Series: **PHYSICS**

Solomon I. Khmelnik
ORCID: https://orcid.org/0000-0002-1493-6630

New equations for the spinning top

Annotation

It is pointed out that at present there is no complete theory of the top that answers all questions. A complete mathematical description of the top is given. This article includes a translation of the author's [7] with corrections and additions. New well-known experiments that currently do not have any explanation are considered in detail. The purpose of the study of this toy is to prove the fact that Coriolis force and centrifugal force are real forces that do work.

Contents

1. Introduction

The question of why the top does not fall is constantly raised despite the fact that there is a well-founded theory of how the top works. The questioner intuitively feels that the initial push cannot give the energy that is needed for a long and vigorous rotation. The questioner intuitively feels that there must be a real force that keeps the top from falling. But the theory explains why it spins, and the unspoken sounds like answer is "doesn't fall because it spins." But maybe our intuition is deceiving us and the spinning top actually has enough energy? This issue first of all discussed below.

2. State equations

On fig. 1 shows a spinning top in its simplest form. The spinning top has

- rotation of the top around its own vertical axis with an angular speed ω_1,
- rotation of a top inclined at an angle $\propto$ to the plane around its own axis with an angular speed ω_2,
- precessionon the circumference of a top inclined at an angle $\propto$ to the plane, with an angular speed ω_3.

In Table 1 of Appendix 1 lists the parameters of the state of the top at the initial moment 1 and at the moment 2, when the top is in a position at which the angle $\propto < \pi/2$. At moment 1, there is only rotation around the vertical axis. At moment 2, a precession additionally appears.

Let us write down for moment 2 the equations of the laws of conservation of momentum L and energy W, which do not depend on how and by what forces the top passed into this state:

$$L_2 + L_3 = L_1, \tag{2}$$

$$W_2 + W_3 = W_1. \tag{3}$$

Substituting the equations from Table 1 into equations (1, 2), we get:

$$J_2\omega_2 + J_3\omega_3 = J_1\omega_1, \tag{4}$$

$$\frac{1}{2}J_2\omega_2^2 + \frac{1}{2}J_3\omega_3^2 = \frac{1}{2}J_1\omega_1^2. \tag{5}$$

where

ω_1, ω_2 - angular speeds of rotation of the top at the moment 1 and 2 around the axis, which is the rod,

ω_3 - angular speed of precession around the vertical axis,

J_1, J_2, J_3 - moments of inertia during rotation with speeds $\omega_1, \omega_2, \omega_3$.

L_1, L_2, L_3 - angular momentum during rotation with speeds $\omega_1, \omega_2, \omega_3$.

Moments of inertia

$$J_1 = J_2 = m\frac{1}{2}\pi R^2 \tag{6}$$

The moment of inertia J_3 changes depending on the angle $\propto$ - see Fig.2. For $\propto = 0$ this moment can be taken equal to $J_3 = mh^2$. For $\propto = \frac{\pi}{2}$ this moment is $J_3 = J_2$ by the Steiner theorem. So,

$$J_3(\propto) = \begin{cases} mh^2 & if \ \propto = 0 \\ m\frac{1}{2}\pi R^2 \ if & \propto = \frac{\pi}{2}. \end{cases} \tag{7}$$

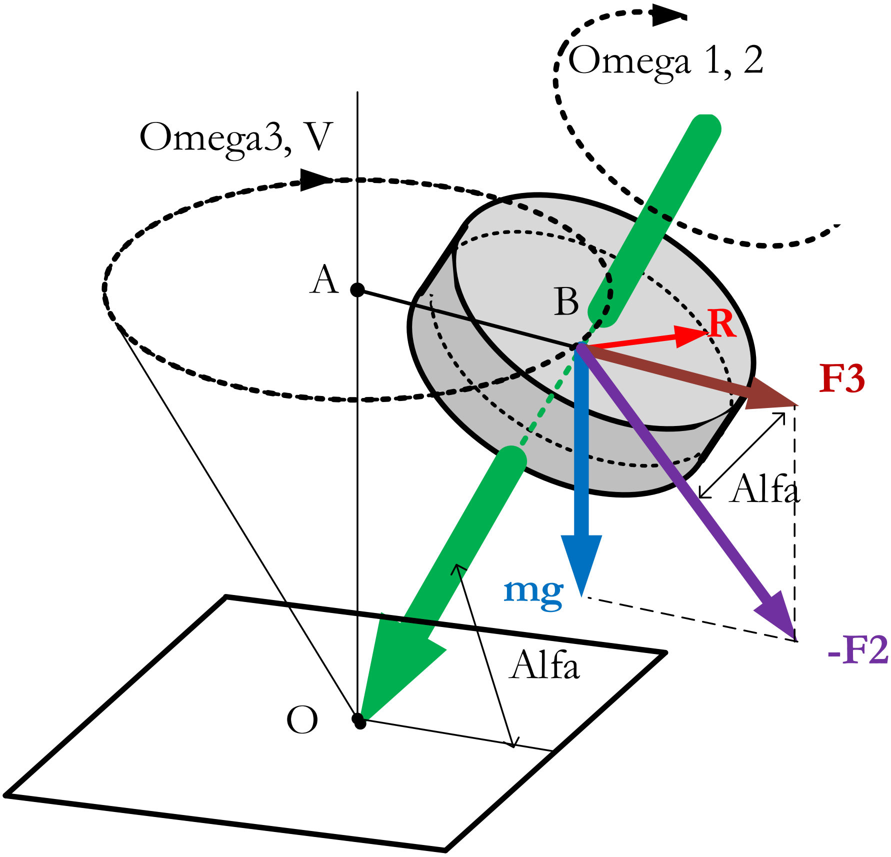

Fig. 1.

We will assume that this function is linear. Then we get:

$$q = J_3/m = h^2 - \left(\frac{2}{\pi}h^2 - R^2\right)\propto. \tag{8}$$

From (4, 6, 8) we find:

$$q\omega_3 = \frac{1}{2}\pi R^2(\omega_1 - \omega_2). \tag{9}$$

Substituting (1, 6, 8, 9) into (5), we find:

$$\frac{1}{4}\pi R^2(\omega_2^2 - \omega_1^2) + \frac{1}{4}q\omega_3^2 = 0 \tag{10}$$

or

$$\pi R^2(\omega_2^2 - \omega_1^2) + q\omega_3^2 = 0. \tag{11}$$

Combining (9, 11), we find:

$$\pi R^2(\omega_2^2-\omega_1^2)+\left(\frac{1}{2}\pi R^2(\omega_1-\omega_2)\right)^2/q=0$$

or

$$4\pi R^2(\omega_2^2-\omega_1^2)+\pi^2 R^4\,(\omega_1-\omega_2)^2/q=0$$

or

$$a(\omega_2^2-\omega_1^2)+b(\omega_1-\omega_2)^2=0 \tag{12}$$

or

$$a=4\pi R^2, b=\pi^2 R^4/q. \tag{13}$$

Solving (12), we find:

$$\omega_2^2(a+b)-2b\omega_1\omega_2+\omega_1^2(-a+b)=0$$

$$\omega_2=\frac{1}{2(a+b)}\left(2b\omega_1\pm\sqrt{4b^2\omega_1^2-4(a+b)\left(\omega_1^2(-a+b)\right)}\right)$$

$$=\frac{1}{2(a+b)}\left(2b\omega_1\pm\sqrt{4b^2\omega_1^2-4\left(b^2-a^2\right)\omega_1^2}\right)$$

$$=\frac{1}{2(a+b)}\left(2b\omega_1\pm\sqrt{4a^2\omega_1^2}\right)$$

$$\omega_2=\frac{(b\pm a)}{(a+b)}\omega_1. \tag{14}$$

Physically, there is a solution of the form

$$\omega_2=\frac{(b-a)}{(a+b)}\omega_1. \tag{15}$$

From (15, 9) we find:

$$q\omega_3=\frac{1}{2}\pi R^2(\omega_1-\omega_2)=\frac{1}{2}\pi R^2\left(\omega_1-\frac{(b-a)}{(a+b)}\omega_1\right)$$

$$=\frac{1}{2}\pi R^2\frac{2a}{(a+b)}\omega_1=\pi R^2\frac{4\pi R^2}{(a+b)}\omega_1=4\pi^2 R^4\frac{1}{(a+b)}\omega_1$$

or

$$\omega_3=4\pi^2 R^4\frac{1}{(aq+b)}\omega_1=4\pi^2 R^4\frac{1}{(aq+\pi^2 R^4)}\omega_1=$$

$$4\pi^2 R^4\frac{1}{(4\pi R^2 q+\pi^2 R^4)}\omega_1=\pi R^2\frac{1}{(q+\pi R^2/4)}\omega_1$$

or

$$\omega_3=\pi R^2\frac{1}{(q+\pi R^2/4)}\omega_1. \tag{16}$$

Consider the particular cases shown in Fig. 2. At the same time, from (8, 13, 15, 16) we find:

$$q(\propto)\approx\begin{cases}h^2 & if\ \ \propto\approx 0\\ \frac{1}{2}\pi R^2 & if\ \ \propto\approx\frac{\pi}{2}\end{cases}. \tag{17}$$

$$b(\propto) \approx \begin{cases} \pi^2 R^4/h^2 & if \ \propto \approx 0 \\ 2\pi R^2 & if \ \propto \approx \frac{\pi}{2} \end{cases} \tag{18}$$

$$\omega_2(\propto) \approx \begin{cases} \frac{(h^2-4\pi R^2)}{(h^2+4\pi R^2)}\omega_1 & if \ \propto \approx 0 \\ -\frac{1}{3}\omega_1 & if \ \propto \approx \frac{\pi}{2} \end{cases}. \tag{19}$$

$$\omega_3(\propto) = \begin{cases} \frac{\pi R^2}{(h^2+\pi R^2/4)}\omega_1 & if \ \propto \approx 0 \\ \frac{4}{3}\omega_1 & if \ \propto \approx \frac{\pi}{2} \end{cases} \tag{20}$$

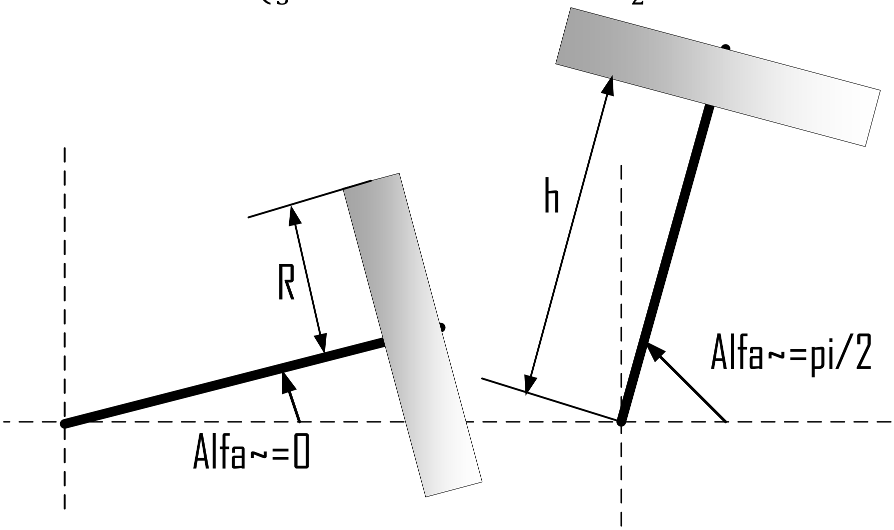

Fig. 2.

3. Forces acting on the spinning top

The resulting formulas show the change in the speed of its own rotation and the precession speed of the top in the process of falling - the change in the angle $\propto$ from $\frac{\pi}{2}$ to 0. The top goes into an inclined position under the action of force of gravity mg. But at the same time, the Coriolis force and the centrifugal force act on the top, depending on the rotation speeds and therefore changing their value depending on $\propto$. These forces counteract the force of gravity and therefore the spinning top falls very slowly. Obviously, with such deceleration, the source of the Coriolis force and centrifugal force consumes energy. Therefore, there is an energy source for this force. But this assumption is hampered by the persistent modern notion that the Coriolis force is a fictitious force. A fictitious force cannot deliver energy...

But consider these forces. Coriolis force

$$F_2 = -2m\omega_2 \times v, \qquad (1)$$

where v is the linear speed of precession; this is the speed of movement of p. B on the radius AB, which rotating with an angular speed ω_3 (see Fig. 1):

$$v = \omega_3 h\cos(\propto). \qquad (2)$$

In addition, a centrifugal force acts on the top, directed on horizontally AB from the center:

$$F_3 = m\omega_3^2 h\cos(\propto). \qquad (3)$$

At each moment of time, the forces acting on the top are F_2, F_3 and the force of gravity

$$F_4 = -mg, \qquad (4)$$

which directed vertically down. Thus, at each moment of time, the total force acts on the top

$$F_s = F_2 + F_3 + F_4. \qquad (5)$$

The horizontal and vertical projections of this force will be denoted as F_{sx} and F_{sy}, respectively. The force F_{sy} creates an overturning moment of rotation of the top around the pivot point, equal to

$$M = hF_{sx}\sin(\propto) - hF_{sy}\cos(\propto). \qquad (6)$$

In addition, the force F_s creates a pressure force on the pivot point, directed along the bar and equal to

$$T = -hF_{sx}\cos(\propto) - hF_{sy}\sin(\propto). \qquad (7)$$

Using these formulas, one can find the listed forces and the moment of rotation, as a functions of the angle $\propto$, at known speeds, as a functions of the angle $\propto$, found in the previous section

There is a certain angle $\propto_0$ at which the top takes a stable position, maintaining this angle of inclination $\propto_0$ for a long time. For $\propto_0=\propto_0$ the function $M(\propto) = 0$ and the derivative

$$dM(\propto)/_{d\propto} < 0.$$

These conditions allow us to find $\propto_0$ from the graph of the function $M(\propto)$.

4. Examles

The following figures show the following functions of the argument $\propto$:
F2x is horizontal projection of the Coriolis force,
F2y is vertical projection of the Coriolis force,
F3 is horizontal centrifugal force,
FSx is total horizontal force,

FSy is total vertical force,
M is torque,
T is the pressure force on the hinge in the direction of the bar,
om1, om2, om3 are the angular velocities $\omega_1, \omega_2, \omega_3$ respectively.

The color of the graph line is indicated in brackets: (b) - blue, (r) - red, (g) - green.

Example 1.

In this example, the functions listed above are found at h=0.6; R=0.15; ω_1=6.5, m=1 - see fig. 1. The dotted vertical in the third window highlights the point where $\propto_0 = 0.33$. At this point, the function

$M(\propto) = 0$ and its derivative $dM(\propto)/_{d\propto} < 0$.

Therefore, at this point the top is in a stable position.

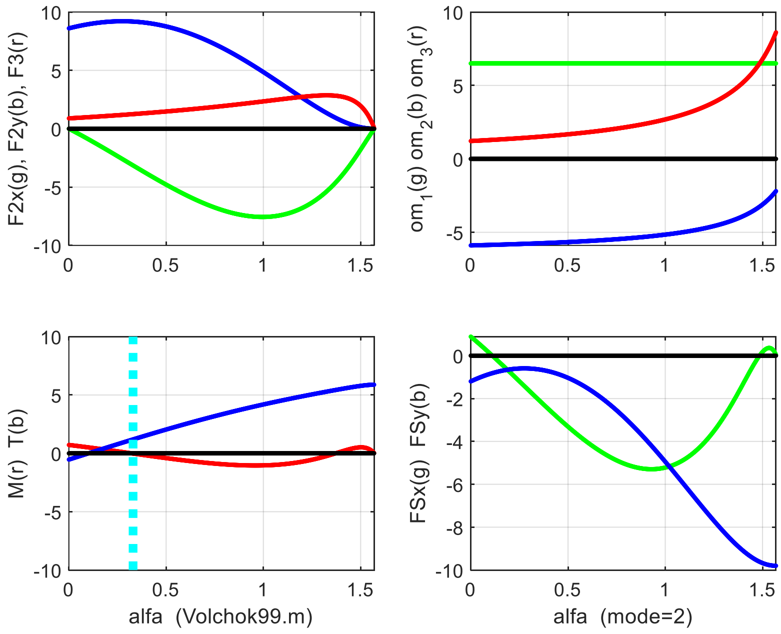

Fig. 1.

Example 2.

In this example, the functions listed above are found at h=0.8; R=0.15; ω_1=5.5, m=1 - see fig. 2. The dotted vertical in the third window highlights the point where $\propto_0 = 1.04$. At this point, the function

$M(\propto) = 0$ and its derivative $dM(\propto)/_{d\propto} < 0$.

Therefore, at this point the top is in a stable position.

The practical implementation of such cases, which are considered in examples 1 and 2, are ordinary top toys. Another option is shown in Fig. 3 of [6].

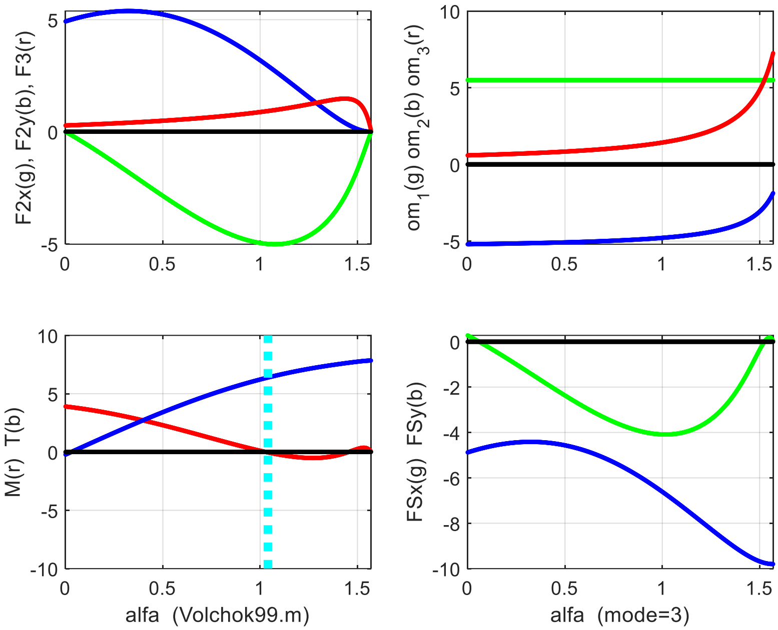

Fig. 2.

Fig. 3.

Example 3.

In this example, the functions listed above are found at h=0.8; R=0.2; ω_1=60, m=1 - see fig. 4. The dotted vertical in the third window highlights the point where $\propto_0 = \frac{\pi}{2}$. At this point, the function

$M(\propto) = 0$ and its derivative $dM(\propto)/_{d\propto} < 0$.

Therefore, at this point the top is in a stable position.

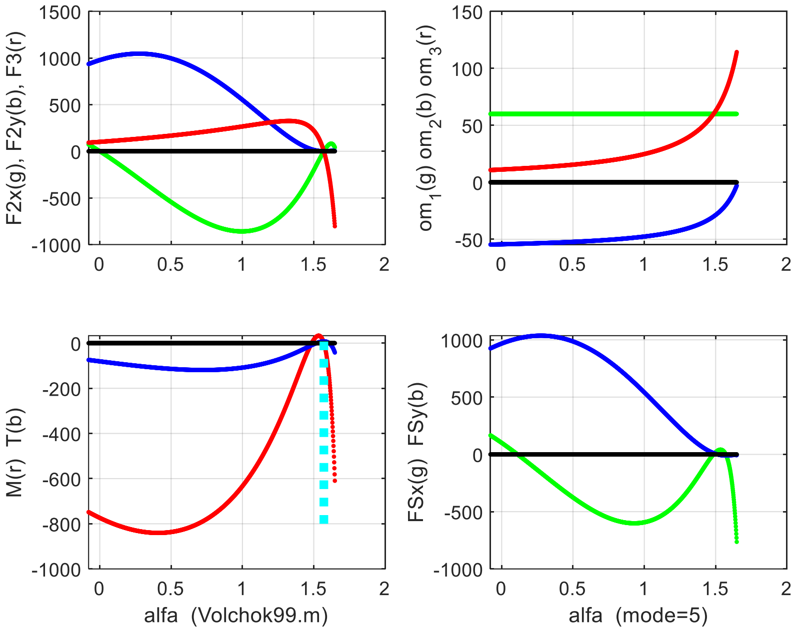

Fig. 4.

The practical implementation of this case, i.e. when the barbell of top is in a <u>vertical</u> position, and the disk is <u>horizontal</u>, can be found on the Internet - see fig. 5. It can be seen that the top in this case hangs motionless in the air, i.e. the <u>vertical force acting on the top is zero</u>. In our example, you can also see that the vertical force acting on the top is zero - see the F2x force graph in the first window.

Fig. 5.

Example 4.

In this example, the functions listed above are found at h=0.18; R=0.2; ω_1=26, m=1 - see fig. 6. The dotted vertical in the third window highlights the point where $\propto_0 = 0$. At this point, the function

$$M(\propto) = 0 \text{ and its derivative } {dM(\propto)}/{d\propto} < 0.$$

Therefore, at this point the top is in a stable position.

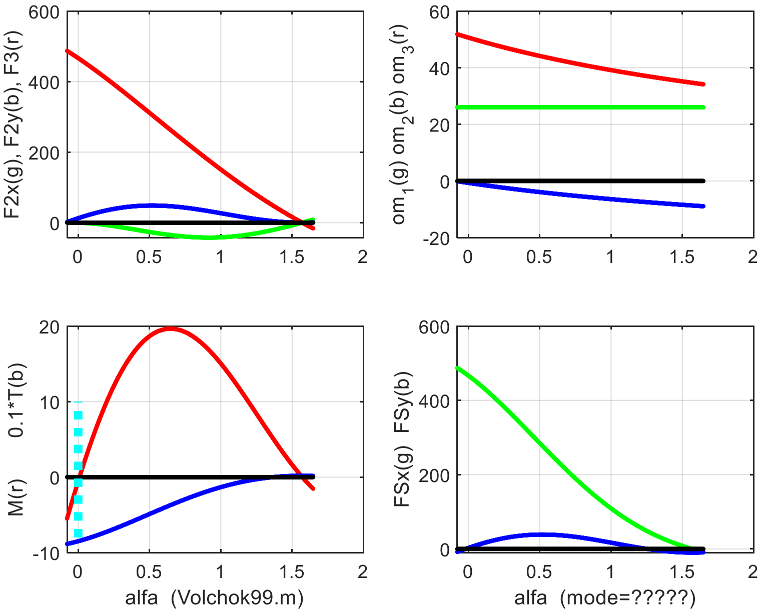

Fig. 6.

The practical implementation of this case, i.e. when the barbell of top is in a horizontal position, and the disk is vertical, is considered in many publications - see, for example, fig. 6 from [3], fig. 7 from [4], [9]. It can be seen that the top in this case rotates on a horizontal rod. Measurements in [4] show that its top weight is zero. There are no explanations. In our example, you can also see that the vertical force acting on the top is zero - see the F2x force graph in the first window. Thus, both practically and theoretically it is shown that in this position the top is weightless.

Such an experiment is also considered in [8] with reference to [9]. The article begins by stating that for such a device "*the detailed mechanics of which are still an enigma*". The author of this article has developed three new Euler equations that are much longer than those found in textbooks. The resulting nonlinear equation is modeled in the MATLAB system to obtain and visualize a numerical solution. Under certain conditions, providing small oscillations of the gyroscope axis (maximum oscillation of eight degrees in the angle of inclination) near the horizontal plane through the fulcrum, linearization is performed, which is successfully compared with the above-mentioned nonlinear numerical solution. The author argues that the numerical solution under certain conditions «*is crucial to the debate about whether such an engine may produce a net thrust, or not. A relevant paradox is resolved*». The question of where the source of forces is is up in the air, as is the device in question.

In our example, you can also see that the vertical force acting on the top is zero - see the F2x force graph in the first window. Thus, both practically and theoretically it is shown that in this position the top is weightless.

Fig. 7.

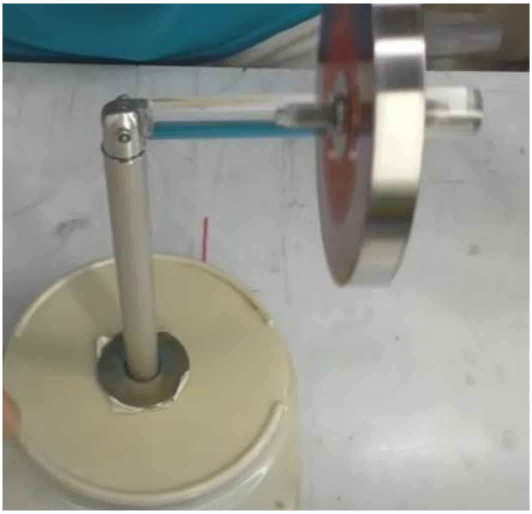

Fig. 8.

5. Conclusions

So, the action of Coriolis forces and centrifugal forces leads to the fact that

- the top is kept from falling for a long time,
- there is (not always) such an inclined position of the top, in which the Coriolis force balances the horizontal centrifugal force and the vertical force of gravity;
- in this position, the kinetic energy of the top remains unchanged and equal to the kinetic energy obtained during the start;
- the fall of the top from the indicated stable position is caused by a gradual decrease in its kinetic energy due to braking;

Such an action of the Coriolis forces and centrifugal forces is possible only if they can do work, i.e. are real powers. This proves the reality of these forces. On the other hand, a mathematical proof of this fact is given in [1]. It shows that these forces can be justified as a consequence of Maxwell's equations for gravitomagnetism, and the energy source for these forces is the Earth's gravitational field. But even in the absence of such evidence, there are many doubts about the assertion that these forces are fictitious [2].

So, the article gives a mathematical description of the top, which is still missing, which uses the known facts of mechanics and the notion of the Coriolis force, which is NOT accepted in mechanics, as a real force. Also, this explanation does NOT use gyroscope theory. However, explanations have been obtained for numerous experiments that have so far remained inexplicable.

The author is repeatedly pointed out the MEPhI experiment [5], in which the fictitiousness of the Coriolis force is convincingly proved - see fig. 1. Consider this proof. But first of all, I want to note that I am not criticizing the lecturer, nothing personal. The experimenter is one of the best teachers in Russia at one of the best physics institutes in Russia.

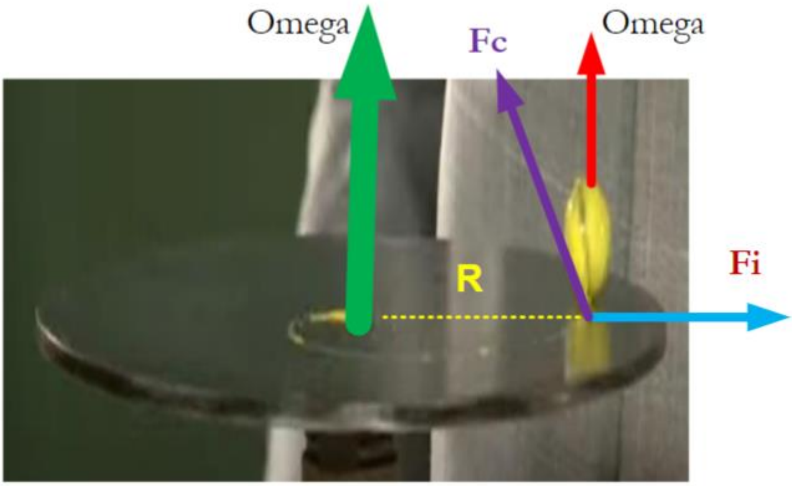

Fig. 1.

The disk rotates with an angular speed ω. The ball is pushed out by the experimenter from the central hole along the radius R of the disk and moves along the radius under the action of the inertial force F_i with a linear speed v. In this case, the Coriolis force F_c, perpendicular to the radius R, acts on it. As a result, the ball describes a spiral, moving in the direction opposite to the rotation of the disk (therefore, we cannot suspect that the disk is pulling it with the force of friction). The force that pulls the ball tangentially is the Coriolis force:

$$F_c = -2m\omega \times v, \quad (1)$$

The question is, where did this power come from?

Further, the experimenter argues that this force appeared because the ball moves in a coordinate system associated with a rotating disk, and the angular speed of the disk ω enters formula (1). The experimenter is one of the best teachers in Russia at one of the best physics institutes in Russia. Modern Physics speaks through him.

We can suggest a modification of the experiment. Let there be a thin plane above the disk and let the ball lie on this plane. At the same time, we completely exclude the mechanical influence of the disk. Only the coordinate system of the disk remains. So, we push the ball and bring the rotating disk to the plane. In this case, the Coriolis force appears, moving the ball. No fraud! No wonder, because there really is no power. The question of where this power came from is superfluous. Such is nature, says physics. Physics is fine with that. But how can a physicist accept such an explanation?! It would be more honest to admit that physics has no explanation and it must be sought. Or, following the example of Mach, to assume the influence of celestial bodies.

But in this experiment, the explanation is much simpler. The ball, lying in the central hole, rotates together with the disk with an angular speed ω and continues to rotate after the central impact of the experimenter. The speed ω in formula (1) is the speed of the ball, not the disk.

It is the lack of a clear answer to the question "where does the power come from?" and led to the emergence of such a theory: it was necessary to find the answer so that the students respected the teachers! It would be possible not to build hypotheses about the nature of this force (as Newton did with the force of inertia). But the times at that time were, apparently, not the same. And off we go. The force was not recognized as real, but fictitious and incapable of doing work. The following physicists had to show miracles of ingenuity in order to find the coordinate system due to which the Coriolis force appeared, and the source of energy that works for it. This issue is considered in detail by Astakhov in [2].

Appendix 1

Table 1

Angular speed	Moment of inertia	Angular momentum	Kinetic energy
ω_1	$J_1 = \frac{1}{2}mR^2$	$L_1 = J_1\omega_1$	$W_1 = \frac{1}{2}L_1\omega_1 = \frac{1}{4}J_1\omega_1^2 = \frac{1}{8}mR^2\omega_1^2$
ω_2	$J_2 = J_1$	$L_2 = J_2\omega_2$	$W_2 = \frac{1}{2}L_2\omega_2 = \frac{1}{4}J_2\omega_2^2 = \frac{1}{8}mR^2\omega_2^2$
ω_3	$J_3 = mq$, где $q = h^2 - \left(\frac{2}{\pi}h^2 - R^2\right) \propto$ - см. (1.8)		
		$L_3 = J_3\omega_3$	
			$W_3 = \frac{1}{2}L_3\omega_3 = \frac{1}{4}J_3\omega_3^2 = \frac{1}{4a}mR^2\omega_3^2$

Here we determine the parameters of the state of the spinning top at moments 1 and 2. In Table. 1 shows the basic formulas, where the following notation is accepted

$\propto$ is the angle of inclination of the spinning top to the rolling plane,
m is spinning top mass,
g is the acceleration of gravity,
mg is gravity,
R is the spinning top radius,
h is the height of the spinning top - see the segment OB in fig. 1,
ω_1 is angular speed of spinning top rotation around the vertical diameter at moment 1,
ω_2 is angular speed of spinning top rotation around the vertical diameter at moment 2,
ω_3 is angular speed of precession at moment 2,
v is linear speed of precession,
L is angular momentum,
J is moment of inertia,
W is energy.

References

1. Khmelnik S.I. Four forces in mechanics. https://www.academia.edu/80854442/, https://doi.org/10.5281/zenodo.7004069
2. Астахов А.А. Физика. Порядок вещей или осознание знаний, 2006, https://www.litres.ru/a-a-astahov/fizika-dvizheniya-alternativnaya-teoreticheskaya-mehanika-ili-osoznanie-znaniya/chitat-onlayn/
3. Anti-Gravity Wheel? https://www.youtube.com/watch?v=GeyDf4ooPdo
4. И. Белецкий. ГИРОСКОП ТЕРЯЕТ ВЕС ? https://www.youtube.com/watch?v=FwrlRpC8BDA
5. Шарик, катящийся по вращающейся платформе, НИЯУ МИФИ, https://www.youtube.com/watch?reload=9&v=LkrmALM8TsA
6. Летающий спинер, https://www.youtube.com/watch?v=rDDfKVjjG2g
7. Хмельник С.И. Новые уравнения для волчка. Доклады независимых авторов, ISSN 2225-6717, 2020, 50(1), 65–74. https://doi.org/10.5281/zenodo.4047902

8. Christopher G. Provatidis. Forced precession of a gyroscope and its application to Laithwaite's engine. European Journal of Physics, Volume 42, Number 3, https://iopscience.iop.org/article/10.1088/1361-6404/abce88/meta
9. Professor Eric Laithwaite gives a demonstration of a large gyro wheel, https://www.youtube.com/watch?v=JRPC7a_AcQo