ seems to be written in terms of a cause $F(x)$ and effect $p(x)$, because momentum is written in terms of a change i.e. d/dt . His third law, however, describes two causes i.e. equal and opposite causes (forces) leading to two effects. We argue that a constant momentum, the effect of a force, may also act as a "force" if divided by a constant time i.e. $\text{change in } p / dt(\text{constant}) = \text{Force}$. A conservative external force may be written in terms of the derivative of a potential i.e. $-dV/dx$, but we suggest that a quantum free particle is linked with two distributions $\cos(px)$, $\sin(px)$ which describe an internal force/acceleration for even a constant p . The two combine in two dimensions i.e. $\exp(ipx)$ to create a type of probability which has a constant modulus i.e. treats each x point equally. Each x point is treated equally because there is no external force $-dV/dx$, but we argue that there is still an internal one linked with probability and spatial distributions i.e. $\cos(px)$, $\sin(px)$. If an impulse is applied to a quantum free particle to change it from p_1 to p_2 , the change in energy is completely described in terms of momentum i.e. $p_2^2/2m - p_1^2/2m$. There is no need to consider space, but at the same time there is a change in the two distributions $\cos(px)$ and $\sin(px)$ due to the change in p .

Thus a constant p is linked with two spatial distributions which have physical consequences i.e. two slit interference etc. There is an internal distribution in space. If a quantum particle is placed in a potential $V(x)$, there is now both an external potential, but also an internal one associated with each $\exp(ipx)$ if one assumes an ensemble of $\exp(ipx)$'s is created i.e. $W(x) = \text{wavefunction} = \sum_p a(p) \exp(ipx)$. The point we make is that both the external and internal potentials are linked with x distributions, thus applying d/dx effects both, but Newton's conservation of energy $KE(x) + V(x) = E$ deals with changes in kinetic energy only linked to the external potential $V(x)$. Formally, one may write: $-1/2m d/dx dW/dx / W + V(x) = E$, but we argue that external and internal potentials "interact" and we try to describe the situation considering $V(x)W(x)$ to demonstrate the interplay of internal and external forces from the point of view of the internal force (probability) $\exp(ipx)$ and statistical mechanical loss of information i.e. equilibrium.

Quantum Free Particle

Newton's second law demonstrates cause and effect i.e.

$$dp/dt = \text{Force}(x) \quad \text{or} \quad p(x,t) \quad ((1a))$$

$$((1)) \text{ may be written as: } d/dx p^2/2m + d/dx V(x) = 0 \quad ((1b))$$

Force is the cause and p the effect, so force is related to a change in momentum. Thus force and a change in momentum may be associated with a spatial distribution. Newton's third law, however, speaks of two causes, namely equal and opposite reaction forces.

We suggest a constant momentum, created by a force e.g. $dp = \text{Force } dt$, may also have an x distribution showing that force information is retained. This momentum is closely related to a force because if a particle with p strikes an object, it may impart an impulse. Thus we think of even a free particle momentum p as being like a force, but we need to distinguish between internal and external force. Momentum p is associated with an internal force and if it is constant, this suggests no external force acting on it. Thus $\exp(ipx)$, the wavefunction of a free particle shows two internal force related distributions $\cos(px)$, $\sin(px)$, but the modulus 1 indicates no external force trying to change the particle's momentum.

What happens if an external potential $V(x)$ is introduced, as in a bound state? Matters become a little different from Newtonian mechanics because now a free particle itself has changes in x which would normally be reserved for acceleration due to an external force.

Consider a quantum free particle moving to the right. Its momentum and kinetic energy at any x point are p and $pp/2m$ even though there are two x probability distributions which seems to represent an overall fixed pattern of fluctuations even though the particle moves on average according to $x = p/m \cdot t$ with t and x measured by an external clock and ruler. The fixed fluctuation pattern describes an internal ruler gradation. Thus momentum and energy of the single particle are defined strictly in terms of momentum even though two spatial distributions exist related to internal force. If one gives an impulse to a free particle such that $p \rightarrow p_1$, then the change in energy is $p_1 p_1 / 2m - pp/2m$, yet the spatial distributions change from $\cos(px), \sin(px)$ to $\cos(p_1 x), \sin(p_1 x)$. Given that $\cos(px), \sin(px)$ are like probabilities, one may consider a bound state containing $\exp(ipx)$ and $\exp(-ipx)$ with equal weights leaving on average the x distribution $2\cos(px)$. This $\cos(px)$, however, is related to the internal force, but one may create energy averages i.e. $pp/2m$ averages using $\cos(px)$ for different p 's. In such a case at least some of the change in x is due to the internal force fluctuations associated with $\cos(px)$ and cannot be considered as due to an external force $-dV/dx$. We consider this in the next section.

Single Particle Quantum Bound State

As noted above, a quantum free particle has two x distributions associated with internal force, namely $\cos(px)$ and $\sin(px)$. The probability form is $\exp(ipx)$ so the modulus 1 shows no external force exists. Thus one may say that p and $pp/2m$ exist at each x point just as in a classical case. The distributions $\cos(px), \sin(px)$, however, have physical consequences i.e. at the peak $\cos(px)=1$ one has the highest probability of finding the particle i.e. measuring p and $pp/2m$. Thus the notion of a wavelength is physically real. The issue is that an x distribution in Newtonian mechanics is usually associated with an external force, but there is none here. " d/dx " derivatives appear in Newtonian equations, and if applied to a problem involving quantum free particles, it seems they will "pick up" x fluctuations due to the internal force. This internal force, however, is separate from the external one which changes classical kinetic energy. Thus, we think it is important to consider the interplay of the external and internal forces in a quantum single particle bound state. We note that for a single particle:

$$\langle p \rangle = -id/dx \exp(ipx) / \exp(ipx) \quad \text{and} \quad \langle pp/2m \rangle = -1/2m d/dx d/dx \exp(ipx) / \exp(ipx) \quad ((2))$$

The constant p and $pp/2m$ values are directly linked to the probability distributions $\cos(px)$ and $\sin(px)$. It is also clear that $\exp(ipx)$ has a modulus of 1 indicating no external force. If one considers a bound system, there is no "time" within a wavelength and there is a probability distribution for the particle to be at one x or another. If the wavelength is on the order of the system length, then one must consider " p " and " $-p$ " together, but their sum yields:

$$2\cos(px) \quad ((3))$$

$((3))$ is x dependent, but related to the internal force fluctuations. At the same time it is linked to p and $-p$ and an external force is needed to convert one into the other and vice versa.

Thus the average kinetic energy:

$$KE_{ave}(x) = pp/2m \cos(px) / \cos(px) = pp/2m \quad ((4))$$

If one, however, retains $\cos(px)$, the internal force distribution, one may see two canceling effects:

$$[d/dx (pp/2m \cos(px))] \cos(px) - pp/2m \cos(px) (-p\sin(px)) / [\cos(px)\cos(px)] \quad ((5))$$

The second term is due to the change in $W(x)$ the wavefunction from one x point to another i.e. accounts for the internal force fluctuations. The first term is the change in the relative average kinetic energy which includes both internal and external force considerations. Thus $-dV/dx$, the external force, is equivalent to a difference of terms $((5))$ with the second seemingly being the spatial fluctuation changes in x due to the internal force. In the case of $((4))$ and $((5))$ this seems to be trivial, but is not for a full range of different p values. In such a case:

$$KE_{ave}(x) = \{\text{Sum over } p \quad pp/2m a(p)\exp(ipx)\} / W(x) \quad ((5))$$

where $a(p)$ are weights. Taking d/dx one again subtracts:

$$\text{Term subtracted} = KE(x) (i (-idW/dx / W)) \quad ((6))$$

This we argue represents changes in average kinetic energy fluctuations which have nothing to do with the external force $-dV/dx$ which one average accelerates the average kinetic energy $KE_{ave}(x)$. This perhaps explains why average momentum $-dW/dx / W$ is equivalent to the root mean square momentum associated with the external force (i.e. classical type momentum under an external force $-dV/dx$).

One may note that $-dW/dx/W$ multiplied by density $W(x)W(x)$ and integrated yields 0 so "average" momentum $dW/dx/W$ changes in sign in different x regions i.e. is somewhat periodic. In other words, this does not seem to be the result of the "usual" external forces which appear in problems. Again we note that $pp/2m \exp(ipx)$ changes in space, but this is compensated by $d/dx \exp(-ipx)$ because overall $pp/2m$ does not change in space i.e. is not being acted upon by an external potential.

External - Internal Force Interplay

A free particle $\exp(ipx)$ has a constant kinetic energy at each x point i.e. $p^2/2m$, but $p^2/2m \cos(px)$ varies in x . This variation in x does not signify an external force, but rather an internal one which does not change kinetic energy. This is why $\cos(px)$ is divided by the normalization $\cos(px)$. In other words one has a $-1/2m \frac{d}{dx} \frac{dW}{dx}$ relative to a $W(x)$.

The time-independent Schrodinger equation is:

$$-1/2m \frac{d}{dx} \frac{dW}{dx} / W + V(x) = E_n \quad ((7))$$

As it stands mathematically $V(x)$ is the classical potential, but classically it accelerates a particle from one dx to another. Quantum mechanically this cannot be the picture because $\exp(ipx)$ is associated with a wavelength \hbar/p which is much larger than dx . If different $\exp(ipx)$'s are present, there is a range of \hbar/p 's. Given this is the case, the physical interpretation of the workings of the potential $V(x)$ are not clear from the manner in which ((7)) is written. In a Newtonian bound state, one seeks a $p(x)p(x)/2m$ such that when added to $V(x)$ one obtains a constant at each x , namely total energy.

If one multiplies ((7)) by $W(x)$ one sees that each x is not represented by a constant E , but by E multiplied by an average probability $W(x)$. $W(x)$ includes both spatial fluctuations due to the internal forces and changes due to the external force (as $a(p)$ is governed by $V(x)$). The interesting term, we argue is:

$$V(x)W(x) \quad ((8))$$

This is clearly different from $V(x)$ and perhaps provides an indication of how potential energy acts in the presence of the wavelike internal force. We first write $V(x) = \sum_k V_k \exp(ikx)$ because $\exp(ipx)$ is a repeated set of probability crests and troughs representing spatial invariance. In other words, we try to express the external potential (force) in terms of the internal one which is associated with the form $\exp(ipx)$. Such an $\exp(ipx)$ may receive an impulse hit at any x changing to $\exp(ip_1 x)$. This may be written as:

$$\exp(ipx) \exp(ip(\text{change } x)) = \exp(ip_1 x) \quad ((9))$$

((9)) is of the form of a statistical AND probability combination which loses the information of the two individual p values, but only retains the sum i.e. p_1 . This is reminiscent of a statistical equilibrium.

V_k is the kinetic energy associated with the wavelike external potential probability form $\exp(ikx)$. One may note that unlike the case $P(p/x) = a(p)\exp(ipx)/W(x)$ one does not "normalize" $V(k)$. It is strictly an external force. It may simply be cast in the form of the internal force $\exp(ipx)$.

Now there are many ways to create $\exp(ip_1 x)$ involving different V_k 's (i.e. potential energy pieces). Thus from the point of view of the internal energy probability structure $\exp(ipx)$ one considers all potential energy pieces V_k with the associated $a(p-k)$ weights to see how $V(x)$ renders potential energy to $\exp(ipx)$. The kinetic energy is simply $p^2/2m a(p)\exp(ipx)$. Thus:

$$p^2/2m a(p)\exp(ipx) + \sum_k V_k a(p-k) \exp(ipx) = E_n \exp(ipx) \quad ((10))$$

Thus considering each free particle internal energy form $\exp(ipx)$, one may display its interaction with the potential energy $V(x)$ as shown in ((10)). This is very different from the Newtonian picture of acceleration for which one does not consider internal energy in the first place. V_k is associated with momentum k and a distribution $\exp(ikx)$, but when multiplied by $a(p-k)\exp(i(p-k)x)$ its distribution is changed to $\exp(ipx)$. Thus there is already a shift in distribution of the external force ($\exp(ikx)$) when it combines with the internal force distribution $\exp(ipx)$. We call ((10)) the interplay between the internal and external forces for a given p dependent internal force.

Two Slit Interference Example

As a second example, consider an $\exp(ipx)$ (internal force probability) approaching a two slit setup with the slits about a wavelength apart. The internal force probability interacts with the external force potential, but one does not know the exact form of $V(x)$. One suggests it carries equal weights for each slit and shifts the direction of p , but not its magnitude. Because one has linear invariance, one has $\exp(i p_1 \cdot x_1) + \exp(i p_2 \cdot x_2)$ where $p_1 = p$ unit vector along x_1 and $p_2 = p$ unit vector along x_2 . The vectors x_1 and x_2 meet at the same x point on a screen so at that point $x_1=x_2$, but there is interference which changes the x distribution. This is due to the interplay of the internal force $\exp(ipx)$ with the potential $V(x)$ in exactly the same way that $V(x)$ interplays with $W(x)$ to create a new periodic function $W(x)$ which is a weighted average of the $\exp(ipx)$ s with the weights $a(p)$ accounting for the way in which $V(x)$ changes the kinetic energy with x . Thus there is a combination of internal and external force distributions. The very weight of the internal force distributions $\exp(ipx)$ ensures that their average, normalized by $W(x)$ yields the classical kinetic energy at x .

Conclusion

In conclusion, we argue that unlike the Newtonian case, a constant momentum represents an internal force distribution(s) such that there is no external force distribution $-dV/dx$ (i.e. p is constant). This requires two internal force distributions creating $\exp(ipx)$ such that the modulus is 1. The modulus of 1 indicates all x points have the same probability because there is no $-dV/dx$ force disrupting this. Thus the presence of x in $\exp(ipx)$ represents a distribution associated with an internal force which does not appear in Newtonian mechanics.

It is, however, possible for a quantum particle to form a bound state with an external potential $V(x)$ (i.e. external force $-dV/dx$). This means both external and internal forces are present and affect one another. A free particle has the same p and $p^2/2m$ at each x point, but at the same time two distributions $\cos(px)$, $\sin(px)$ at each x . Thus average kinetic energy = $p^2/2m = p^2/2 \exp(ipx) / \exp(ipx)$. d/dx kinetic energy = $p^2/2m \exp(-ipx) d/dx \exp(ipx) + p^2/2m \exp(ipx) d/dx \exp(-ipx)$. The second term brings in a negative sign which gives the appearance of subtracting off the internal force distribution effects (which appear as a normalization constant) so that the

resulting difference may be matched with the external force. In this case, it is 0, but for $W(x) = \sum_p a(p) \exp(ipx)$ and $V(x)$ it is not.

We also note that the time-independent Schrodinger equation: $-\frac{1}{2m} \frac{d}{dx} \frac{dW}{dx} / W + V(x) = E_n$, written in this form, displays $V(x)$ in a classical mechanical form which is difficult to interpret. We wish to "see" the interplay between the internal and external forces. To do so, we simply multiply by $W(x)$ and examine $V(x)W(x)$ from the perspective of the internal force because we know its probability form i.e. $\exp(ipx)$. Thus we write $V(x) = \sum_k V_k \exp(ikx)$. We then note the statistical feature that $\exp(i(p-k)x) \exp(ikx) = \exp(ipx)$. In other words an internal force distribution associated with momentum $p-k$ combines with an external force form $\exp(ikx)$ carrying V_k of potential energy to create a new distribution $\exp(ipx)$.

Given the statistical nature of the combination, the result depends on the momentum sum i.e. $p-k + k = p$. This is reminiscent of statistical equilibrium. Thus there are many ways to achieve this same sum and all of these represent interplay between the internal and external forces in order to create the result $\exp(ipx)$ which is associated with the kinetic energy $pp/2m$. Thus: $a(p) \frac{pp}{2m} \exp(ipx) + \exp(ipx) \sum_k V_k a(p-k) = E_n a(p) \exp(ipx)$. The same E_n holds for all p . The external potential $V(x)$ governs the choice of $a(p)$, but $\exp(ipx)$ is still present representing the internal force because it is interplay between the two which yields the Schrodinger equation.

In other words the Newtonian equilibrium: $KE(x) + V(x) = E_n$ is built upon the internal force fluctuations.