

Quantum Average Momentum Compared with Classical Statistical Pressure Part III

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Newtonian force is based on a change in momentum in time, but momentum is a vector and so may change in magnitude and direction. In many cases, the change is of one form or the other. For example, for a bound particle the change is in kinetic energy i.e. in magnitude, while for a particle moving at constant speed in a circle or pressure in a Maxwell-Boltzmann gas, the change is in direction. For an MB gas, the probability for p and $-p$ are the same so the average momentum at any point x is 0 and thus chunks of the gas do not move, even though a single particle in the gas moves. The pressure, on the other hand, is the average of $2pv$ ($2p$ for impulse, v for flux) and so pressure for p and $-p$ do not cancel.

A quantum free particle is based on $\exp(ipx)$ which does not contain time and is a dynamical probability. This is both an eigenfunction of the momentum operator $-i\hbar/dx$ and the kinetic energy operator $-\hbar^2/2m d^2/dx^2$, yet as pointed out in Part II, there are two pieces ($\cos(px), \sin(px)$) with a spatial density and hence a kind of pressure/force. These, however, combine to give the sense of constant motion in one direction. We argue this is due to a dynamical probability, unlike a static balance of pressures/forces found in an MB gas or Newtonian statics. There is a density with pressure distribution $\cos(px)$, but this is continuously changing because the particle moves in one direction with constant p . Thus the two pressure distributions “balance” so to speak to create a constant p . Interference of $\exp(ipx)$ and say $\exp(-ipx)$ may “bring out” a “static” spatial density i.e. $2\cos(px)$, but behind the scene the dynamical probabilities still exist. It is these which “hold” the $\cos(px)$ together which appears strange from a static point of view. For example, a quantum bound state has a static $W^*(x)W(x)$ spatial density at the same time that the single particle has an average classical kinetic energy $KE(x)=E_n-V(x)$. Interference also gives rise to the notion of an impulse-pressure in quantum as one ignores the $i\sin(px)$ and $i\sin(-px)$ terms.

Creating a pressure scenario with p and $-p$ leads to two considerations. First, there is an impulse-pressure (defined without flux) at each x point and secondly, this impulse-pressure changes with x . The time-independent Schrodinger equation is formulated in terms of energy (i.e. a scalar): $KE(x)+V(x)=E_n$ where $KE(x)=\{-\hbar^2/2m d^2/dx^2\} W$ and $W=\sum_p a(p)\exp(ipx)$. Thus, as in Newtonian mechanics, one thinks of $-dV/dx$ solely accelerating a particle. The quantum particle however, is described by an ensemble $W(x)$ which has pressure. This pressure is due to a combination of the form $\exp(ipx)$ and $V(x)$, so $V(x)$ does not simply change root mean square speed. $\exp(ipx)$, by itself, creates pressure when interference occurs as in a two slit scenario.

We try to examine these ideas in this note.

Free Quantum Particle

Newtonian mechanics equates force with a change of momentum in time, but momentum is a vector and so both its magnitude and direction may change. For a bound state classical problem, one thinks only of kinetic energy changing and deals with the magnitude. A particle moving in a circle in two dimensions, however, has constant speed yet still exhibits acceleration

i.e. there is a force. The mathematical form of this scenario is very similar to $\exp(ipx)$, the “probability” or square root flux of a quantum free particle. In particular, the particle moving in a circle is two dimensional and each dimension x, y is described in terms of time through $\cos(wt)$, $\sin(wt)$ where w is the angular frequency. There is an acceleration in time in x and y which is periodic. In classical physics, one may associate spatial density with the time spent in a little dx piece (1). Acceleration (speed change) implies a change in time and hence a change in density. Thus there is a classical change in density along x and y , both of which are periodic.

$\exp(ipx)$, which is an eigenfunction of both the spatial generator d/dx multiplied by $-i$ and the kinetic energy operator, is of a similar structure, but the two dimensions are not two spatial dimensions. There is only one spatial dimension x linked with a density $\cos(px)$, but the second dimension takes the place of time showing how density changes in time or in other words the direction of motion. Thus we argue that just like the particle moving in a circle, there are two density distributions which each represent acceleration even though the overall momentum is constant.

In other words, instead of thinking of a static balance of forces as in a Maxwell-Boltzmann gas or Newtonian statics, we consider a dynamical balance of forces. The spatial density-pressure of $\cos(px)$ is balanced by a force linked with $\sin(px)$ due to its change in time which is continuously happening. A particle is not a point, but for a given p a spatial distribution with a length of $.5 hbar/p$. In other words, the probability $\exp(ipx)$ or square root flux is “all about motion” with a built-in action-reaction force mechanism, we argue.

This differs from Newtonian mechanics for which one has x at a given t . With $\exp(ipx)$, one has two distributions representing two different times indicating how density changes. In order to do this, there needs to be an internal ruler which is created by the wavelength $hbar/p$. This ruler length, the associated x dependent probability and associated force/acceleration allow for interactions not seen in the Newtonian one dimensional $p(x)$ which changes in magnitude.

In particular, impulse-based pressure emerges when interference occurs. We note that $\exp(ipx)$ does not contain time. This is why two spatial densities are needed in the first place. Thus, while the particle moves on average as $x=p/m t$, inside a wavelength one cannot follow the particle in time, but rather has an x dependent probability scheme.

Consider the case of two slit interference with the slits being about a wavelength apart. An interaction occurs of a p vector, not magnitude, nature. $\exp(ipx)$ is like a probability, but now there are two paths, through one slit and the other, i.e. two vector r directions with p lying along each, but with the same p magnitude i.e.

$$W(x) = \exp(ip \cos(a_1)x + i p \sin(a_1)y) + \exp(ip \cos(a_2)x + i p \sin(a_2)y) \quad ((1))$$

In such a case $W(x)$ is still an eigenfunction of the kinetic energy operator $-1/2m[d/dx d/dx + d/dy d/dy]$, but no longer of the momentum operator $\{-id/dx, -id/dy\}$. Given that there is no time in ((1)), we argue that instead of calculating pressure as the average of $p v$ as in a Maxwell-Boltzmann gas, one may calculate it using $(1/\text{time}) * \text{Integral } F dt$ i.e. $(1/\text{time}) * \text{impulse}$. For example, at $y=0$, one may use $-id/dx$ to find the average momentum which we consider equivalent to impulse and multiply by $\cos(a_1)\cos(a_1)p^2/m$ as $1/\text{time}$. In other words there is an impulse pressure which is a function of x . Thus not only does this impulse pressure

exist at a point x , but it changes with x in accordance with spatial density (although formally this is W^*W). Given that spatial density/probability must be conserved this leads to a hump pattern.

Originally a free particle with $\exp(ipx)$ (two spatial densities-pressures) combined so as to “create” a constant p interacts with a two slit apparatus (precisely because of the internal spatial density-pressure pattern). The interaction breaks the scalar appearance of $\exp(ipx)$ (eigenfunction of both kinetic energy and momentum) so that a visible “static” spatial density appears together with a visible x -dependent impulse-pressure.

This notion has implications on the quantum bound state.

Quantum Bound State

The quantum bound state is formulated in terms of a Newton’s scalar or magnitude of “ p ” type of problem, in other words in terms of a change in average kinetic energy with x . In particular, the time independent Schrodinger equation is:

$$KE(x) + V(x) = E_n \quad \text{where} \quad KE(x) = \{-1/2m \, d/dx \, dW/dx\} / W \quad \text{and} \quad W(x) = \text{Sum over } p \, a(p)\exp(ipx) \quad ((2))$$

This indicates that the notion of the free particle, associated with spatial invariance and $\exp(ipx)$, is key. $\exp(ipx)$, however, contains two internal spatial density-pressure regions which vary within a wavelength. Thus the idea of kinetic energy changing into potential energy and vice versa with x cannot be the whole picture as it is the Newtonian case.

In a quantum bound state, unlike the Newtonian case, there is no time and so one has both p and $-p$. This further follows from the notion that there is no time within a wavelength, and the wavelength length is comparable to the length of the bound system (at least where most of the probability is i.e. within classical turning points). Given that average kinetic energy $\{\text{Sum over } p \, a(p)p/2m \exp(ipx)\} / W(x)$ changes with x , there is “Newtonian” type particle motion at each x , yet at the same time there is spatial density and impulse pressure due to the average of p at x . As a result, we argue that ((2)) should be considered in terms of both $KE(x)$ and $\langle p \rangle = |-idW/dx| / W$.

If one only considers that $KE(x)$ changes, one has a strictly Newtonian problem, but this does not account for the changing spatial density $W(x)$ [which leads to $P(x)=W^*(x)W(x)$]. We argue, as in Part II, that the spatial density indicates:

1. An impulse type pressure at each x .
2. An impulse type pressure which changes with x .

The condition 2 is not due to a change in the free variable “particle number” as in a Maxwell-Boltzmann gas which has many particles which may adjust their positions in response to a force. In the quantum bound state, the “static-seeming” impulse-pressure itself is a combination of:

1. Interference (especially of p and $-p$)
2. The form $\exp(ipx)$

3. $V(x)$

In Part II, we considered a quantum particle in a well with infinite potential walls. In such a case there is no $V(x)$ in the interior, but there is still a visible spatial density (due to the interference of p and $-p$'s) and also an impulse-pressure, proportional to $dW/dx / W$ which changes with x . Thus even though ((2)) holds, in quantum mechanics one considers quantities multiplied by a spatial density $W^*(x)W(x)$. The solution of the infinite well problem is an eigenfunction of kinetic energy, but not of momentum as one has p and $-p$ and this "interference" yields the "static-seeming" spatial density and impulse-pressure. (Note: All p 's exist in this problem and interfere to create a $W(x)=C\sin(pave\ x)$ within the box and $W(x)=0$ outside.)

In a Maxwell-Boltzmann gas, pressure which is $\text{density}(x)RT$ balances (on two sides of a tiny box) $-dV(x)/dx * \text{density}(x)$. In the infinite well problem there is no $V(x)$ so what balances the impulse-pressure linked to the changing spatial density? One may note that spatial density, pressure already exists in $\exp(ipx)$ and is balanced to give an impulse of p and kinetic energy of $pp/2m$ at any x . $W(x)=C\sin(pave\ x) = 1/2i \{ \exp(i\ pave\ x) - \exp(-i\ pave\ x) \}$. (Here we use $pave$ quantities for simplicity.) Thus each term is balanced on average (through the dynamical balance of $\cos(px)$ with $\sin(px)$), but through interference the appearance of a static spatial density emerges.) Thus one does not need an external force brought in to enforce the $\sin(px)$ shape. It is already provided by $\exp(ipx)$ and $\exp(-ipx)$.

Influence of $V(x)$ on Impulse-Pressure

In the above section we argued that for an infinite well potential there is no $V(x)$ in the interior, yet there is still an overall spatial density associated with $\sin(pave\ x)$ (officially the square of this). How can such a spatial type density exist without external forces holding the density in this shape? We argued that the dynamical probability form already contains these forces as the density $\cos(px)$ and the changing density $\sin(px)$ compensate for each other. This probability is dynamical, not static. $\cos(px)$ is a pressure, but this pressure is continually changing in x (the particle moves in one direction) so a balance is maintained and one has constant p .

In the case of a potential $V(x)$ one has the Newtonian change in average $KE(x)$ with x , but at the same time the $a(p)$ values in $W(x) = \text{Sum over } p\ a(p)\exp(ipx)$ are fixed by $V(x)$ and so influence the impulse-pressure proportional to:

$$|-idW/dx / W| \quad ((3))$$

Thus the spatial density and impulse-pressure ((3)) are influenced by $V(x)$. Thus a quantum single bound state is linked both to features. First, there is the usual Newtonian conservation of energy which holds for the average of $pp/2m$ for any energy level. Second, the dynamical probability form (which contains its own balanced pressure scheme and spatial density) interferes to also create an average static appearing density and impulse-pressure proportional to ((3)) i.e. the average momentum. Thus in a quantum bound state, ((3)) is not the square root of kinetic energy (multiplied by a constant). Both the scalar notion of force (acting on $prms(x)$) and the vector notion due to $\exp(ipx)$ (which is based on vector ideas) hold and should be considered, we argue.

Conclusion

In conclusion, we argue that $\exp(ipx)$, the dynamical probability (square root flux) and eigenfunction of the spatial generator $-i\hbar d/dx$ (and also kinetic energy operator $-1/2m d/dx d/dx$), contains two dimensions of acceleration/spatial distribution just like a Newtonian particle moving in a circle with constant speed. We argue that a moving quantum particle is not a point, but has a distribution (spatial) within a half wavelength \hbar/p . This suggests a pressure or force, so why is the speed constant? We suggest that $\exp(ipx)$ is a dynamical probability (even if it creates static-looking bound state densities.) One cannot examine one distribution at a time, but must simultaneously consider two to show the direction of motion. Both contain a spatial density-pressure, but these offset each other (action-reaction) to create constant momentum.

If interference occurs, a static-appearing density emerges, but actually there is still a dynamical balance occurring i.e. $\cos(px) = .5(\exp(ipx) + \exp(-ipx))$. Both \exp terms have balance even though a $\cos(px)$ type density appears and there is no "external" force holding this density in place. Quantum mechanics deals with a dynamical probability $\exp(ipx)$ which differs from the view of a point.

In the case of a quantum bound state, it consists of an ensemble of $\exp(ipx)$ s which is a key idea because spatial densities associated with impulse-pressure may occur, but at the same time an average kinetic energy balances with a $V(x)$ to give an E_n at each x . It may be noted that in this case, however, $V(x)$ does not just balance $KE(x)$, but helps determine the form of the impulse-pressure together with the $\exp(ipx)$ s.

References

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