# Same Last Digits of Consecutive Primes

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## Notation

N is a Whole Number
P is a prime number
The last digit is the digit that is in the one's place of any whole number.
Divisible means being able to divide into smaller whole numbers.
A possible prime is any number that ends in a 1, 3, 7, or 9
Function R(N) computes the Nth prime
LN is the list of all whole numbers
G is any possible prime gap

## Theorem

It was found in 2016 that prime numbers had an unexpected bias. The last digit of any prime was unlikely to repeat itself. Even more unexpected is that repetitions of 3 and 7 are more likely than repetitions of 9 and 1. The reason for this phenomenon is unknown, but I think I have found the solution. This bias towards not repeating is due to the fact that gaps of 10N are less likely than gaps of 2N, 4N, 6N, and 8N, and they are also less likely to happen because they are spread out across 4 numbers, while the other numbers are spread out between 3 numbers.

# Proof

## 1. Possible Primes are eliminated evenly

It is known that for N^2, The last Digit of N^2 is based on the last digit of N.

| Last Digit Of N | Last Digit of N^2 |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | 4 |
| 3 | 9 |
| 4 | 6 |
| 5 | 5 |
| 6 | 6 |
| 7 | 9 |
| 8 | 4 |
| 9 | 1 |

But the last digit of a prime cannot be 0, 2, 4, 6, or 8, (besides 2) because then it would be divisible by 2, and the last digit of a prime number cannot be 5, because then it would be divisible by 5. So the only possibilities for the last digit of prime numbers are 1, 3, 7, and 9. I will contract our table to reflect this.

| Last Digit of P | Last Digit of P^2 |
|---|---|
| 1 | 1 |
| 3 | 9 |
| 7 | 9 |
| 9 | 1 |

This means that any P^2 can only end on 1 or 9. We can expand the table to include Values of P^2+NP

| Last Digit Of P | Last Digit Of P^2 | P^2+ P | P^2+ 2P | P^2+ 3P | P^2+ 4P | P^2+ 5P | P^2+ 6P | P^2+ 7P | P^2+ 8P | P^2+ 9P | P^2+ 10P |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 |
| 3 | 9 | 2 | 5 | 8 | 1 | 4 | 7 | 0 | 3 | 6 | 9 |
| 7 | 9 | 6 | 3 | 0 | 7 | 4 | 1 | 8 | 5 | 2 | 9 |
| 9 | 1 | 0 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |

As you can see, there is a nice cyclic pattern. It is also seen that any number, P^2+(2N+1)P is divisible by 2, so they cannot be prime. Meaning we can contract the table once more.

| Last Digit Of P | Last Digit Of P^2 | P^2+2P | P^2+4P | P^2+6P | P^2+8P | P^2+10P |
|---|---|---|---|---|---|---|
| 1 | 1 | 3 | 5 | 7 | 9 | 1 |
| 3 | 9 | 5 | 1 | 7 | 3 | 9 |
| 7 | 9 | 3 | 7 | 1 | 5 | 9 |
| 9 | 1 | 9 | 7 | 5 | 3 | 1 |

This table shows that possible primes are eliminated evenly. What about values below P^2? Let us make another table.

| Last Digit Of P | Last Digit Of 2P | Last Digit Of 3P | Last Digit Of 4P | Last Digit Of 5P | Last Digit Of 6P | Last Digit Of 7P | Last Digit Of 8P | Last Digit Of 9P | Last Digit Of 10P | Last Digit Of 11P |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 |
| 3 | 6 | 9 | 2 | 5 | 8 | 1 | 4 | 7 | 0 | 3 |
| 7 | 4 | 1 | 8 | 5 | 2 | 9 | 6 | 2 | 0 | 7 |
| 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | 9 |

In this table, values divisible by 2 are always at 2NP, so we can contract it.

| Last Digit Of P | Last Digit Of 3P | Last Digit Of 5P | Last Digit Of 7P | Last Digit Of 9P | Last Digit Of 11P |
|---|---|---|---|---|---|
| 1 | 3 | 5 | 7 | 9 | 1 |
| 3 | 9 | 5 | 1 | 7 | 3 |
| 7 | 1 | 5 | 9 | 3 | 7 |
| 9 | 7 | 5 | 3 | 1 | 9 |

This table also shows that possible primes are eliminated evenly. Therefore we can conclude that all primes eliminate every possible prime ending evenly

## 2. 1 and 9 occur less often than 3 and 7

Let us have a look at the 8 primes between 0 and 20.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 |

1/8 of the digits end with 1. 2/8 of the digits end with 3. 2/8 of the digits end with 7. 1/8 of the digits end with 9. With the other 2/8 being 2 and 5, which stop there. 3 and 7 are more apparent than 1 and 9 because 9 is a multiple of 3 and 1 is not prime. Let us look at the 4 primes between 20 and 40.

| | | | |
|---|---|---|---|
| 23 | 29 | 31 | 37 |

1,3,7 and 9 are the ending digits of an equal amount of primes. 3 and 7 are again more apparent than 1 and 9. Let us look at the 5 primes between 40 and 60.

| | | | | |
|---|---|---|---|---|
| 41 | 43 | 47 | 53 | 59 |

1/5 of the digits end with 1. 2/5 of the digits end with 3. 1/5 of the digits end with 7. 1/5 of the digits end with 9. 3 is now favored over 7, and both 3 and 7 are favored over 9 and 1. Let us look at the 5 primes between 60 and 80.

| | | | | |
|---|---|---|---|---|
| 61 | 67 | 71 | 73 | 79 |

2/5 of the digits end with 1. 1/5 of the digits end with 3. 1/5 of the digits end with 7. 1/5 of the digits end with 9. Now the most favorable to least is 3,7,1,9. Let us look at the 3 primes between 80 and 100.

| | | |
|---|---|---|
| 83 | 89 | 97 |

0/3 of the digits end with 1. 1/3 of the digits end with 3. 1/3 of the digits end with 7. 1/3 of the digits end with 9. Now the last digit of primes is favored as 3,7,9,1.

And because possible primes are eliminated evenly, we can assume that the last digits of primes, 1,3,7, and 9, are evenly distributed after those initial 8. But because those initial few have more 3s and 7s than 1s and 9s, 3s and 7s will be more than 50% of prime last digits, no matter how many terms you look at. That explains the bias to 3s and 7s.

## 3. Bias towards non-repetition.

| Last Digit Of P | Last Digit Of P^2 | P^2+2P | P^2+4P | P^2+6P | P^2+8P | P^2+10P |
|---|---|---|---|---|---|---|
| 1 | 1 | 3 | 5 | 7 | 9 | 1 |
| 3 | 9 | 5 | 1 | 7 | 3 | 9 |
| 7 | 9 | 3 | 7 | 1 | 5 | 9 |
| 9 | 1 | 9 | 7 | 5 | 3 | 1 |

| Last Digit Of P | Last Digit Of 3P | Last Digit Of 5P | Last Digit Of 7P | Last Digit Of 9P | Last Digit Of 11P |
|---|---|---|---|---|---|
| 1 | 3 | 5 | 7 | 9 | 1 |
| 3 | 9 | 5 | 1 | 7 | 3 |
| 7 | 1 | 5 | 9 | 3 | 7 |
| 9 | 7 | 5 | 3 | 1 | 9 |

Looking at the repetition of the last digit of P happens at P*(10N+1) or P+10N when N≥1

This happens with P*(10N+1) because P*10 will give you P moved to the left 1 decimal, making the last digit a 0, and then when you add P one more time, that 0 is replaced by the last digit of P. P*(10N+1) cannot be prime because it is divisible by 10N+1.
However, P+10N could be prime. For example, 11+10 is 21, which is composite, but 11+20 is 31, which is prime. 3+10 is 13, which is prime, 3+20 is 23, which is prime, and 3+30 is 33, which is composite. But if P+10N is divisible by any P, then the number is composite. This means that any prime pair that has the same last digit is P and P+10N when P+10N is prime and all numbers between P and P+10N are composite.

So how often are P and P+10N next to each other? Let us find out.

LN is divisible by 2 1/2 of the time. LN is divisible by 3 1/3 of the time. LN is divisible by 4 1/4 of the time LN is divisible by 5 1/5 of the time. This pattern goes on forever, being generalized LN is divisible by N exactly 1/Nth of the time when LN is the list of all whole numbers. But it's not just 1/2 or 1/3 of the whole numbers that are composite. What happens when these values mix?

Well, we can say that 1/2 of all values of 3N are divisible by 2, so we can conclude that 3 only affects primality 1/6 of the time. Therefore we can say that any N is divisible by 2 or 3 2/3 of the time. We can now say that 2/3 of all values of 5 are divisible by 2 or 3, so we can conclude that 5 only affects primality 1/15 of the time. Therefore we can say that any N is divisible by 2, 3, or 5 11/15 of the time. And again, a pattern develops. For values from P(1) to P(N), Those primes eliminate several possible primes that grow in size when more values are added, but the growth gets smaller to higher values. This value can be expressed as a summation.

$$A(N) = \sum_{M=2}^{N} ( 1/R(M)*1/R(M-1))$$

When function R(N) computes the Nth prime for values greater than 1, and R(N) when N is 0 is equal to 1. What this function computes is the ratio of the numbers divisible by R(1) through R(N) below R(1)*R(2)...R(N-1)*R(N) to the product of numbers R(1) through R(N). For example, A(3)=11/15. 2*3*5 is 30, so I will adjust the value to 22/30. This number now means that there are 22 Numbers under 30 that are divisible by 2,3 and 5. We can change A(N) to reflect this change and count all numbers divisible by 2,3 and 5 <= 30

$$B(N) = \sum_{M=1}^{N} (1/R(M)*1/R(M-1)) * \prod_{O=1}^{N} (R(O))$$

B(N) now counts all numbers ≤ 30 that are divisible R(1) to R(N). However, this function counts R(1) to R(N) as divisible, which they are, but we want this function to count composite numbers. To get that value, we must subtract N from our function

$$C(N) = -N+\sum_{M=1}^{N} (1/R(M)*1/R(M-1)) * \prod_{O=1}^{N} (R(O))$$

The function above, C(N), now computes the number of composites below the product of all primes from R(1) to  R(N). We can subtract the sum from 1 to get the amount of numbers not divisible by R(1) to R(N). Then we can add N to include all primes from R(1) to R(N) as prime.

$$D(N) =( 1-\sum_{M=2}^{N} (1/R(M)*1/R(M-1))) * \prod_{O=2}^{N} (R(O))$$

The function above, D(N), now calculates the number of primes below the product of all primes from R(1) to the R(N).  D(N)-1 will calculate the number of prime gaps. A subtraction of 1 is necessary because there is no gap behind 2, but there is one behind every other prime. We can now take the value of F(N)=(C(N)/(D(N)-1)) to find the average gap between 2 and R(N). As N increases, It can be said that F(N) also increases. Let us make a table that shows the value of F(N) for growing values of N

| F(1) | F(2) | F(3) | F(4) | F(5) | F(6) | F(7) | F(8) | F(9) | F(10) |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 2.44 | 2.90 | 3.26 | 3.47 | 3.60 | 3.69 | 3.76 | 3.81 |

| F(11) | F(12) | F(13) | F(14) | F(15) | F(16) | F(17) | F(18) | F(19) | F(20) |
|---|---|---|---|---|---|---|---|---|---|
| 3.84 | 3.86 | 3.88 | 3.90 | 3.91 | 3.92 | 3.93 | 3.94 | 3.95 | 3.95 |

| F(21) | F(22) | F(23) | F(24) | F(25) |
|---|---|---|---|---|
| 3.96 | 3.96 | 3.97 | 3.97 | 3.97 |

The value of F(N) increases as N increases, and because there are an infinite number of primes, this number will get larger, but never be uncountable because there will always be another prime. We know that F(N) cannot be infinity,  so we can assume that it includes all numbers from 1 to LIM N→∞.  However, The list of all whole numbers, LN, has an infinite amount of digits, and 1/10 of LN is divisible by 10, so we can say that a (1/10)-H of the average gap between primes are divisible by 10 as H→∞. We can say an average prime gap is divisible by 1/N  (1/N)-H the time. So we can say that an average gap of 10N, because gaps of 10N make the same last digits possible, happens (1/10N)-H of the time. Let us make a table that shows the last digit based on distance from a prime.

| Last digit of prime | Last digit of number after a gap of 2 | Last digit of number after a  gap of 4 | Last digit of number after a gap of 6 | Last digit of number after a gap of 8 | Last digit of number after a gap of 10 |
|---|---|---|---|---|---|
| 1 | 3 | 5 | 7 | 9 | 1 |
| 3 | 5 | 7 | 9 | 1 | 3 |
| 7 | 9 | 1 | 3 | 5 | 7 |
| 9 | 1 | 3 | 5 | 7 | 9 |

We can see that the last digit of a prime will repeat about 1/10 of the time.

## 4. 1 and 9 will be the last digit of P+G more than 7 and 3

For any last digit of P, The last digit of P+2 has a 1/4 chance of being 3,5,9 or 1, and a 0/4 chance of being 7. Since there is a 1/2 chance that the average prime gap is 2, we can say that, for any last digit of P, The last digit of P+2 has a 1/8 chance of being 3,5,9 or 1 and prime and a 0/8 chance of being 7 and prime. This can be generalized.

For any last digit of P, The last digit of P+G has a 1/4 chance of being 1+G, 3+G, 7+G, and 9+G, and a 0/4 chance of being 5+G. Since there is a 1/G chance that the average prime gap is G, we can say that for any last digit of P, P+G has a (1/G*1/4) chance of being  1+G, 3+G, 7+G, and 9+G, and prime and a 0/4*1/G chance of being 5+G and prime. We can find the avg. last digit of P+G by adding together every G from 2 to 2N.

After any P, the last digit of P+G when G is 2 to 10 will be 1 and prime about 39/160, or .243, of the time, 3 and prime about 61/240, or .145, of the time, 7 and prime about 77/480, or .160, of the time, and 9 and prime about 107/480, or .223 of the time. We can see that the most likely last digit for prime gaps less than or equal to 10 is 1, followed by 9, then 7, then 3. Since there is a pattern to the numbers 1+G, 3+G, 7+G, and 9+G, and a 0/4 chance of being 5+G relative to each gap, we can say with certainty that, for any last digit of P, the chance that the last digit of P+G is always going to go in order from highest to lowest, 1,9,7,3. This is possible because, after P+10, digits repeat, so these numbers stay in the same order when G increases by 10.

# 5. Application

So, from a culmination of this paper, we can say that there are more 3s and 7s than 9s and 1s, repetition of digits happens about 1/10 of the time, and the last digits that are most likely to follow any P are 1,9,7,3. We also find that the most likely numbers to come after 1 are 3,7,9,1 respectively, the most likely numbers to come after 3 are 7,9,1,3 respectively, the most likely numbers to come after 7 are 9,1,3,7 respectively, and the most likely numbers to come after 9 are 1,3,7,9 respectively. We can assign weights to these values in a table. Notice that after a last digit of 3 appears, the values are weighted 2 less, this is because there is a 5 after 3, therefore making the following values less likely.

| | | | | |
|---|---|---|---|---|
| 1 | 3 | 7 | 9 | 1 |
| Weighted 1 | 10 | 6 | 4 | 2 |
| 3 | 7 | 9 | 1 | 3 |
| Weighted 3 | 8 | 6 | 4 | 2 |
| 7 | 9 | 1 | 3 | 7 |
| Weighted 7 | 10 | 8 | 6 | 2 |
| 9 | 1 | 3 | 7 | 9 |
| Weighted 9 | 10 | 8 | 4 | 2 |

That is a good weighted table, but we are missing some things. If it is followed by 1 or 9, add 1, and if it is followed by 3 or 7, subtract 1, because 1 and 9 are the most likely to come after, and 3 and 7 are the least likely. We can also add a 1 if it starts with 3 and 7 and subtract 1 if it starts with 1 and 9 because 3 and 7 are the most likely start and 1 and 9 are the least likely.

| | | | | |
|---|---|---|---|---|
| 1 | 3 | 7 | 9 | 1 |
| Weighted 1 | 8 | 4 | 4 | 2 |
| 3 | 7 | 9 | 1 | 3 |
| Weighted 3 | 8 | 8 | 6 | 2 |
| 7 | 9 | 1 | 3 | 7 |
| Weighted 7 | 12 | 8 | 6 | 2 |
| 9 | 1 | 3 | 7 | 9 |
| Weighted 9 | 10 | 6 | 2 | 2 |

We can arrange these values in a table to reflect how likely a digit is to come after 1, 3, 7, or 9

| | | | |
|---|---|---|---|
| 1,1 - 2 | 1,3 - 8 | 1,7 - 4 | 1,9 - 4 |
| 3,1 - 6 | 3,3 - 2 | 3,7 - 8 | 3,9 - 8 |
| 7,1 - 8 | 7,3 - 6 | 7,7 - 2 | 7,9 - 12 |
| 9,1 - 10 | 9,3 - 6 | 9,7 - 2 | 9,9 - 2 |

Now I will separate it into different prime gaps.

| | | | | |
|---|---|---|---|---|
| Gap of 2 | 1,3 - 8 | 7,9 - 12 | 9,1 - 10 | |
| Gap of 4 | 3,7 - 8 | 7,1 - 8 | 9,3 - 6 | |
| Gap of 6 | 3,9 - 8 | 1,7 - 4 | 7,3 - 6 | |
| Gap of 8 | 1,9 - 4 | 9,7 - 2 | 3,1 - 6 | |
| Gap of 10 | 1,1 - 2 | 3,3 - 2 | 7,7 - 2 | 9,9 - 2 |

This table shows that gaps of 2 are most common, followed by 4, 6, 8, 10. It also shows that, not only are prime gaps of 10N less likely, but that there are more possible last digits for prime gaps of 10N than with any other gap, so the percentage of a prime gap of 10N will always be smaller than gaps of 10N+2, 10N+4, 10N+6 and 10N+8 because 10N cannot end in 5. After all, no prime can have its first digit be 5 besides 5.

## 6. Conclusion

Therefore it can be concluded that the bias towards not repeating is due to the fact that gaps of 10N are less likely than gaps of 2N, 4N, 6N, and 8N, and they are also less likely to happen because they are spread out across 4 numbers, while the other numbers are spread out between 3 numbers.

# References


[1] Oliver, R. J., & Soundararajan, K. (2016). Unexpected biases in the distribution of consecutive primes. *arXiv*. https://doi.org/10.1073/pnas.1605366113

[2]  A. Ash, L. Beltis, R. Gross, and W. Sinnott. Frequencies of successive pairs of prime residues. Exp. Math., 20(4):400–411, 2011.

[3] Numberphile. (2016, May 4). *The Last Digit of Prime Numbers - Numberphile* [Video]. YouTube. https://www.youtube.com/watch?v=YVvfY_lFUZ8

[4] Peculiar pattern found in ‘random’ prime numbers. (2016, March 14). *Nature*. https://www.nature.com/articles/nature.2016.19550?error=cookies_not_supported&code=3e2a3b8f-d19d-4c22-bcad-cc347136c51b