Cascade Theory for Turbulence, Dark Matter and SMBH Evolution

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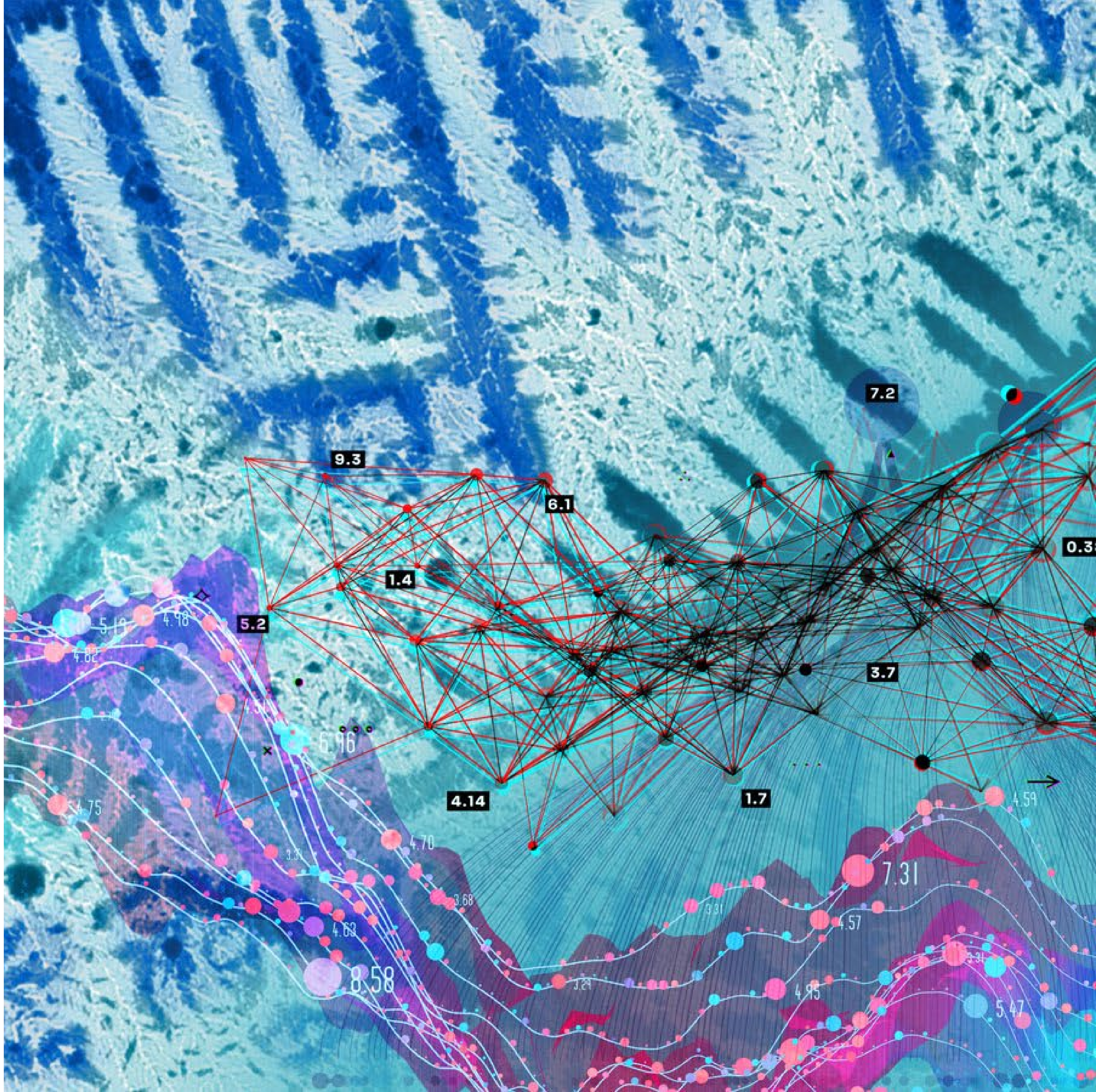


Outline

- Introduction
- Turbulence vs. the flow of dark matter: similarities and differences?
- Inverse mass cascade in dark matter flow
  - Random walk of halos in mass space and halo mass function
  - Random walk of dark matter in real space and halo density profile
- Energy cascade in dark matter flow
  - Universal scaling laws from N-body simulations and rotation curves
  - Dark matter properties from energy cascade
  - Uncertainty principle for energy cascade?
  - Extending to self-interacting dark matter
- Velocity/density correlation/moment functions
- Maximum entropy distributions for dark matter
- Energy cascade for the origin of MOND acceleration
- Energy cascade for the baryonic-to-halo mass relation
- Energy cascade for SMBH-bulge coevolution

Relevant datasets are available at:

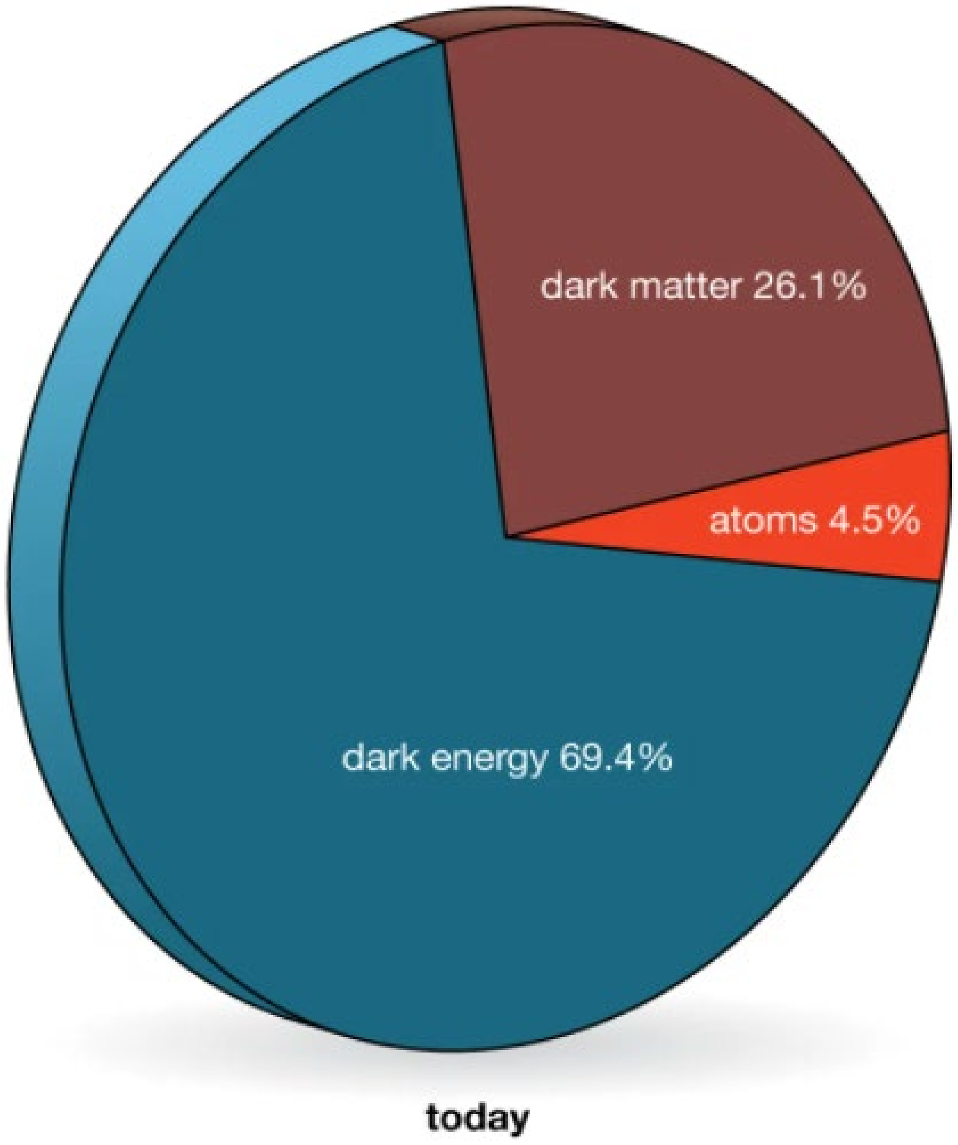
"A comparative study of dark matter flow & hydrodynamic turbulence and its applications"
http://dx.doi.org/10.5281/zenodo.6569901

Introduction

- Dark matter: 85% of the total matter.
- Dark matter flow (DMF): the widest presence in the universe.
- Hydrodynamic turbulence: the most familiar flow in our daily life.
- What are the similarities and differences?

During the pandemic, we find a time to think about and leverage this comparison for better understanding the nature of dark matter (DM) flow and DM properties.

Content of universe



What is dark matter?

No definite answer.

What it should not be?

- No electric charge
- No color charge (strong interactions)
- No strong self-interaction
- No fast decay: stable and long-lived
- Not any particles in standard model of particle physics

What it should be?

- Non-baryonic
- Cold (non-relativistic)
- Collisionless
- Dissipationless (optically dark)
- Sufficiently smooth with a fluid-like behavior

What is the nature of dark matter flow (DMF)? A special example of non-relativistic, self-gravitating, collisionless fluid dynamics (SG-CFD)

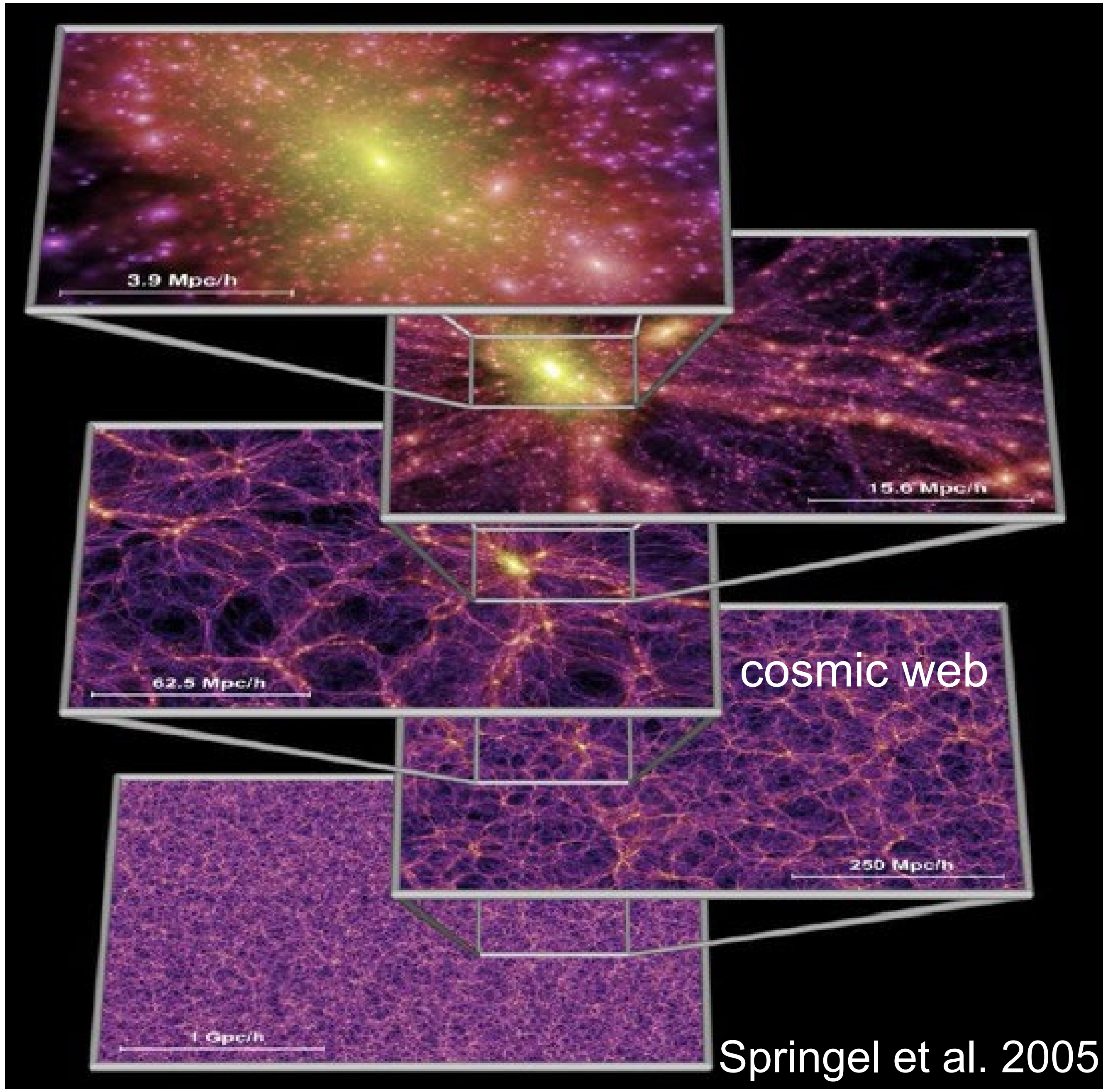
Might be a new opportunity for fluid dynamics contributing to the dark matter mystery, the biggest quest of contemporary astrophysics.

Brief timeline for dark matter research (~100 years)

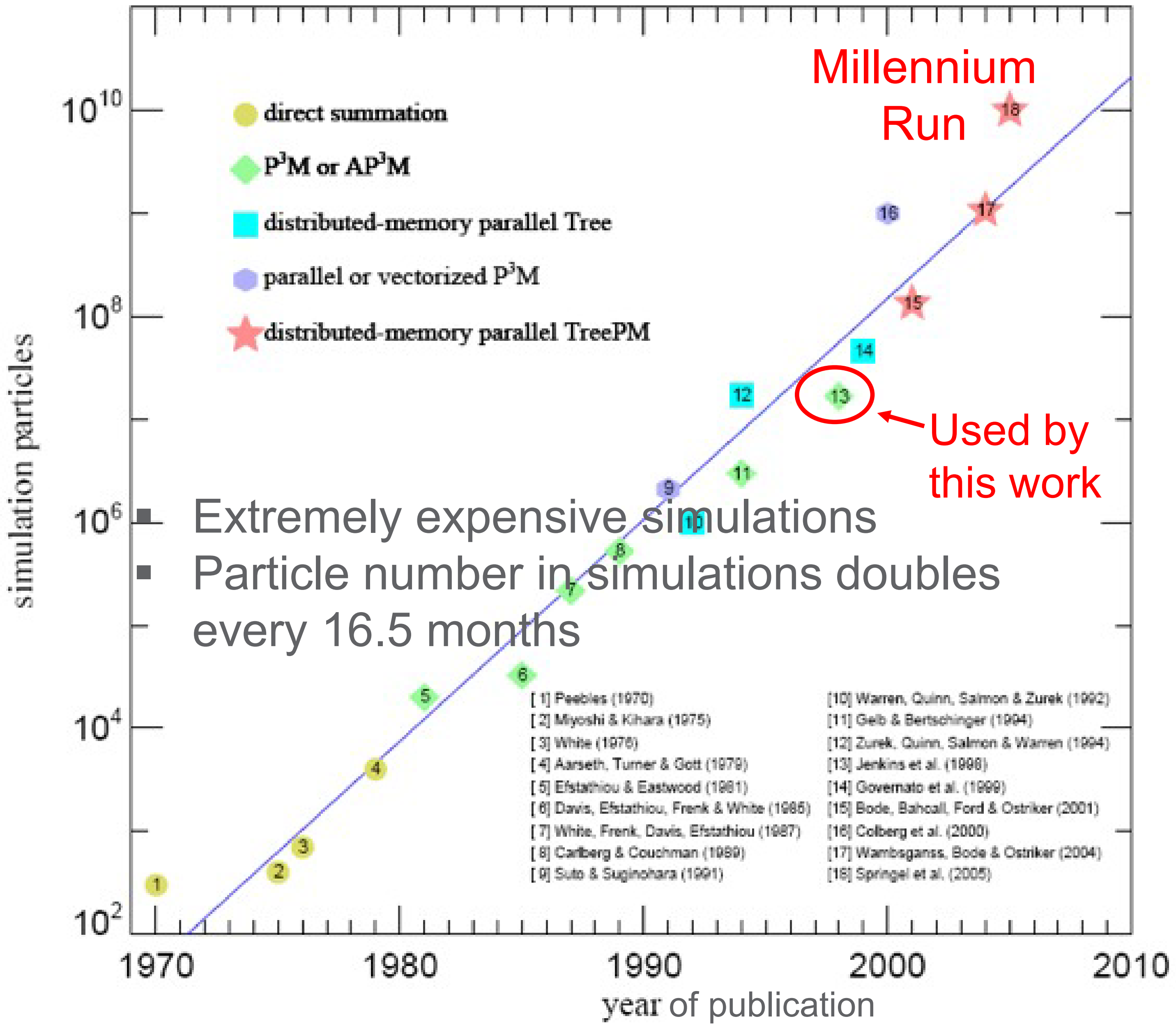
- **1930s** — Zwicky: Discovery of galaxy cluster velocity ~1000km/s
- **1960s** — Discovery of the CMB
- **1970s** — Rubin: Discovery of flat galaxy rotation curves
- **1980s** — Cold Dark Matter (CDM) model proposed; MOND theory;
- **1992** — CMB fluctuations from COBE Confirms CDM prediction
- **1998** — Evidence for Dark Energy and accelerating expansion: Type Ia supernova
- **2000** — ΛCDM as the standard cosmological model
- **2003-2008** — WMAP and LSS data Confirm ΛCDM predictions
- **2013-2018** — Planck data of CMB anisotropies Confirm ΛCDM predictions
- **2022** — James Webb Space Telescope
- **2030s** — Discovery of dark matter particles??

COBE: COsmic Background Explorer (NASA)
WMAP: Wilkinson Microwave Anisotropy Probe (NASA)
LSS: Large Scale Structure (LSS) of the universe
CMB: Cosmic Microwave Background
ΛCDM: dark energy + cold dark matter (double dark)
Planck: European Space Agency (ESA)

Cosmological N-body simulations for the flow of DM



Springel et al. 2005



- Extremely expensive simulations
- Particle number in simulations doubles every 16.5 months

Dehnen and J. I. ReadThe European Physical Journal Plus, 126, 55

N-body simulations in this study

Run	Ω_0	Λ	h	Γ	σ_8	$L(Mpc/h)$	N_p	$m_p(M_\odot/h)$	$l_{soft}(Kpc/h)$
SCDM1	1.0	0.0	0.5	0.5	0.51	239.5	256^3	2.27×10^{11}	36

- N-body simulations carried out by the Virgo consortium. https://wwwmpa.mpa-garching.mpg.de/Virgo/data_download.html
- Standard CDM power spectrum (SCDM) with matter-dominant gravitational flow.
- Dark matter only simulations
- Similar analysis can be extended to other cosmological models and hydrodynamic simulations.
- All relevant datasets for this work are available at http://dx.doi.org/10.5281/zenodo.6569901

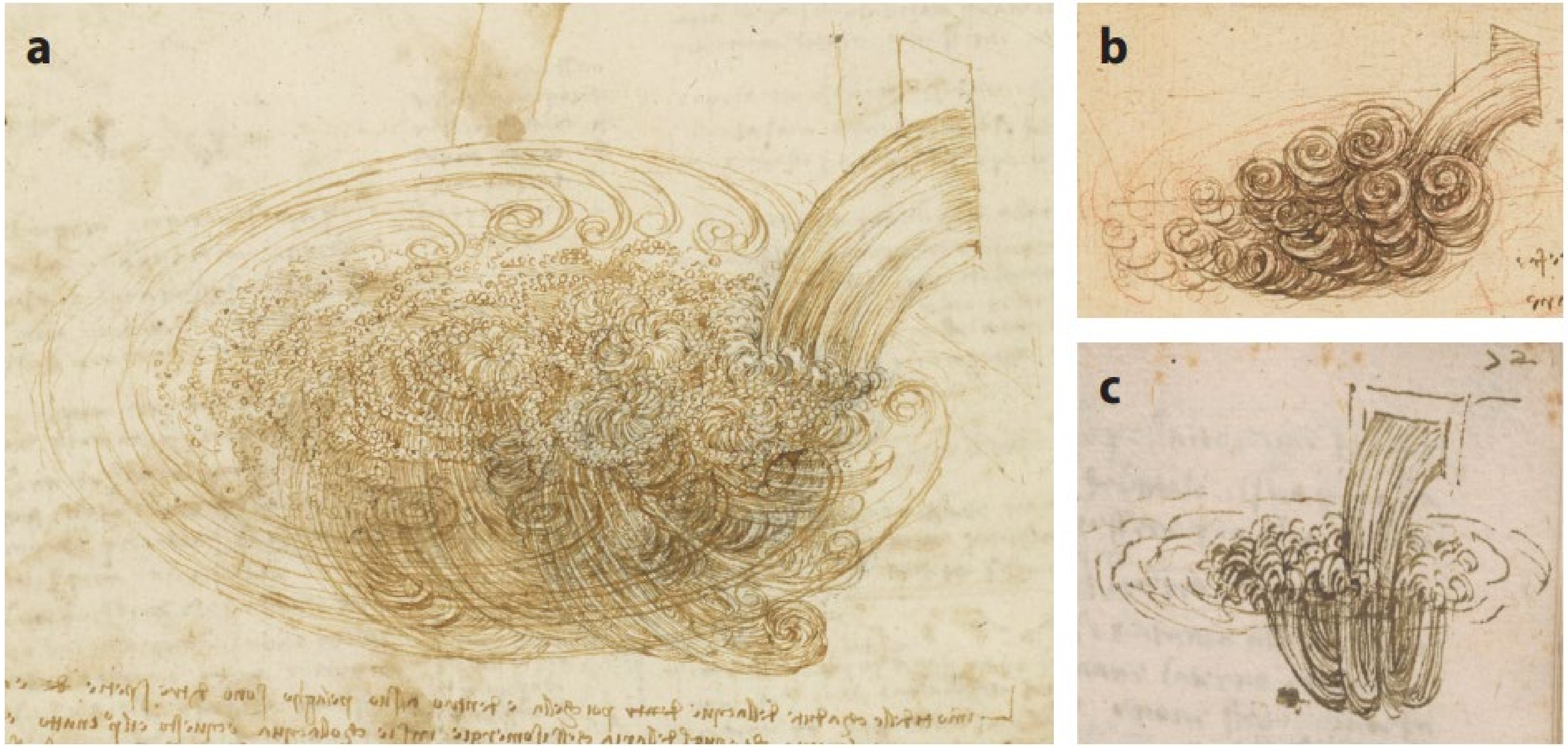
Brief timeline for turbulence research (~500 years)

- **1500s** — da Vinci: Earliest recognition of turbulence as a physical phenomenon
- **1877** — Boussinesq's concept of "eddy viscosity" replacing molecular viscosity
- **1894** — Reynolds number for transition from laminar to turbulence; Reynolds's decomposition (RANS) $v = \langle v \rangle + v'$ Reynolds stress;
- **1925** — Prandtl "mixing-length theory" for eddy viscosity Richardson "energy cascade"
- **1930s** — Taylor's statistical methods involving correlations power spectra; Karman and Howarth
- **1940s** — Kolmogorov: the "K41 theory" -5/3 law
- **1963** — Lorenz propose possible links between "deterministic chaos" and turbulence
- **1967** — Kraichnan's inverse cascade in 2-dimensional turbulence

RANS: Reynolds-averaged Navier-Stokes Equation;

What can we learn from turbulence?

da Vinci's sketch of turbulence (~1500 AD)



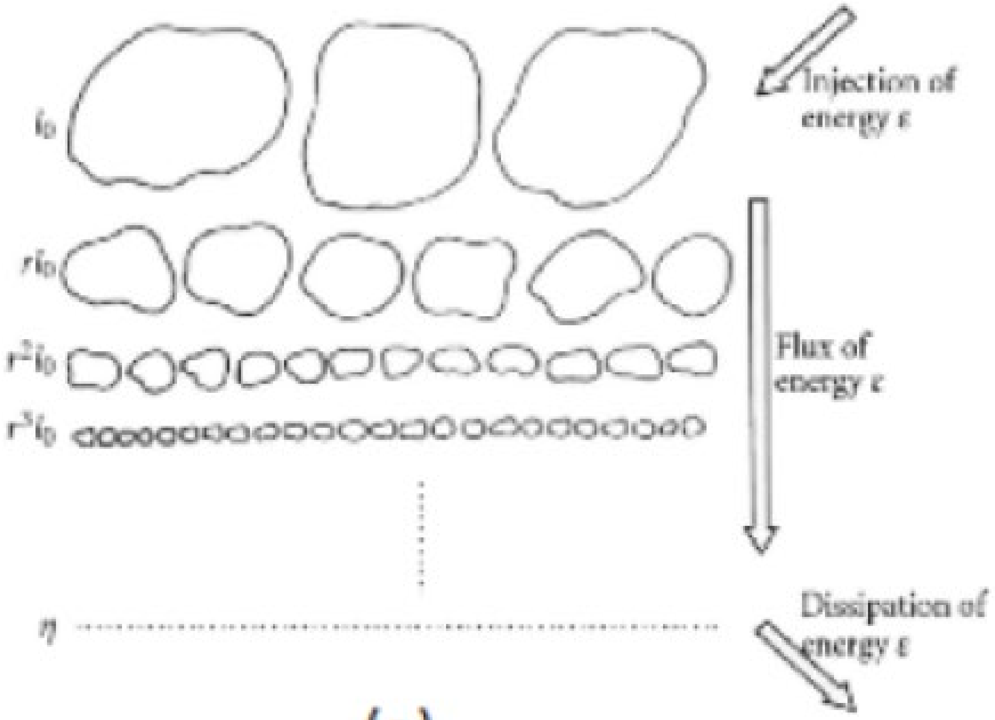
da Vinci sketch of turbulence: plunging water jet

- "turbolenza": the origin of modern word "turbulence"
- The pattern of flow with vortexes in fluid
- The random chaotic nature

". . . the smallest eddies are almost numberless, and large things are rotated only by large eddies and not by small ones, and small things are turned by small eddies and large."

Richardson's direct cascade (1922)

"Big whorls have little whorls, That feed on their velocity;
And little whorls have lesser whorls, And so on to viscosity."



(a)

(b)

Cascade: energy is injected on large scale, propagating across different scales, and dissipated on the smallest scale.

(a) : Cascade of energy, (b) : Lewis Richardson

Key attributes:

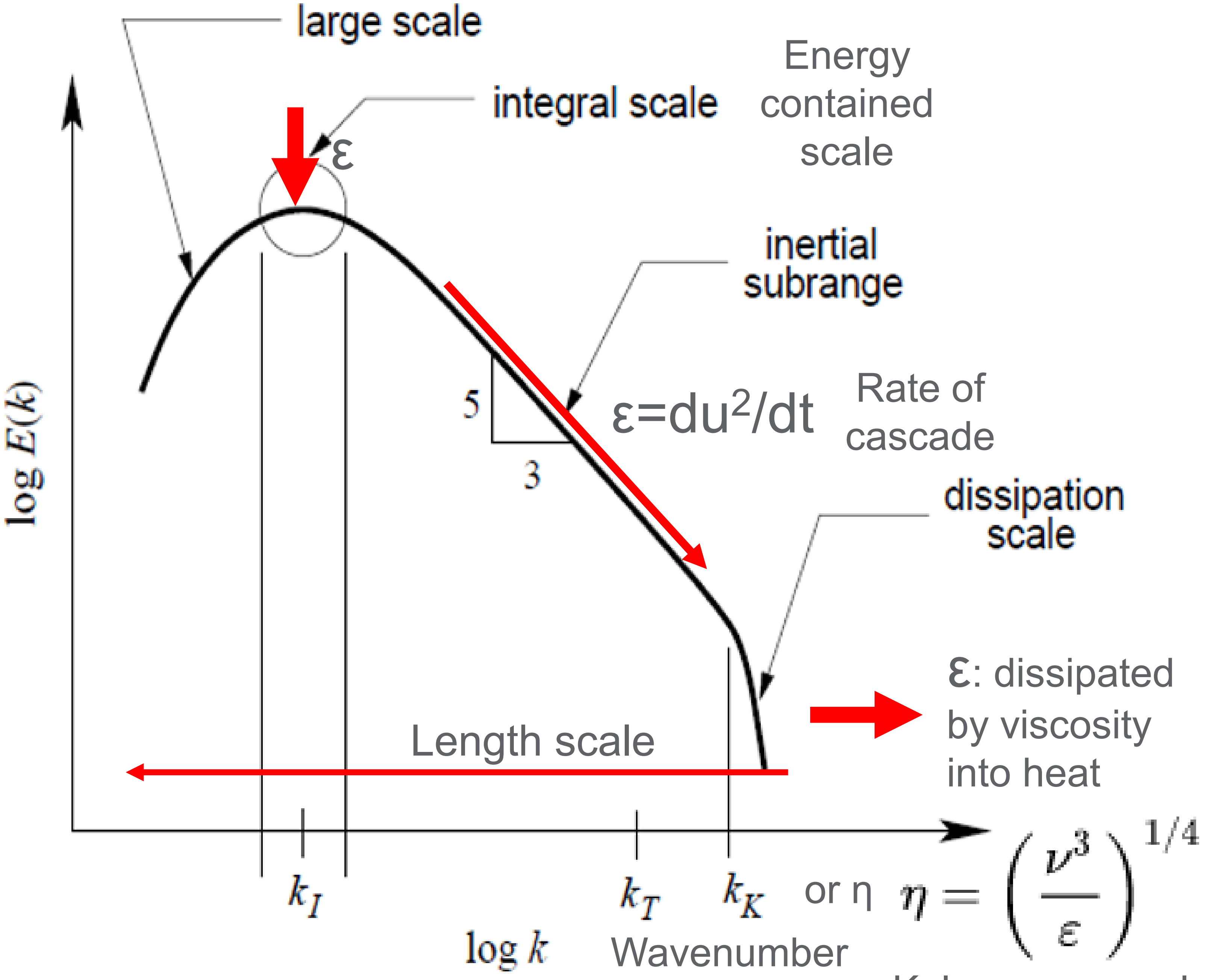
- Disorganized, chaotic, random;
- Nonrepeatability (sensitivity to initial conditions);

- Multiscale: large range of length and time scales;
- Dissipation mediated by viscosity;

- Three dimensionality;
- Time dependence;
- Rotationality (incompressible);
- Intermittency in space and time;

[1] "Weather Prediction by Numerical Process", Richardson, L.F. 1922

Direct energy cascade in turbulence (1940s)



$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$

Kolmogorov scale

- Freely decaying vs. forced stationary
- Integral scale: energy injection
- Inertial range: inertial >> viscous force
- Dissipation range: viscous dominant
- Dissipation scale: determined by kinematic viscosity (m^2/s) and rate of cascade (m^2/s^3)

Is there cascade in dark matter flow?
If yes, how does it initiate, propagate, and die?

Hydrodynamic turbulence vs. dark matter flow

Key attributes of hydrodynamic turbulence:

- Chaotic, random;
- Nonrepeatability (sensitivity to initial conditions);
- Multiscale in length and time scales; Non-equilibrium;
- Intermittency in space and time;
- Dissipative and collisional
- Short-range interaction
- Velocity fluctuation
- Vortex as fundamental building block
- Maximum entropy distribution (Gaussian)
- Incompressible on all scales $\nabla \cdot \mathbf{v} = 0$
  - Divergence-free
  - Constant density
- Energy cascade from large to small length scales
- Vortex stretching responsible for energy cascade
  - Volume conserving
  - Shape changing
  - Uniform density
- Reynolds decomposition
- Reynolds stress for energy transfer between mean flow and random motion (turbulence)
- Closure problem, eddy viscosity, etc…
- Statistical theory: correlation/structure functions
- Scaling laws in inertial range

Key attributes of dark matter flow:

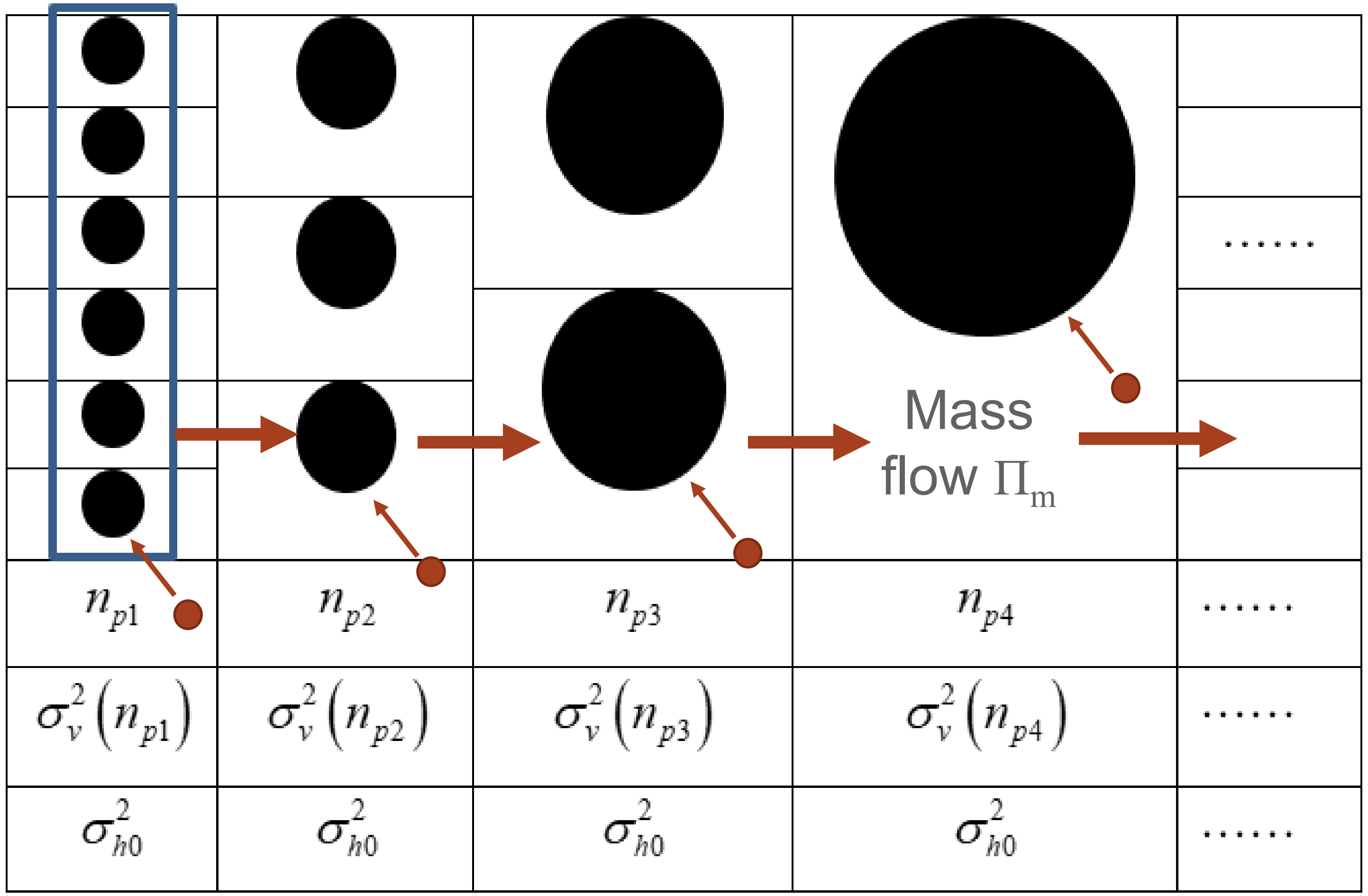
- Chaotic, random;
- Nonrepeatability;
- Multiscale in mass/length/time scales; Non-equilibrium;
- Intermittency in space and time;
- Dissipationless and collisionless
- Long-range gravity
- Velocity & acceleration fluctuation
- Halos as fundamental building block
- Maximum entropy distribution?? (X dist.)
- Flow behavior is scale-dependent (peculiar velocity)
  - Small scale: constant divergence $\nabla \cdot \mathbf{v} = \theta$
  - Large scale: irrotational (curl-free) $\nabla \times \mathbf{v} = 0$
- Mass/energy cascade from small to large mass scales
- Role of halos for energy cascade??
  - Halos are growing, rotating, with nonuniform density
  - Is halo shape changing important?
  - Mass cascade facilitates energy cascade?
- Velocity/acceleration decomposition?
- What facilitates the energy transfer between mean flow and random motion in dark matter??
- Self-closed model (analogue of NS) ?? Closure problem?
- Statistical theory: Kinematic and dynamic relations?
- Scaling laws in dark matter flow?

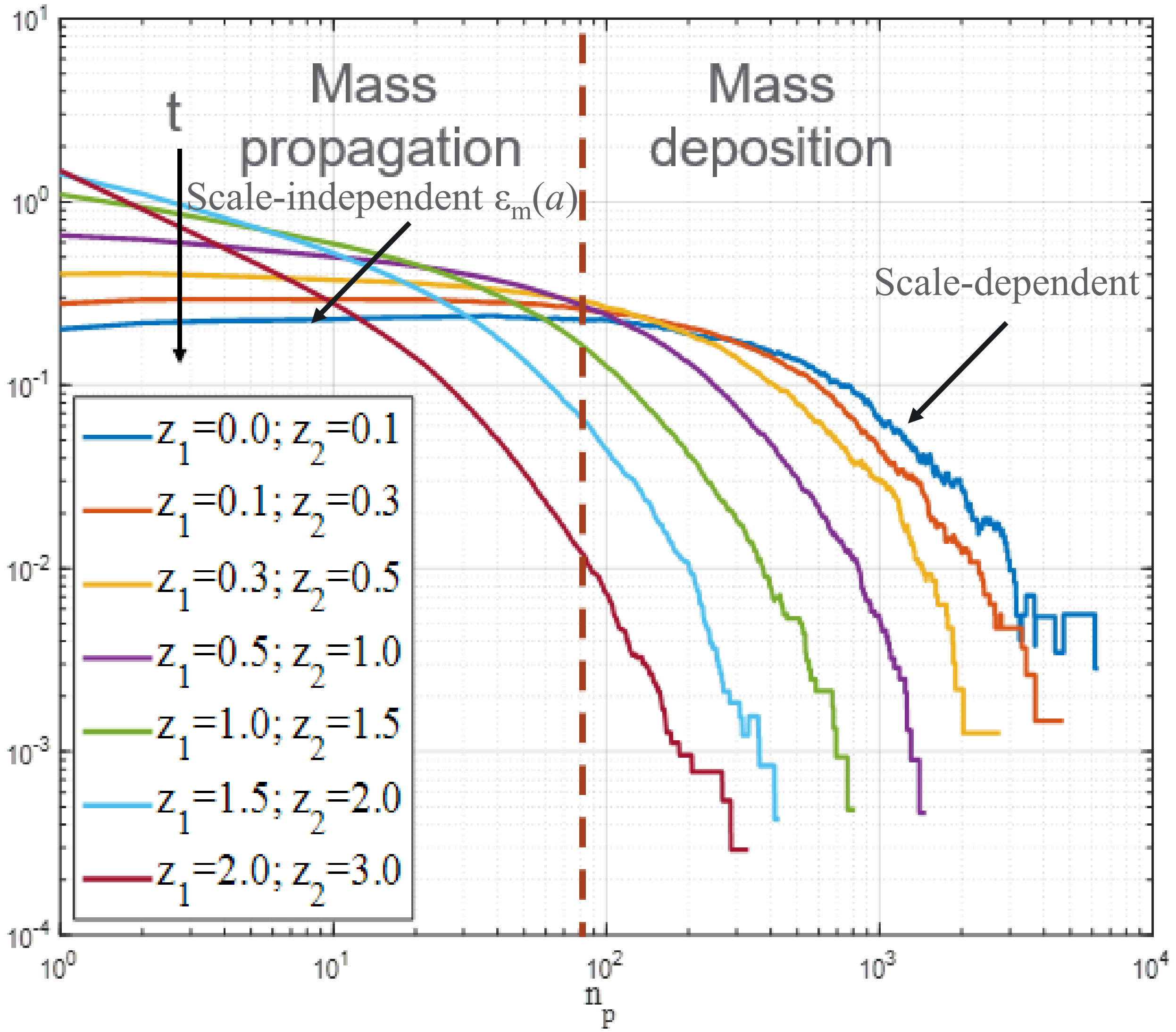
Common features

Critical MOND acceleration a_0?

This talk

Inverse mass cascade in dark matter flow





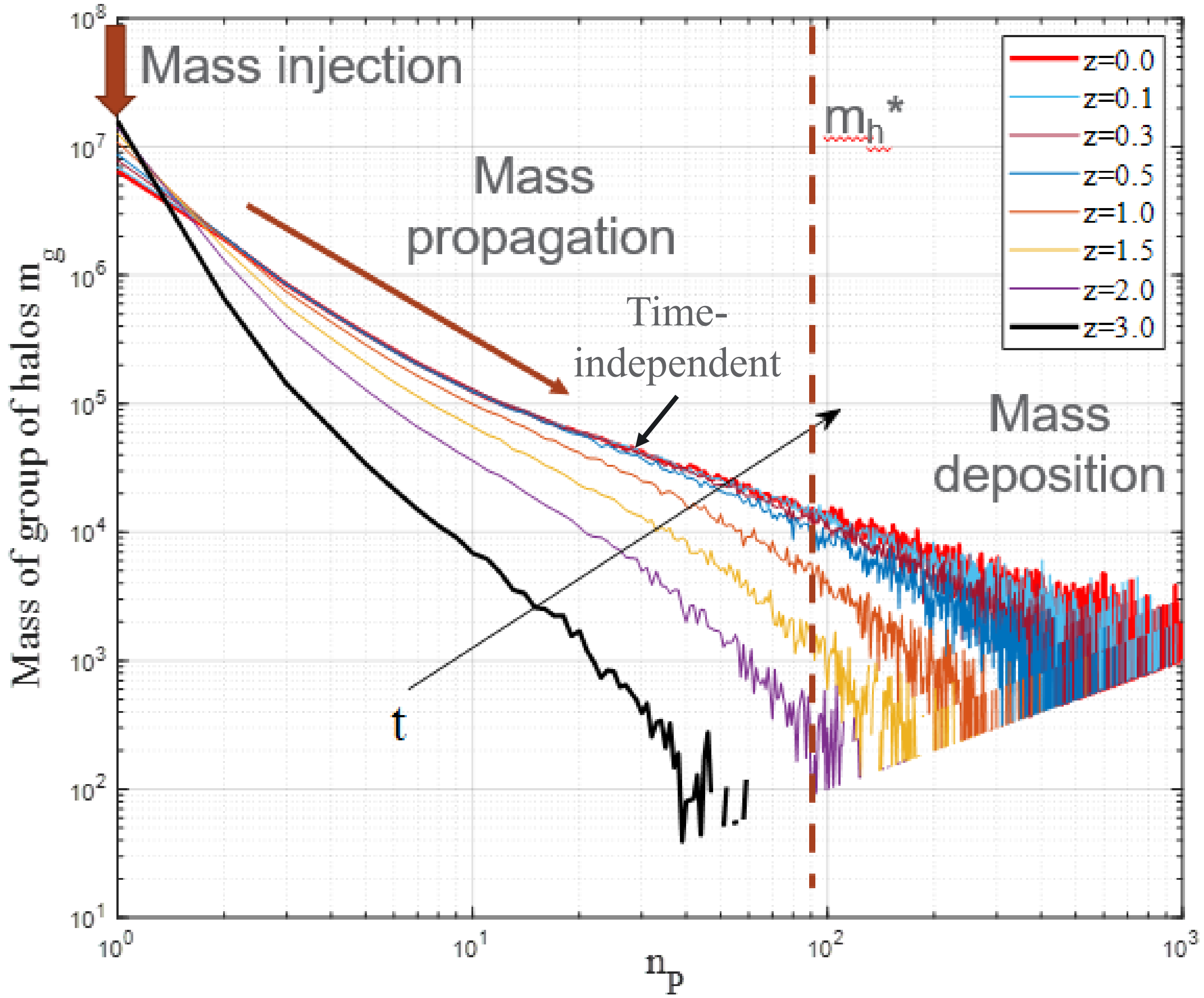
Mass cascade rate $\Pi_m(m_h, a)$ (normalized by Nm_p/t_0)

- Identify all halos of different sizes
- Group halos according to the halo size n_p
- Mass flow across halo groups from small to large mass scale (inverse) through the merging with "single merger"
- Cascade leads to random-walk of halos in mass space

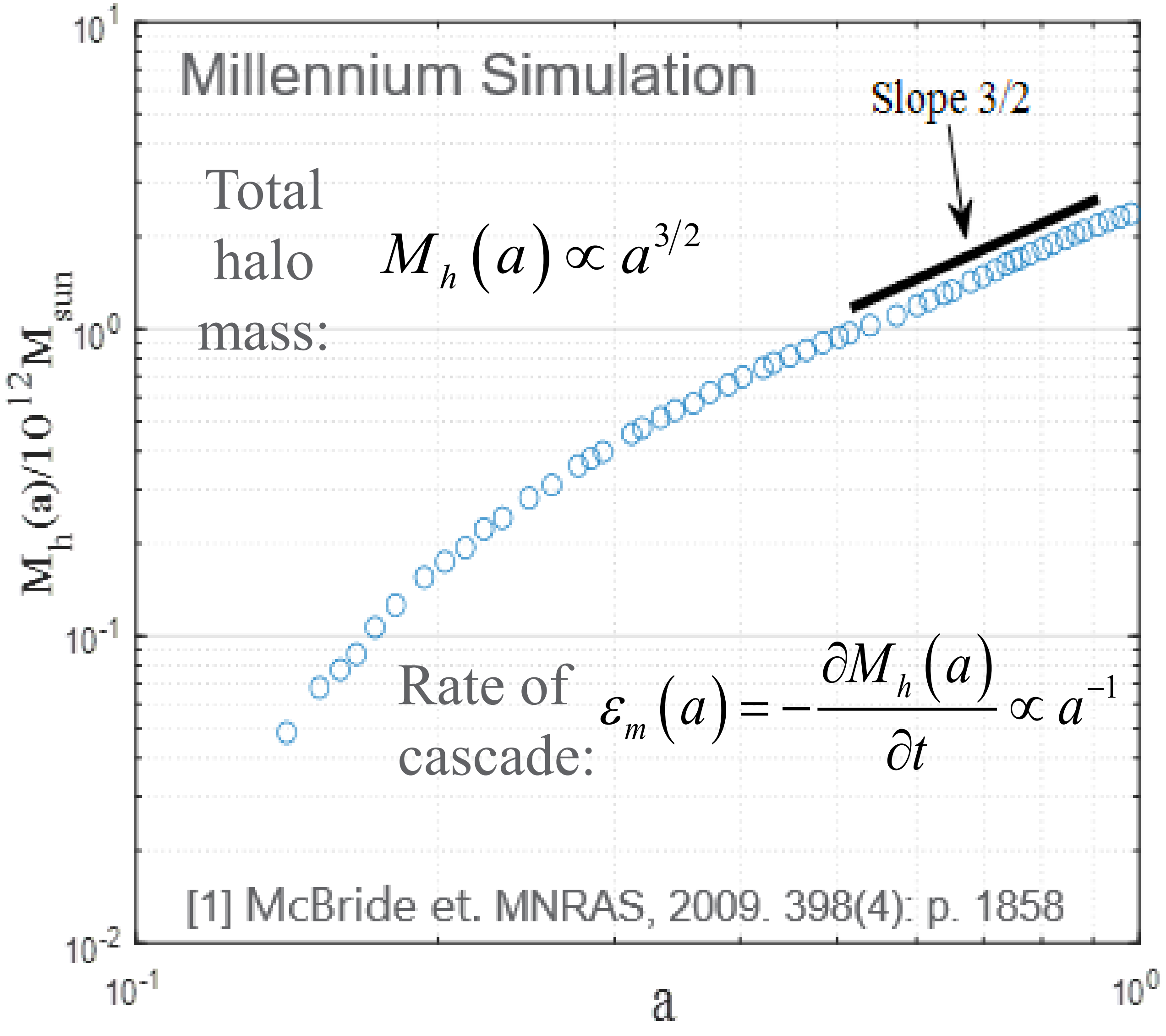
$$\Pi_m(m_h,a) = -\frac{\partial}{\partial t}\left[M_h(a)\int_{m_h}^{\infty} f_M(m,m_h^*)\,dm\right] \Rightarrow \frac{\partial \Pi_m}{\partial m_h} = \frac{1}{m_p}\frac{\partial m_g}{\partial t}$$

Total halo mass   Mass function m_g: Group mass

Halo group mass and time variation of total halo mass

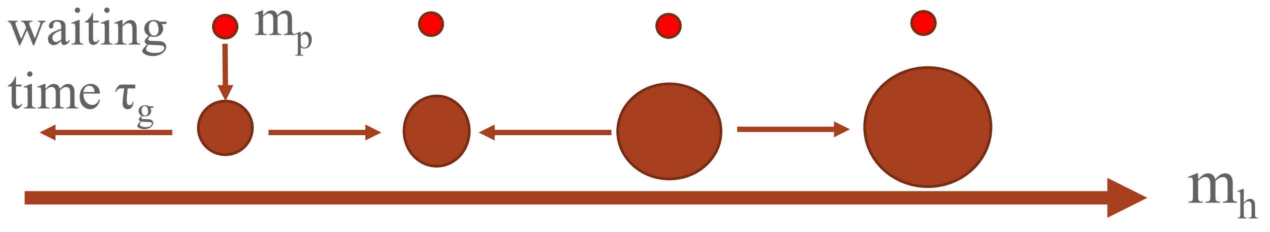


Halo group mass $m_g(m_h, a)$
(time-independent in mass propagation range)



The halo mass for type II halos (the dominant type for large halos, Fig. 2 in ref. [1]) exhibits a power law scaling

Random walk of halos and halo mass function



Merging frequency for halo group: $f_h(m_h, a) \propto \text{halo surface area} \propto n_h m_h^{\lambda}$

$\lambda \sim 2/3$: Exponent for halo surface area.

Characteristic merging time for halo group: $\tau_h(m_h, a) = 1/f_h$

Characteristic merging time (lifetime) for a given halo: waiting time to merge

$$\tau_g(m_h, a) = \underbrace{n_h}_{\text{\# of halos in group}} \tau_h$$

$$\boxed{\tau_g \propto m_h^{-\lambda}}$$ Position-dependent

The exponential distribution of waiting time to merge:

$$P(\tau_{gr}) = \frac{1}{\tau_g}\exp\left(-\frac{\tau_{gr}}{\tau_g}\right)$$

1D Random walk equation in mass space (similar to diffusion):

$$\frac{\partial m_h(t)}{\partial t} = \frac{m_p \xi(t)}{\tau_g(m_h)} = \sqrt{2D_p(m_h)}\varsigma(t)$$

D_p: kg^2/s; ς: Noise s$^{-1/2}$

Position-dependent diffusivity: $D_p(m_h) \propto m_h^{2\lambda}$

Fokker-Planck equation for distribution function:

$$\frac{\partial P_h}{\partial t} = \frac{\partial}{\partial m_h}\left[\sqrt{D_p}\frac{\partial}{\partial m_h}\left(\sqrt{D_p}P_h\right)\right] = D_{p0}\frac{\partial}{\partial m_h}\left[m_h^{\lambda}\frac{\partial}{\partial m_h}\left(m_h^{\lambda}P_h\right)\right]$$

Solving Fokker-Planck Eq. leads to Halo mass function:

$$f_M(m_h, a) = \frac{(1-\lambda)}{\sqrt{\pi\eta_0}}\left(\frac{m_h^*}{m_h}\right)^{\lambda}\frac{1}{m_h^*}\exp\left[-\frac{1}{4\eta_0}\left(\frac{m_h}{m_h^*}\right)^{2-2\lambda}\right]$$

Reduce to Press-Schechter (PS) if λ=2/3 !

(single λ here, how about double λ?)

Double-λ mass function from mass cascade

λ: halo geometry parameter; naturally, we can have two different λ in two different ranges.
λ_1 for mass propagation range (small halos);
λ_2 for mass deposition range (large halos);

$$f_M(m_h,a)=\frac{(1-\lambda)}{\sqrt{\pi\eta_0}}\left(\frac{m_h^*}{m_h}\right)^{\lambda_1}\frac{1}{m_h^*}\exp\left[-\frac{1}{4\eta_0}\left(\frac{m_h}{m_h^*}\right)^{2-2\lambda_2}\right]$$

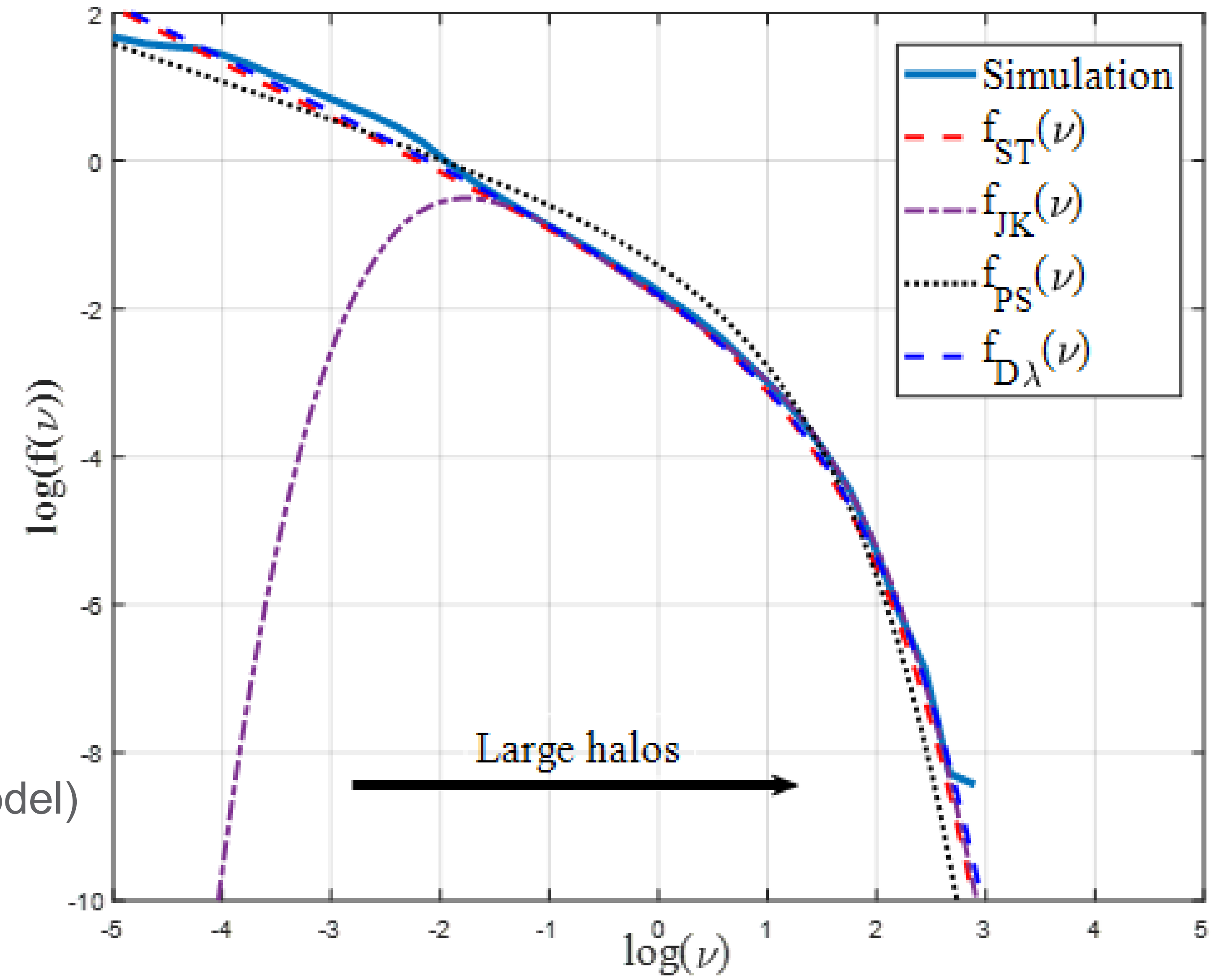
Two-parameter double-λ mass function:

$$f_{D\lambda}(\nu)=\frac{\left(2\sqrt{\eta_0}\right)^{-q}}{\Gamma(q/2)}\nu^{q/2-1}\exp\left(-\frac{\nu}{4\eta_0}\right)\qquad q=\frac{1-\lambda_1}{1-\lambda_2}$$

- PS mass function (require δc and spherical collapse model)
- ST model (modified PS) from ellipsoid collapse
- JK mass function by data fitting
- From simulation $q\approx 0.6$ ➡ $\lambda_1\approx 4/5\quad \lambda_2\approx 2/3$

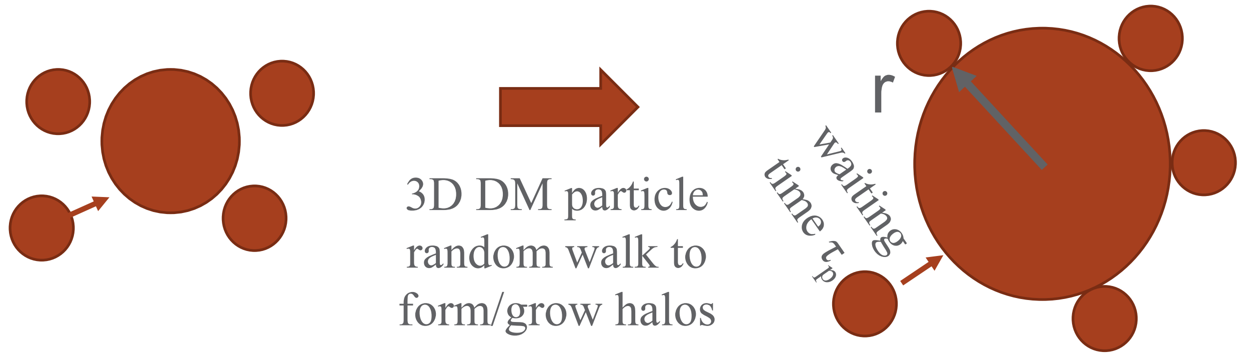
$$f_{PS}(\nu)=\frac{1}{\sqrt{2\pi}}\nu^{-1/2}\exp\left(-\frac{\nu}{2}\right)$$

$$f_{ST}(\nu)=\frac{\left(1+1/(q\nu)^p\right)\sqrt{2q}}{\Gamma(1/2)+2^{-p}\Gamma(1/2-p)}\frac{1}{2\sqrt{\nu}}e^{-q\nu/2}$$



Comparison between different mass functions and N-body simulation

Random walk of DM and double-γ halo densify profile



Waiting time dependent on halo size r (position-dependent):

$$\boxed{\tau_p(r) \propto m_r(r)^{-\lambda} \propto r^{-\gamma}}$$

The larger halo, the shorter waiting time

3D Random walk equation: $$\frac{d\boldsymbol{X}_t}{dt} = \sqrt{2D_P(\boldsymbol{X}_t)}\xi(t).$$

$$\boxed{D_P(\boldsymbol{X}_t) = D_0(t)r^{2\gamma}}$$

Fokker-Planck equation for distribution function:

$$\frac{\partial P_r(\boldsymbol{X},t)}{\partial t} = D_0\frac{\partial}{\partial X_i}\left[r^{\gamma}\frac{\partial}{\partial X_i}\left(r^{\gamma}P_r(\boldsymbol{X},t)\right)\right]$$

$$\boxed{\alpha = 2-2\gamma_2 \quad \beta = \frac{2-2\gamma_2}{2-\gamma_1}}$$

Double-γ halo density profile: $x = r/r_s(t)$

$$\rho_{D\gamma}\left(x \equiv \frac{r}{r_s(t)}\right) = \frac{\alpha\beta^{-(1/\alpha+1/\beta)}}{4\pi\Gamma(1/\alpha+1/\beta)}x^{\frac{\alpha}{\beta}-2}\exp\left(-\frac{x^{\alpha}}{\beta}\right)$$

Reduce to Einasto if α=2β !

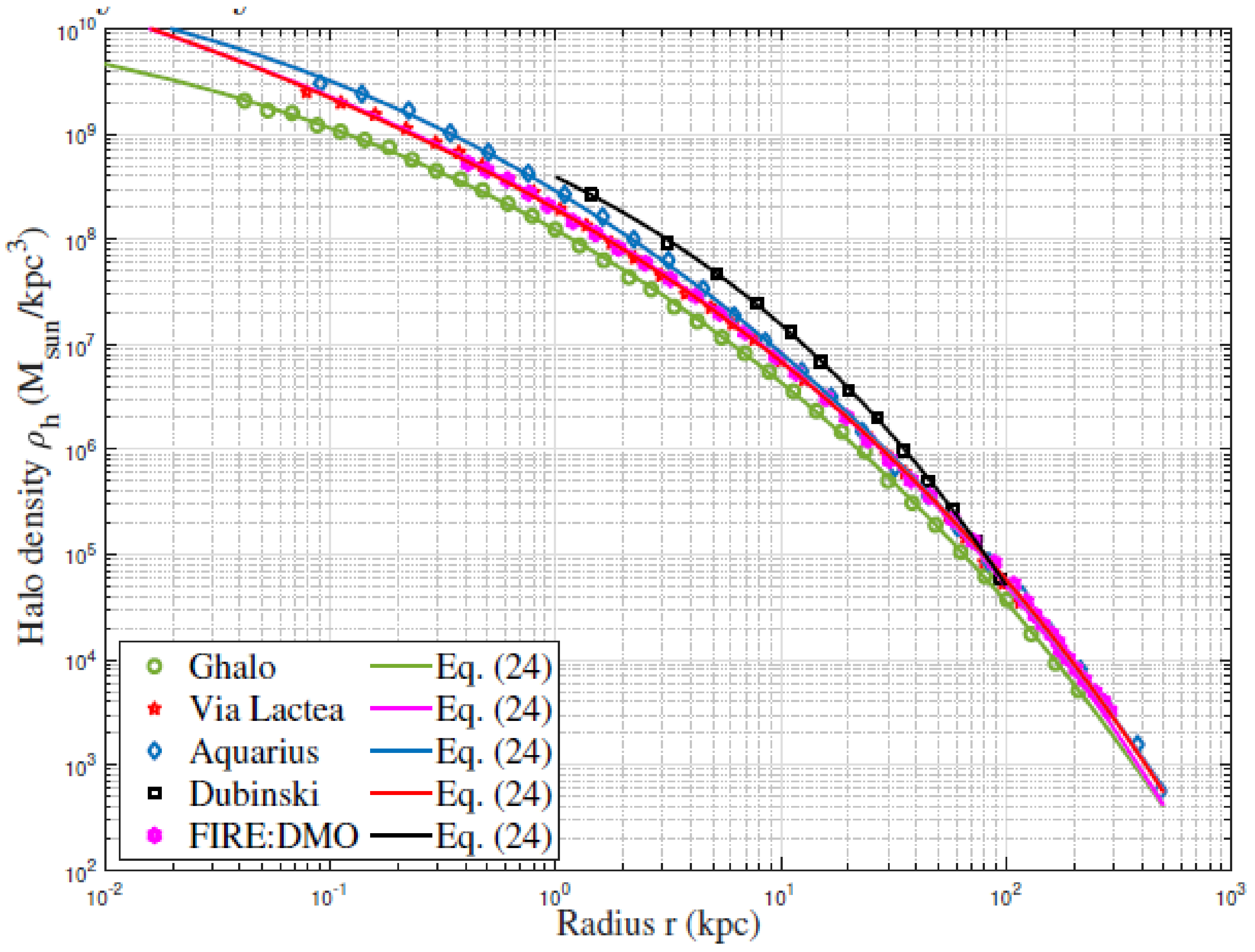


FIG. 2. Halo density profiles for simulated halos: 1) Ghalo [51]; 2) Via Lactea [52]; 3) Aquarius [53]; 4) Dubinski [54]; 5) FIRE:DMO [30]. The double-γ density model (Eq. (24)) was also used to fit all simulated halos for the entire range of r.

Quick Recap I

In (incompressible) hydrodynamic turbulence:

- Energy cascade is well established
  - Direct energy cascade from large to small scales (3D)
  - Inverse energy cascade from small to large scales (2D)
- No mass cascade involved

Halo mass function and density profile

$$f_{D\lambda}(\nu)=\frac{\left(2\sqrt{\eta_0}\right)^{-q}}{\Gamma(q/2)}\nu^{q/2-1}\exp\left(-\frac{\nu}{4\eta_0}\right)$$

$$\rho_{D\gamma}(x)=\frac{\alpha\beta^{-(1/\alpha+1/\beta)}}{4\pi\Gamma(1/\alpha+1/\beta)}x^{\frac{\alpha}{\beta}-2}\exp\left(-\frac{x^\alpha}{\beta}\right)$$

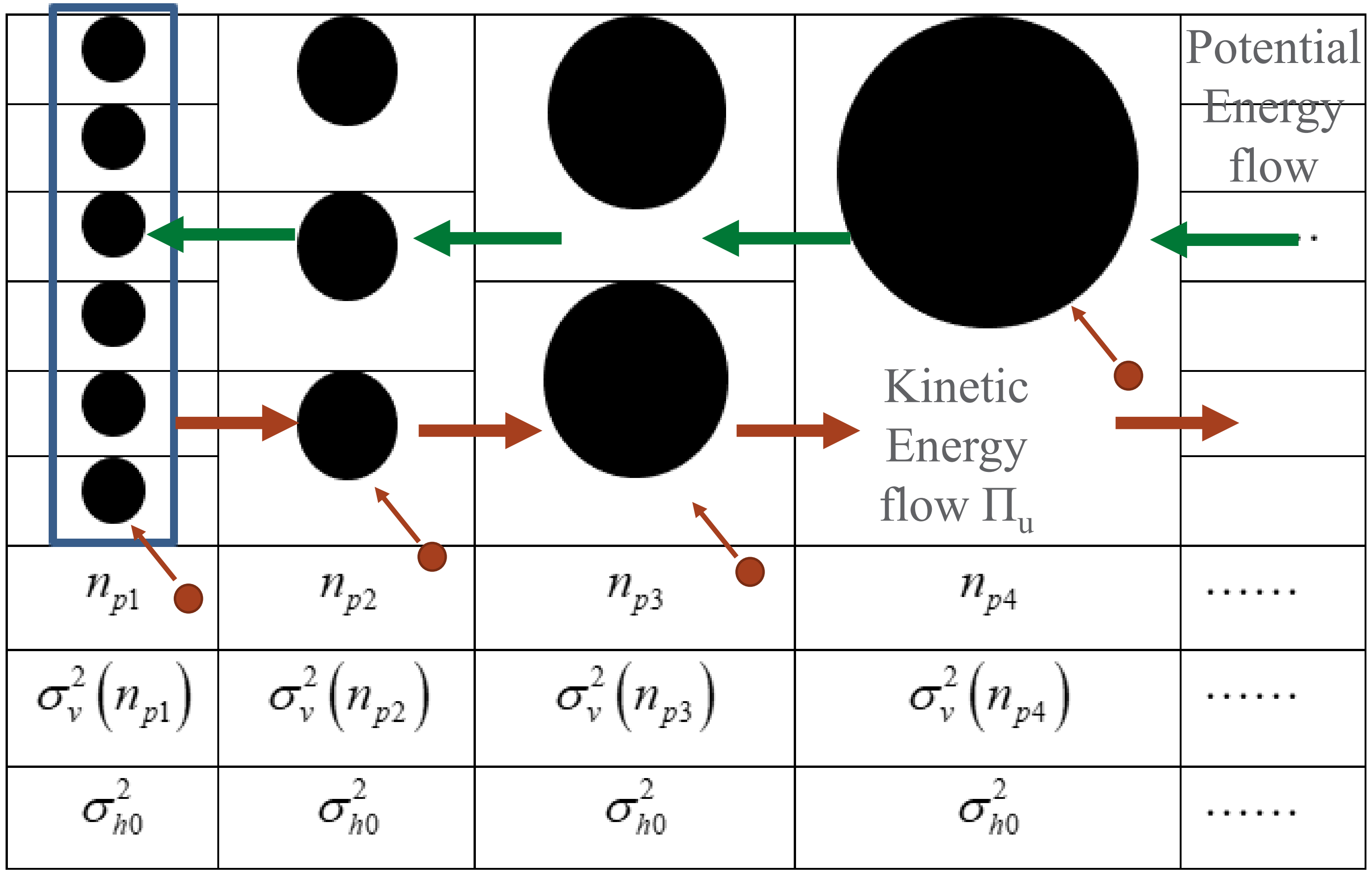
In dark matter flow:

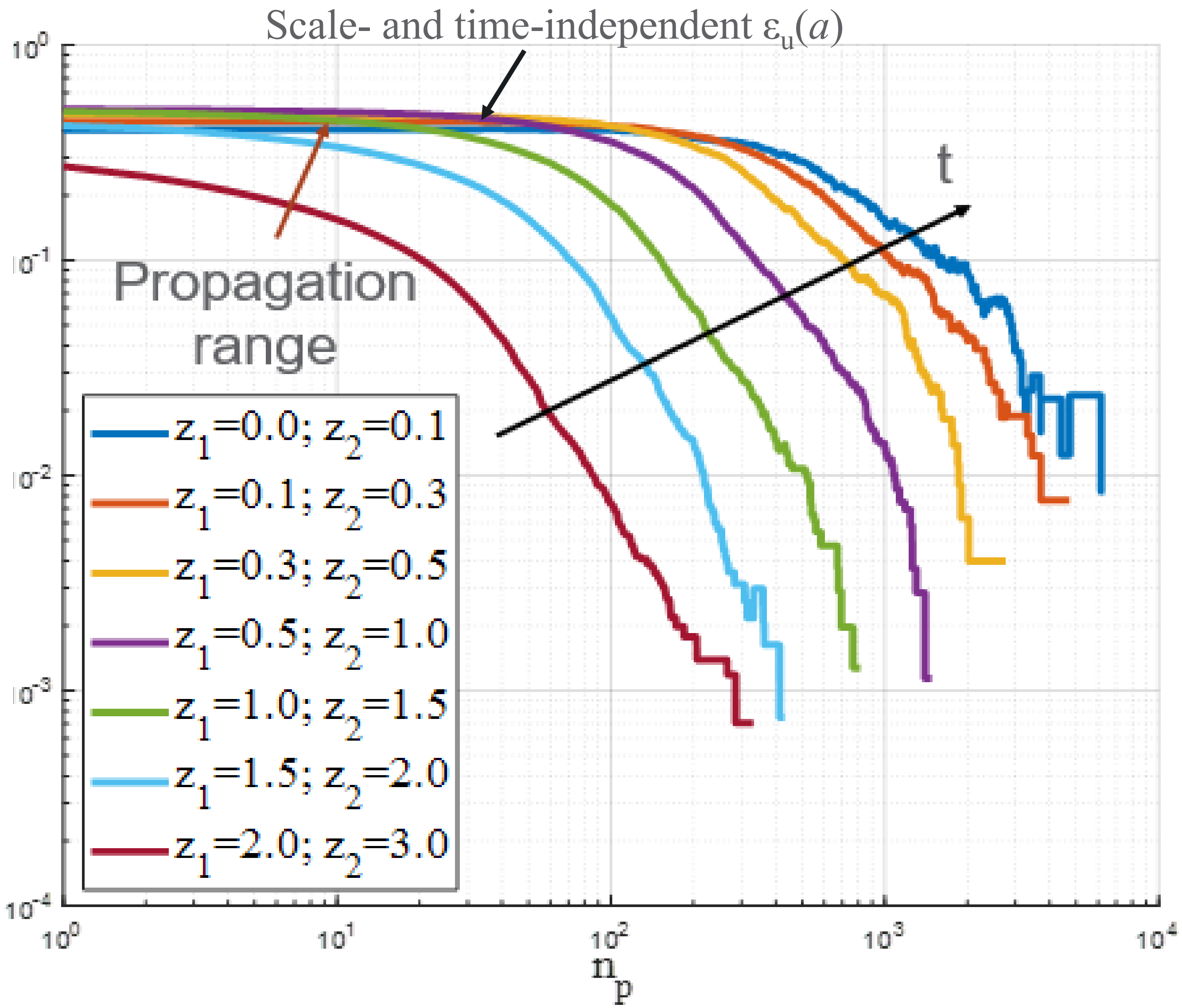
- Inverse mass cascade from small to large scales (rate: ε_m kg/s)
- Mass cascade leads to the random walk of halos in mass space
- Random walk of halos in mass space leads to halo mass function (just like diffusion)

- Random walk of DM particles leads to halo density profile
- Halo density profile and mass function share the same origin.
- Halo density and mass function share similar functional form

- Both random walks involve a position-dependent waiting time (or diffusivity)
- No critical density ratio δc or spherical/ellipsoidal collapse model required

Energy cascade in dark matter flow





Rate of kinetic energy flux function $\pi_u(m_h,a)$

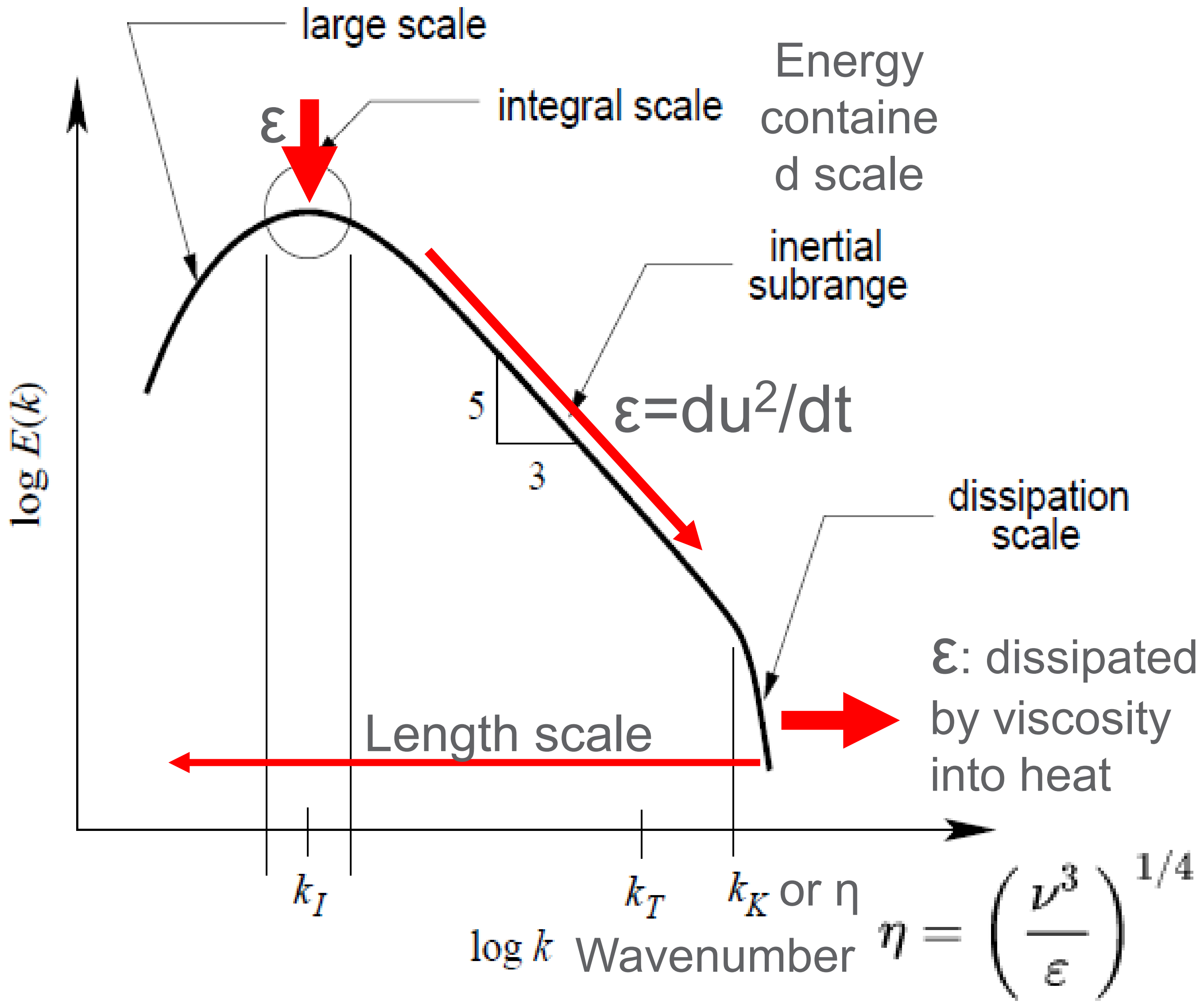
$$\Pi_u(m_h,a) = -\frac{\partial}{\partial t}\left[M_h(a)\int_{m_h}^{\infty} f_M(m,m_h^*)\sigma^2(m,a)dm\right]$$

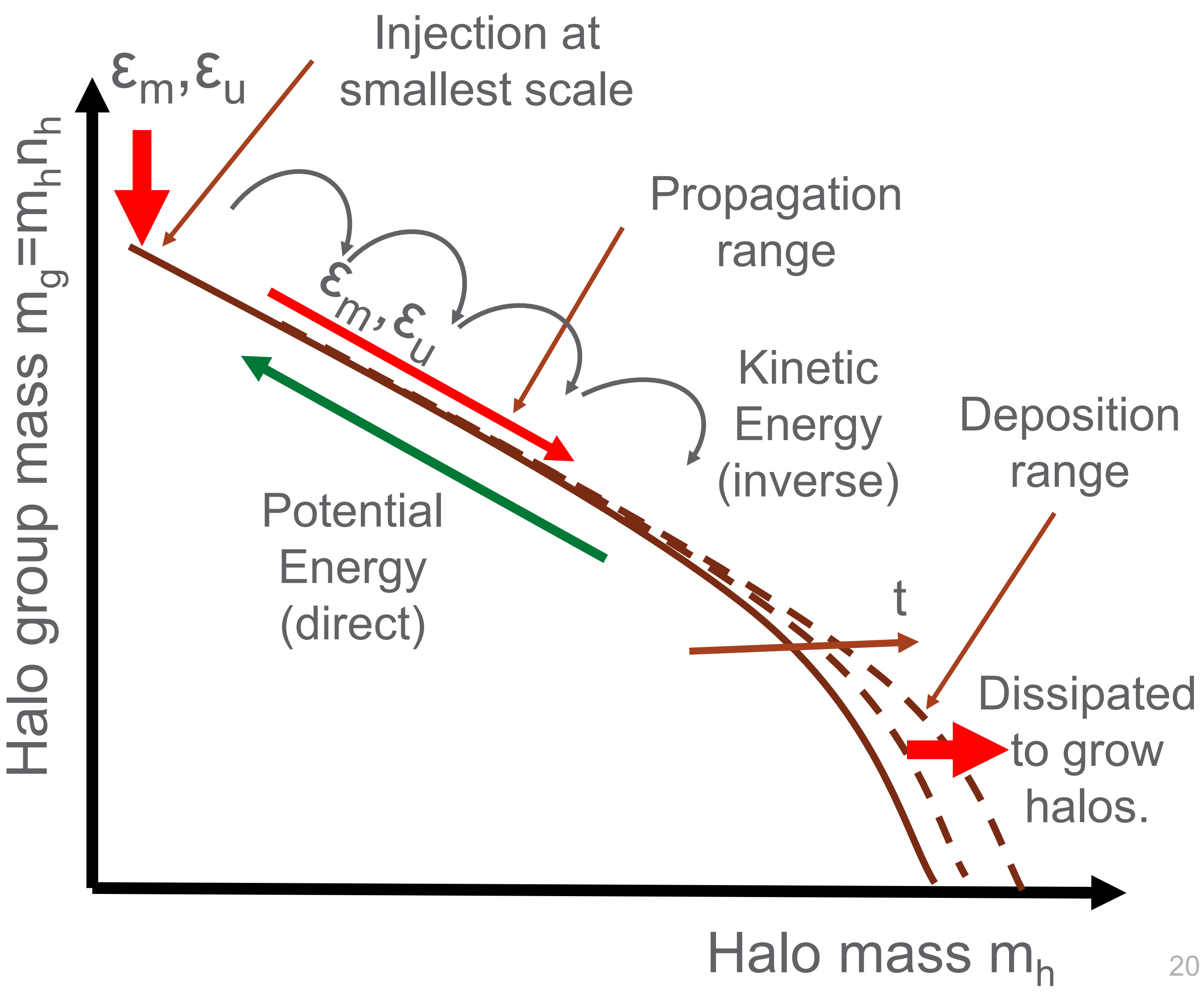
- Identify all halos of different sizes
- Group halos according to the halo size n_p
- Kinetic energy flows from small to large mass scale through the merging with “single merger” (inverse cascade)
- Potential energy flows from large to small scales (direct cascade)

Energy cascade in turbulence and dark matter

Big whirls have little whirls, That feed on their velocity;
And little whirls have lesser whirls, And so on to viscosity.

Little halos have big halos, That feed on their mass;
And big halos have greater halos, And so on to growth.

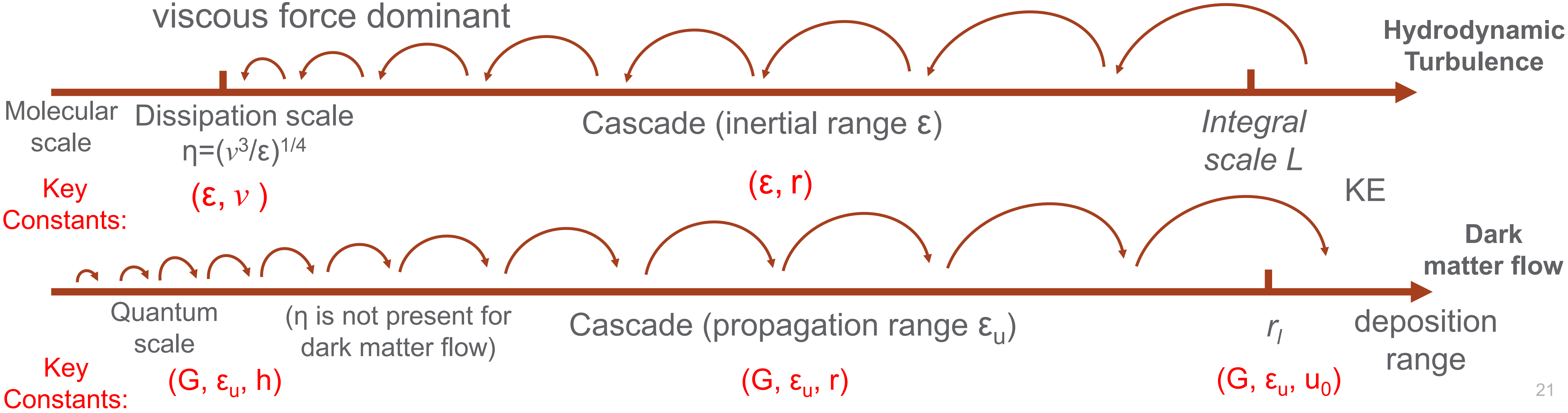




Energy cascade in turbulence and dark matter

Turbulence:
- Freely decaying (rate: ε)
- Direct energy cascade
- Vortex of different scales
- Integral scale: energy injection
- Inertial range:
  inertial >> viscous force
- Dissipation range:
  viscous force dominant

Dark matter flow:
- Freely growing (rate: ε_u): Virial theorem
- Inverse energy cascade
- Halos of different scales
- Collisionless, no dissipation range!
- The smallest length scale is not limited by viscosity.



Constant rate of energy cascade from N-body sim.

$$\frac{\partial E_y}{\partial t} + H\left(2K_p + P_y\right) = 0$$

Cosmic energy Equation (Irvine 1961)

$$\boxed{\begin{aligned} K_p &= -\varepsilon_{\mathbf{u}} t \\ P_y &= \frac{7}{5}\varepsilon_{\mathbf{u}} t \end{aligned}}$$

Power-law for Peculiar kinetic energy K_p

Power-law for potential energy P_y

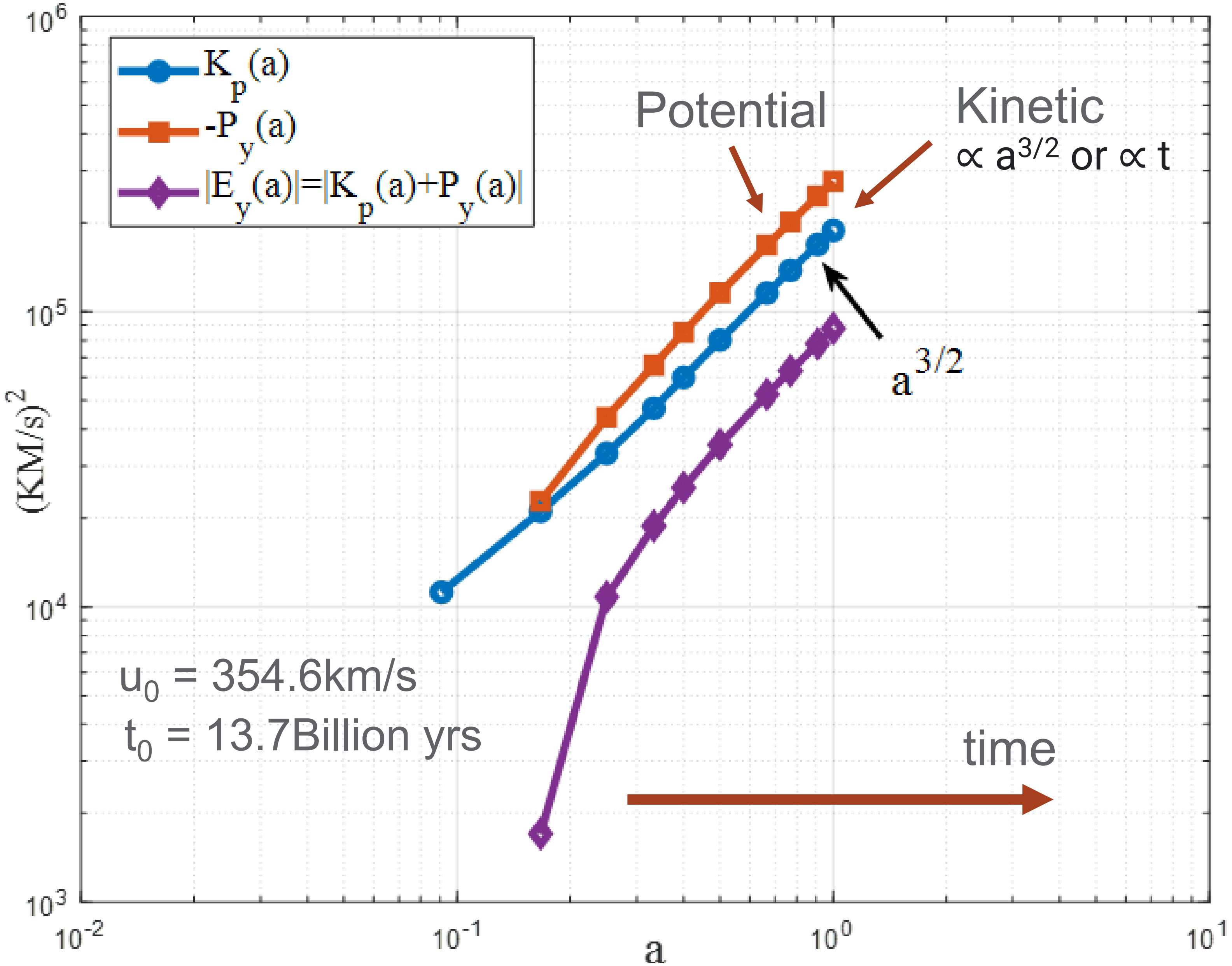
This rate ε_u is both time and scale independent, a fundamental constant!

From N-body simulation: (negative for inverse)

$$\varepsilon_u = -\frac{K_p}{t} = -\frac{3}{2}\frac{u_0^2}{t_0} \approx -4.6\times 10^{-7}\frac{m^2}{s^3} < 0$$

In Earth's atmosphere: $\varepsilon \approx 10^{-3}\, m^2/s^3$

In Galaxy bulge: $\varepsilon_b \approx 10^{-4}\, m^2/s^3$



The time variation of specific kinetic and potential energies from *N*-body simulation.

Pair conservation equation for validation

Pair conservation equation (Peebles 1980) relates the pairwise velocity with density correlation ξ:

$$\frac{\langle \Delta u_L \rangle}{Har} = -\frac{\left(1+\bar{\xi}(r,a)\right)}{3\left(1+\xi(r,a)\right)} \frac{\partial \ln\left(1+\bar{\xi}(r,a)\right)}{\partial \ln a}$$

For large scale in linear regime, average correlation

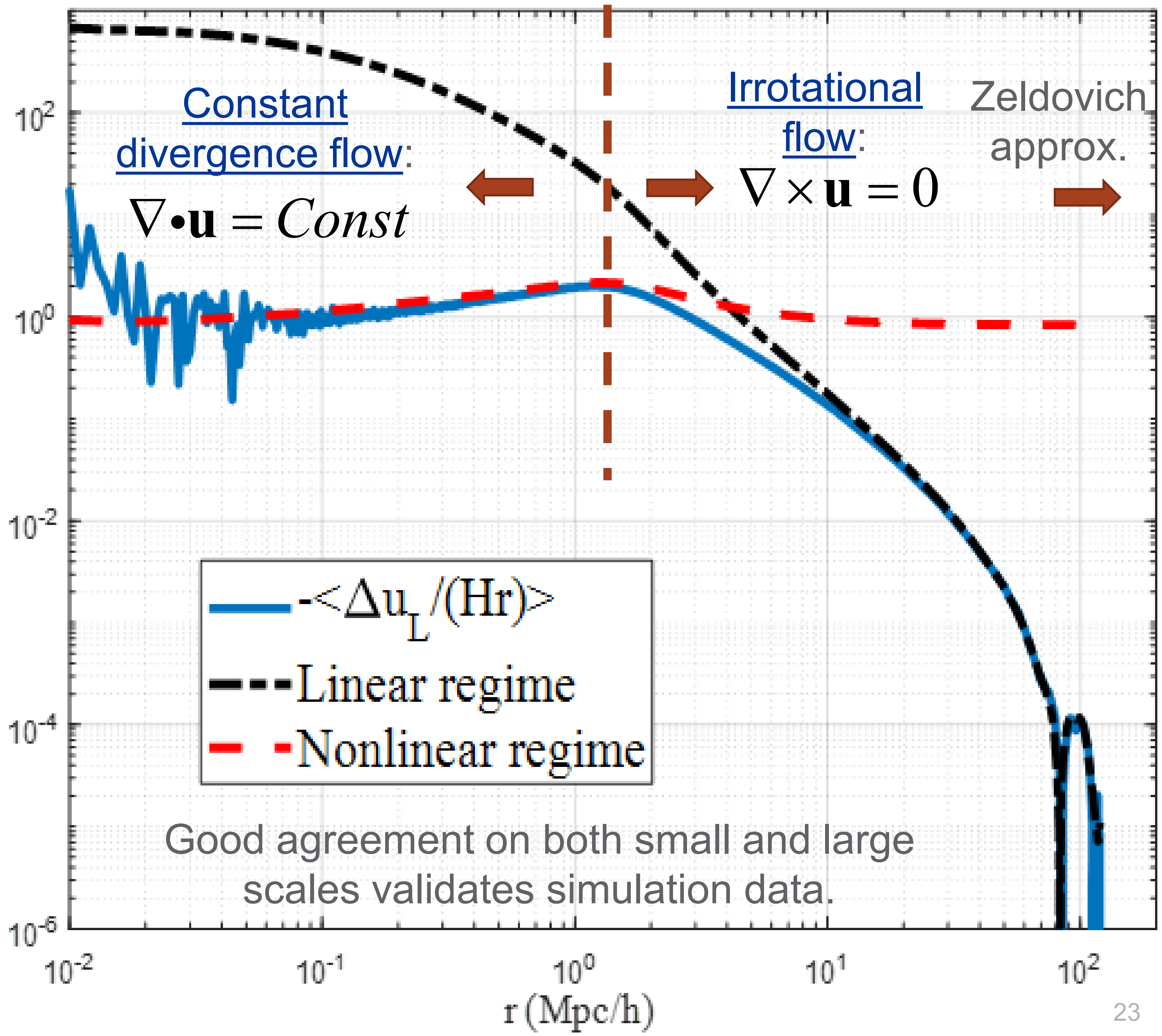
$\bar{\xi} \ll 1$ and $\partial \ln \bar{\xi} / \partial \ln a = 2$

$$\frac{\langle \Delta u_L \rangle}{Har} = -\frac{2\bar{\xi}(r,a)\left(1+\bar{\xi}(r,a)\right)}{3\left(1+\xi(r,a)\right)} \approx -\frac{2}{3}\bar{\xi}(r,a)$$

For small scale in non-linear regime (red dash),

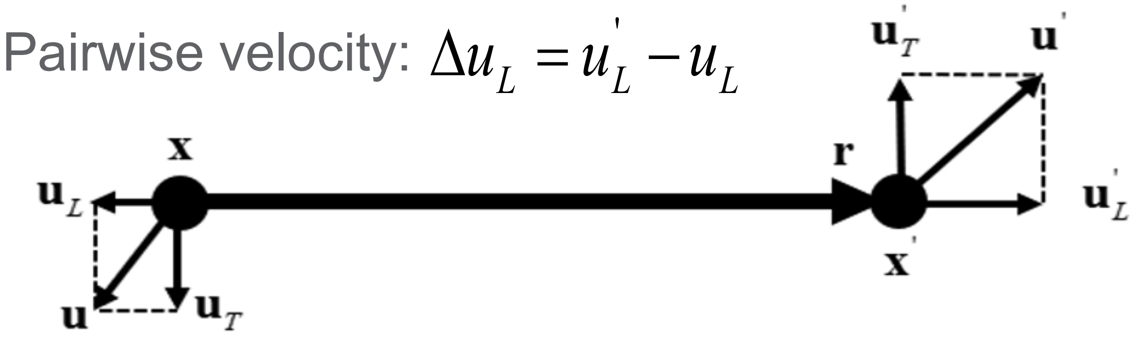
$\xi(r,a) \propto a^{\alpha} r^{\gamma}$ and $\partial \ln \bar{\xi} / \partial \ln a = \alpha$

Stable clustering hypothesis $\frac{\langle \Delta u_L \rangle}{Har} = -1$ ➡ $\alpha = \gamma + 3$



2/3 law for kinetic energy confirmed by N-body sim.

Pairwise velocity: $\Delta u_L = u_L^{'} - u_L$



$$S_2^{lp}(r,a) = \left\langle (\Delta u_L)^2 \right\rangle = \left\langle (u_L^{'} - u_L)^2 \right\rangle$$

Pairwise velocity dispersion (represents the kinetic energy on scale r):

$$S_2^{lp}(r) - 2u^2 = S_{2r}^{lp} = v_r^2 \propto (\varepsilon_u r)^{2/3}$$

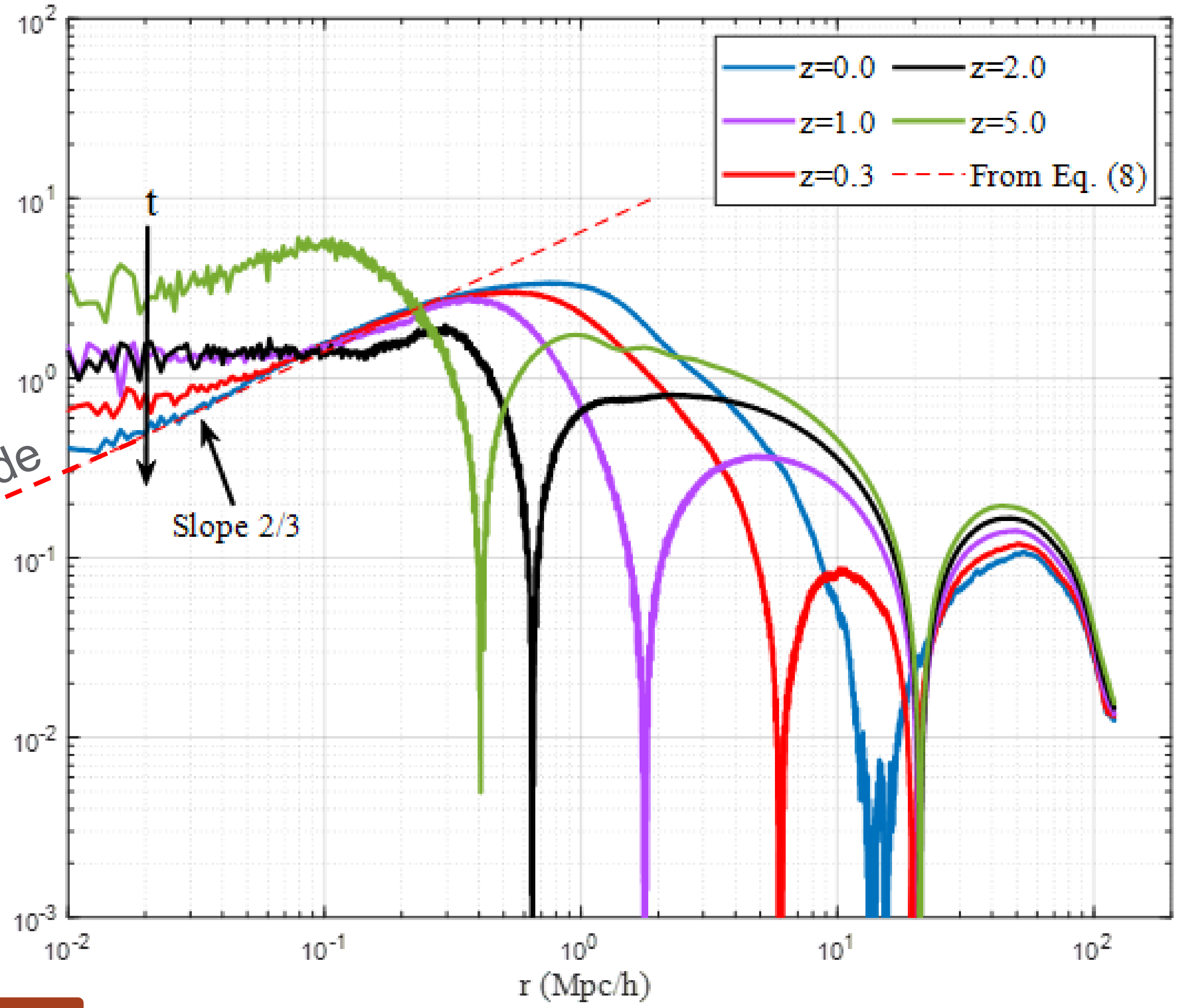
Kinetic Energy

$$(-\varepsilon_u) \propto \frac{v_r^2}{r} v_r = \frac{v_r^2}{r/v_r} = \frac{v_r^3}{r}$$

Acceleration Turnaround time

Due to collisionless: Extend all the way to the smallest scale for dark matter properties

Cascade



On scale r, kinetic energy follows a 2/3 law !

Variation of normalized reduced pairwise dispersion and two-thirds law

5/3 law for halo mass confirmed by N-body sim.

In propagation range, all relevant quantities are determined by G, ε_u, and scale *r*. This predicts:

Mass: $\boxed{m_r = \alpha_r \varepsilon_u^{2/3} G^{-1} r^{5/3}}$ 5/3 law

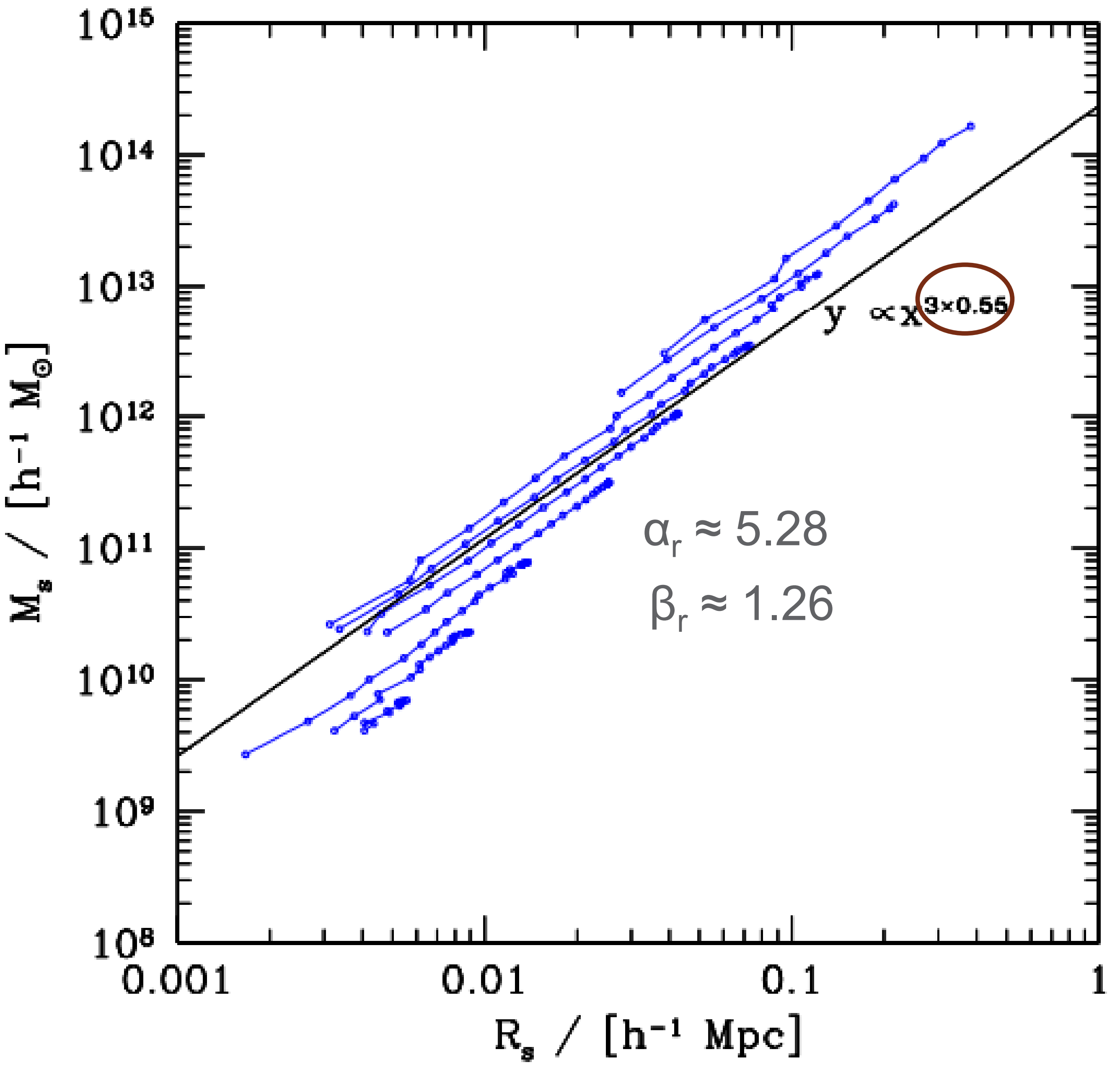
Density: $\rho_r = \beta_r \varepsilon_u^{2/3} G^{-1} r^{-4/3}$ -4/3 law

Kinetic energy: $v_r^2 = \left(\gamma_s \varepsilon_u\right)^{2/3} r^{2/3}$ 2/3 law

Time: $t_r \propto \varepsilon_u^{-1/3} r^{2/3}$

Halo mass m_r enclosed in scale *r* can be obtained from N-body simulations

5/3 law confirmed by N-body simulations



Variation of halo core mass m_r with scale radius r_s follows a 5/3 law (Zhao et al. 2009)

-4/3 law for halo density confirmed by rotation curves

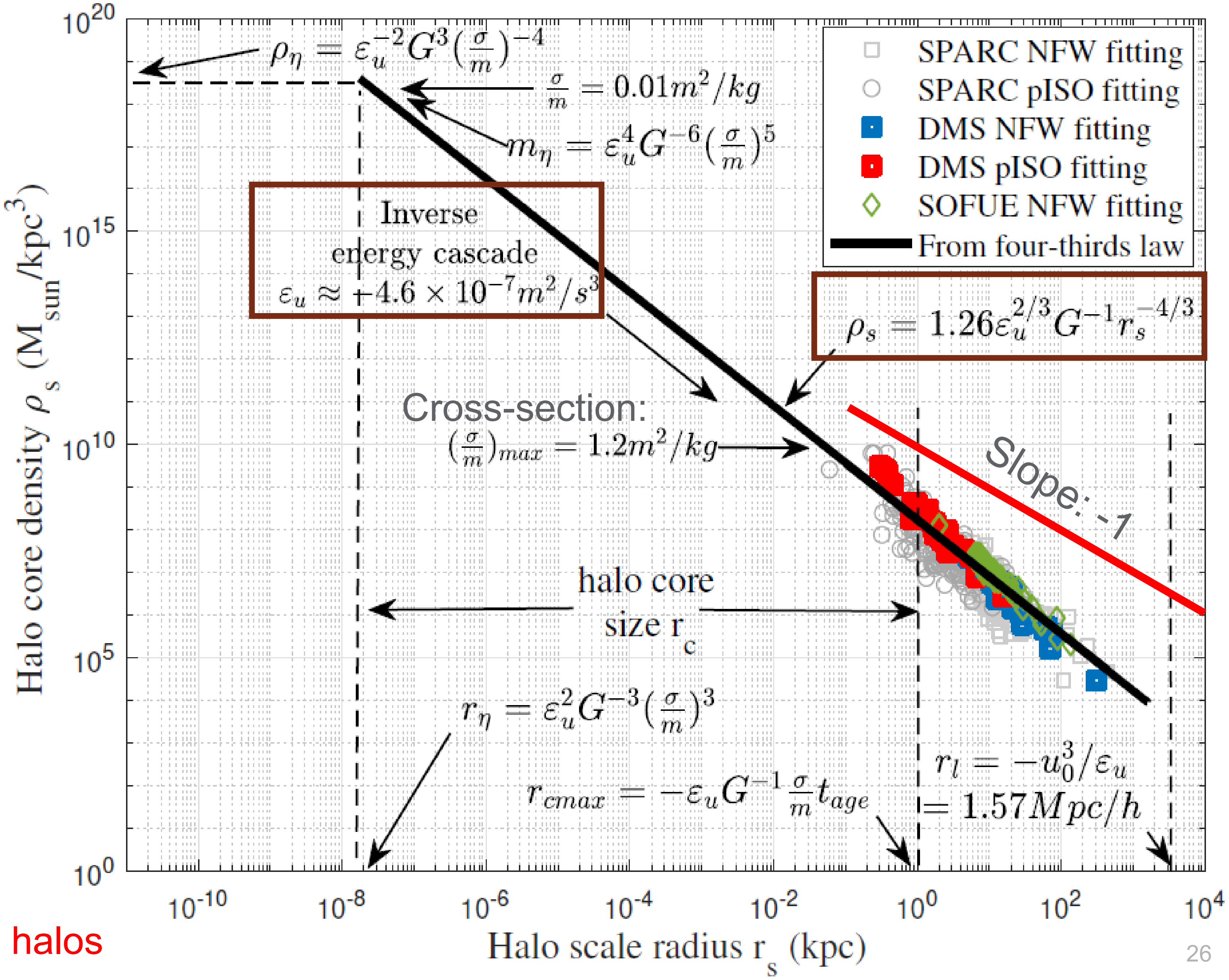
In propagation range, all relevant quantities are determined by G, ε_u, and scale *r*. This predicts:

Mass:	$m_r = \alpha_r \varepsilon_u^{2/3} G^{-1} r^{5/3}$	5/3 law
Density:	$\boxed{\rho_r = \beta_r \varepsilon_u^{2/3} G^{-1} r^{-4/3}}$	-4/3 law
Kinetic energy:	$v_r^2 = \left(\gamma_s \varepsilon_u\right)^{2/3} r^{2/3}$	2/3 law
Time:	$t_r \propto \varepsilon_u^{-1/3} r^{2/3}$	

Halo core density ρ_s and scale r_s radius can be obtained from galaxy rotation curves

-4/3 law confirmed by rotation curves

Cuspy density for fully virialized collisionless DM halos



Quick Recap II

In dark matter flow (DMF):

- Inverse cascade of kinetic energy from small to large scales (constant rate: ε_u m^2/s^3)
- Direct cascade of potential energy from large to small scales
- Two cascade connected by virial theorem

On any scale *r*, energy cascade predicts scaling laws on small scale:
(confirmed by N-body simulations and galaxy rotation curves)

Mass:	$m_r = \alpha_r \varepsilon_u^{2/3} G^{-1} r^{5/3}$	5/3 law
Density:	$\rho_r = \beta_r \varepsilon_u^{2/3} G^{-1} r^{-4/3}$	-4/3 law
Kinetic energy:	$v_r^2 = \left(\gamma_s \varepsilon_u\right)^{2/3} r^{2/3}$	2/3 law
Time:	$t_r \propto \varepsilon_u^{-1/3} r^{2/3}$	

Extend to the smallest scale for collisionless DM

Two hypothesis:

- Dark matter is fully collisionless
- Gravity is the only interaction

On the smallest scale:

$$m_X v_X \cdot l_X/2 = \hbar \quad \text{Uncertainty principle}$$

$$v_X^2 = G m_X / l_X \quad \text{Virial theorem}$$

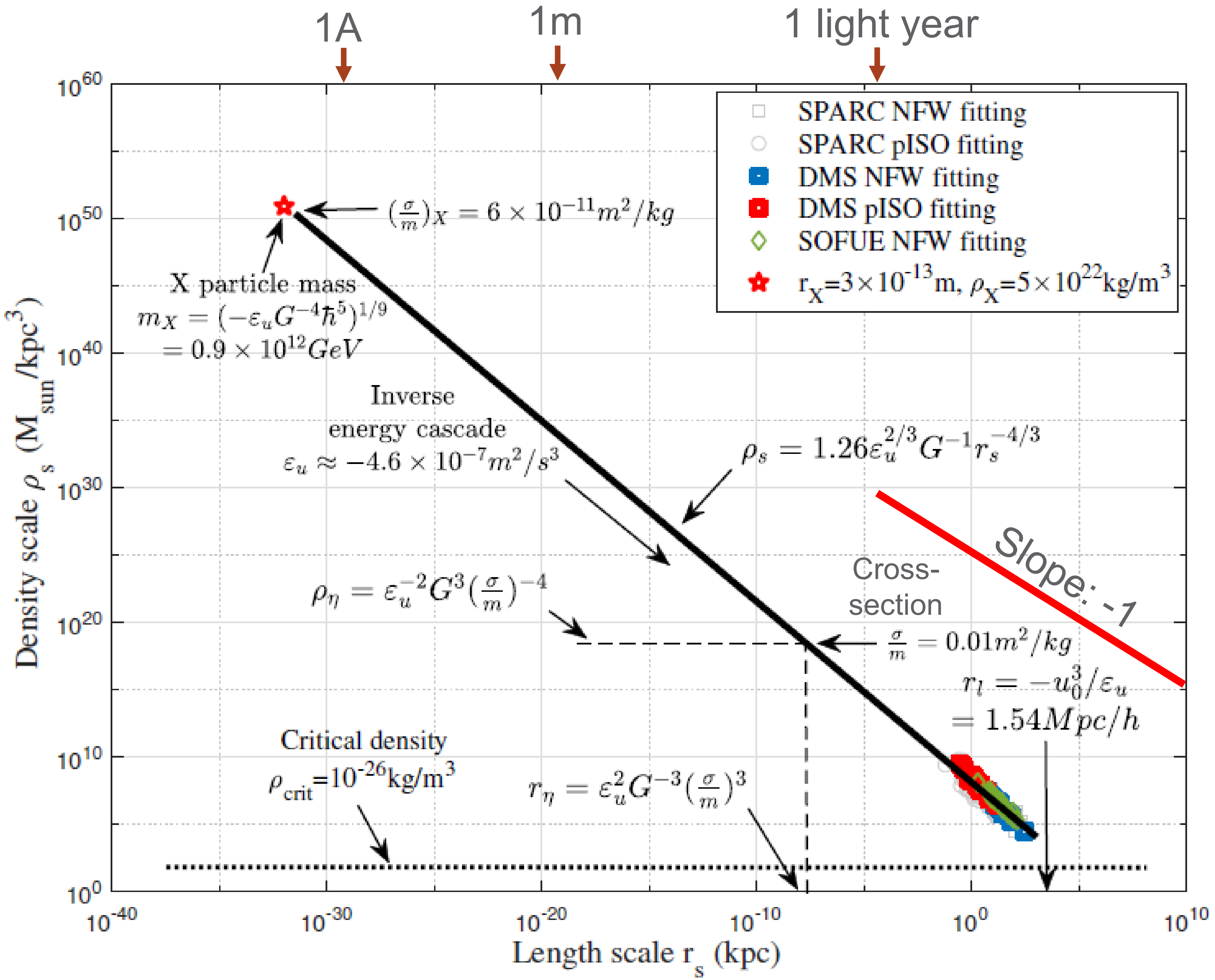
$$(-\varepsilon_u) = v_X^3 / l_X \quad \text{Constant energy cascade}$$

Energy cascade in DMF predicts:

Mass scale: $m_X \propto \left(-\varepsilon_u \hbar^5 / G^4\right)^{\frac{1}{9}} \approx 10^{12} GeV$

Length scale: $l_X \propto \left(-G\hbar/\varepsilon_u\right)^{\frac{1}{3}} \approx 10^{-13} m$

Time scale: $t_X \propto \left(G^2\hbar^2/\varepsilon_u^5\right)^{\frac{1}{9}} \approx 10^{-7} s$



Dark matter particle mass, size, and properties

Density scale: $\rho = m_X / l_X^3 \approx 5.33\times10^{22}\, kg/m^3$ ⟷ Nuclear density: $10^{17}\, kg/m^3$

Power scale (Joule/s): $\mu_X = m_X a_X \cdot v_X = -m_X \varepsilon_u = 7.44\times10^{-22}\, kg\cdot m^2/s^3 = 0.0046\, eV/s$

Energy scale: $\mu_X t_X/4 = \hbar/t_X = \frac{1}{2} m_X v_X^2 = \boxed{0.87\times10^{-9}\, eV}$ ⟷ Rydberg energy of 13.6 eV for the ionization energy of the hydrogen atom

Particle lifetime: $\tau_X = \frac{m_X c^2}{\mu_X} = \boxed{-\frac{c^2}{\varepsilon_u}} = 6.2\times10^{15}\, yr$

If $\tau_X > 13.7\times10^{9}\, yr$ ⟹ $\varepsilon_u < 0.21\, m^2/s^3$

If instantons are responsible for the decay [1]:

$$\tau_X = \frac{\hbar e^{1/\alpha_X}}{m_X c^2} = 6.2\times10^{15}\, yr \Rightarrow \alpha_X \approx \frac{1}{136.85}$$

Dynamic viscosity: $\eta = -\varepsilon_u / G \approx 6900\, Pa\cdot s$ Peanut Butter?

Pressure scale: Number density

$$P_X = \frac{m_X a_X}{l_X^2} = \frac{8\hbar^2}{m_X} \rho_{nX}^{5/3} = 1.84\times10^{10}\, Pa$$

analogue of the degeneracy pressure of Fermi gas

Cross section: $l_X^2 v_X = 4\times10^{-32}\, m^3 s^{-1}$

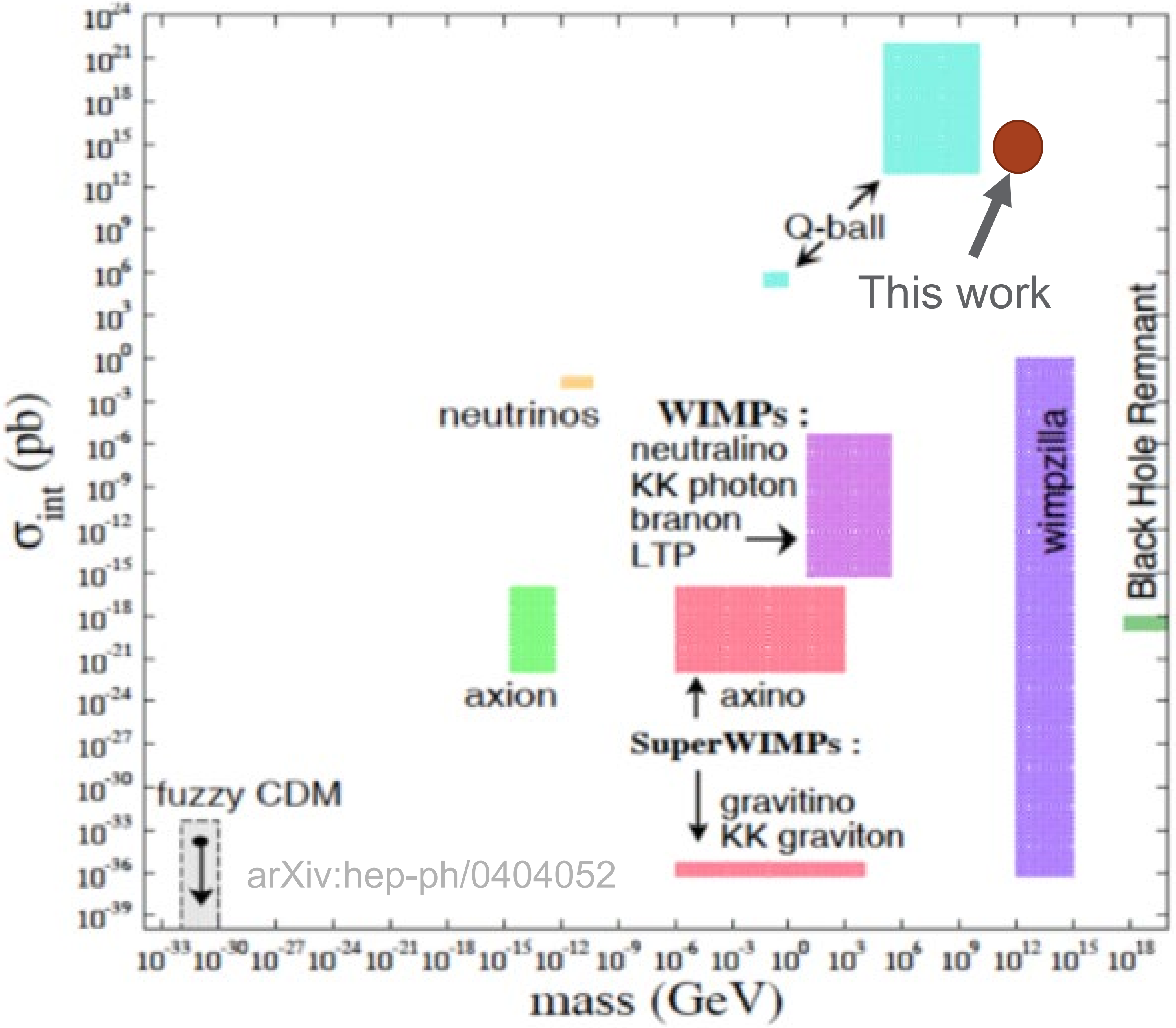
⟷

WIMP miracle: $\langle \sigma v \rangle = 3\times10^{-32}\, m^3 s^{-1}$

Kinematic viscosity for momentum transfer (collisionless): $\upsilon = \eta/\rho \approx 1.3\times10^{-19}\, m^2/s$

[1] Anchordoqui, L.A., et al.,Astroparticle Physics, 2021. 132.

Where is our prediction?



From this prediction:
- Much heavier than WIMP
- Much heavier than axion
- Comparable to Wimpzilla

Two hypothesis:
- DM is fully collisionless
- Gravity is the only interaction

If cannot detect DM at mass of 10^{12}Gev, then
- DM is self-interacting?
- Involve unknown forces?
- How to be consistent with cascade theory?

- Potential flaws in this argument?
- Any impacts on the detection methods?

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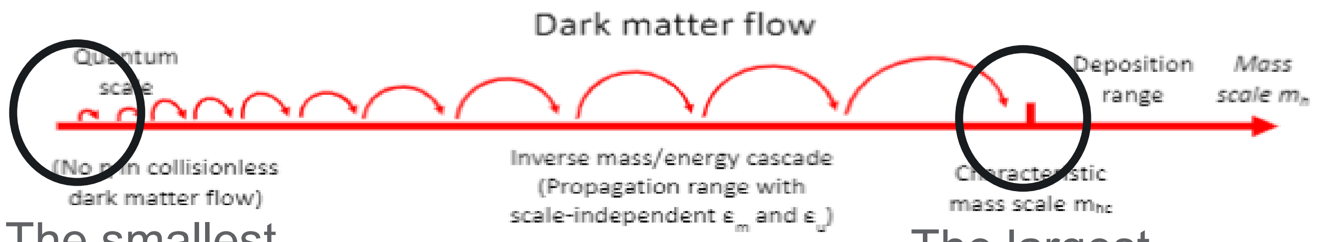
Critical scales in collisionless dark matter flow



The smallest scale

The largest scale

On the smallest scale, three fundamental constants:

Gravitational constant	$G = 6.67\times10^{-11}\, m^3/\left(kg\cdot s^2\right)$
Rate of energy cascade	$\varepsilon_u = -4.6\times10^{-7}\, m^2/s^3$
Planck constant	$\hbar = 1.05\times10^{-34}\, kg\cdot m^2/s$

Simple dimensional analysis predicts:

Mass scale: $m_X \propto \left(-\varepsilon_u \hbar^5/G^4\right)^{\frac{1}{9}} \approx 10^{12}\, GeV$

Length scale: $l_X \propto \left(-G\hbar/\varepsilon_u\right)^{\frac{1}{3}} \approx 10^{-13}\, m$

Time scale: $t_X \propto \left(G^2\hbar^2/\varepsilon_u^5\right)^{\frac{1}{9}} \approx 10^{-6}\, s$

Three fundamental constants on large scale:

Gravitational constant	$G = 6.67\times10^{-11}\, m^3/\left(kg\cdot s^2\right)$
Rate of energy cascade	$\varepsilon_u = -4.6\times10^{-7}\, m^2/s^3$
Velocity dispersion	$u_0 \equiv u\left(a=1\right) = 354.61\, km/s$

Simple dimensional analysis predicts:

Mass scale: $m_L \propto -u_0^5/\left(G\varepsilon_u\right) \approx 9.14\times10^{13}\, M_\odot$

Length scale: $l_L \propto -u_0^3/\varepsilon_u \approx 3.14\, Mpc$

Time scale: $t_L \propto u_0^2/\varepsilon_u \approx 8.7\times10^9\, yr$

Critical scales for self-interacting dark matter

On the smallest length scale:

$$\rho_r(\sigma/m)v_r t_r = 1$$ Elastic scatter

$$v_s^2 = Gm_r(r_s)/r_s$$ Virial theorem

$$-\varepsilon_u = v_s^3/\gamma_s r_s$$ Constant energy cascade

All relevant quantities determined by G, cross-section σ/m and ε_u :

Length or minimum halo core size: $$r_\eta = \varepsilon_u^2 G^{-3}(\sigma/m)^3$$

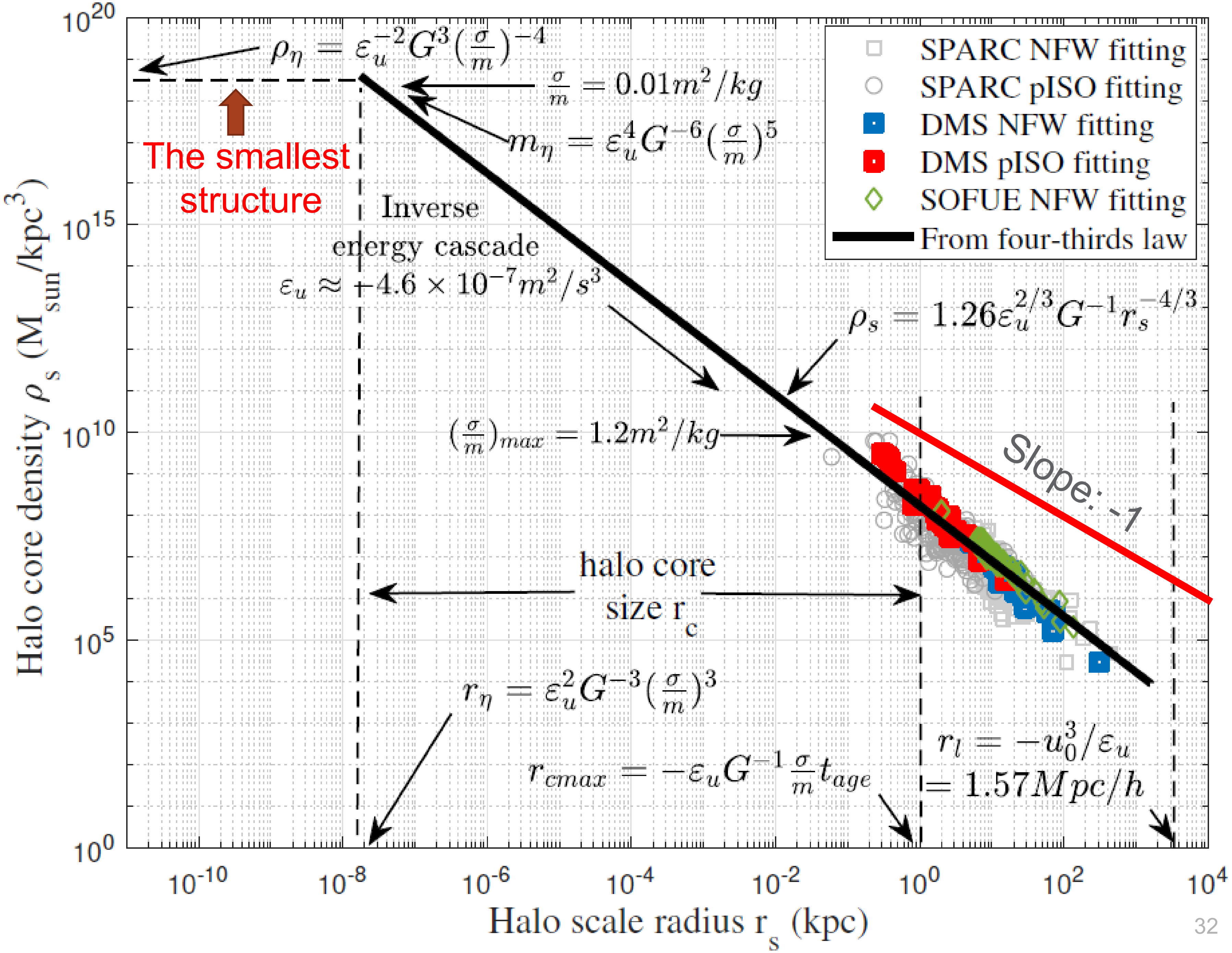
Mass scale: $$m_\eta = \varepsilon_u^4 G^{-6}(\sigma/m)^5$$

Density scale: $$\rho_\eta = \varepsilon_u^{-2} G^3(\sigma/m)^{-4}$$

Maximum halo core size r_{cmax} :

$$\rho_r \frac{\sigma}{m} v_r t_{age} = 1$$ t_{age}: age of Universe;

$$\frac{r_{c\max}}{(\sigma/m)} = -\varepsilon_u G^{-1} t_{age} \approx 10kpc\frac{g}{cm^2}$$



The origin of energy cascade: Uncertainty principle?

Position (**x**), Velocity (**v** = d**x**/dt), Acceleration (**a** = d**v**/dt)

For fully collisionless dark matter:

1) A unique "symmetry" between **x** and **v** in phase space:

- At given **x** , particles can have multiple **v** (multi-stream)
- For given **v,** particles can be at different **x**
- NOT possible for non-relativistic baryons

2) Due to the long-rang gravitational interaction,

- Fluctuations (uncertainty) in **x**
- Fluctuations (uncertainty) in **v**
- Fluctuations (uncertainty) in **a**

3) Two pairs of conjugate variables:

- Position **x** and momentum **p**
- Momentum **p** and acceleration **a**

Wave function for position: $\psi(x)$

Wave function for momentum: $\varphi(p)$

Wave function for acceleration: $\mu(a)$

$$\mu_X = -m_X \varepsilon_u = 7.44\times 10^{-22}\, kg\cdot m^2/s^3$$

$$\boxed{\begin{aligned}\psi(x) &= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\varphi(p)e^{ipx/\hbar}dp\\ \varphi(p) &= \frac{1}{\sqrt{2\pi\mu_X}}\int_{-\infty}^{\infty}\mu(a)e^{ipa/\mu_X}da\end{aligned}}$$

Uncertainty principles: $\sigma_x\sigma_p \ge \hbar/2$ $\quad\sigma_p\sigma_a \ge \mu_X/2$

Postulated uncertainty principle for ***a*** and ***p***
leads to the constant rate of energy cascade: $\varepsilon_u = \mu_X/m_X = a_X v_X$

Quick Recap III

- If DM is fully collisionless:
  - Scaling laws extended to the smallest scale (quantum)
  - Dark matter mass, size, density, pressure, lifetime, cross-section, etc.
  - The origin of cascade: uncertainty principle between momentum and acceleration?

- If DM is self-interacting:
  - The smallest scale determined by G, cross-section σ/m and ε_u
  - Smallest structure size (dependent on σ/m)
  - Maximum core size (dependent on σ/m)
  - Observational constraint for σ/m?

- Suggestions on the current work?

- Suggestions on the future work?

- Suggestion on the potential collaboration?
  - Hydrodynamic simulations?
  - Self-interaction DM simulations?
  - Code, data processing?

Correlation/moment functions from N-body sim.

Velocity correlation:	Longitudinal:	$L_2(r) = \langle u_L u_L' \rangle$	$\langle u_L^2 \rangle$
	Transverse:	$T_2(r) = \langle \mathbf{u}_T \cdot \mathbf{u}_T' \rangle / 2$	$\langle u_T^2 \rangle$
	Total:	$R_2(r) = \langle \mathbf{u} \cdot \mathbf{u}' \rangle = L_2 + 2T_2$	$\langle u^2 \rangle$
Density correlation:		$\xi(r) = \langle \delta \cdot \delta' \rangle$	2nd moment

For incompressible or constant divergence flow (small scale):

$$T_2 = \frac{1}{2r}\left(r^2 L_2\right)_{,r} \qquad R_2 = \frac{1}{r^2}\left(r^3 L_2\right)_{,r}$$

$$L_2(r) = \int_0^\infty E_u(k) \frac{2 j_1(kr)}{kr} dk$$

$$T_2(r) = \int_0^\infty E_u(k) \left(j_0(kr) - \frac{j_1(kr)}{kr} \right) dk$$

*n*th order spherical Bessel function of the first kind: $j_n(kr)$

Kinematic relations

Relations to power spectrum function

Relations to density correlation function

For irrotational flow on large scale:

$$R_2 = \frac{1}{r^2}\left(r^3 T_2\right)_{,r} \qquad L_2 = \left(r T_2\right)_{,r}$$

$$L_2(r) = 2\int_0^\infty E_u(k) \left(j_0(kr) - 2\frac{j_1(kr)}{kr} \right) dk$$

$$T_2(r) = \int_0^\infty E_u(k) \frac{2 j_1(kr)}{kr} dk$$

$$\xi(r,a) = \langle \delta(\mathbf{x}) \cdot \delta(\mathbf{x}') \rangle = \frac{\langle \theta(\mathbf{x}) \cdot \theta(\mathbf{x}') \rangle}{(aHf(\Omega_m))^2}$$

$$= -\frac{1}{(aHf(\Omega_m))^2} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial R_2}{\partial r} \right) \right]$$

Kinematic and dynamics relations for vel. correlation

Table 2. The velocity correlation functions of different order

p	$q=0$	$q=1$	$q=2$	$q=3$	$q=4$	$q=5$
1	$L_{(1,0)}=\langle u_L' \rangle$			$(p-q-1)R_{(p,q+1)}=\frac{1}{r^{p-q}}\left(r^{p-q+1}L_{(p,q)}\right)_{,r}$		
2	$L_{(2,0)}=\langle u_L u_L' \rangle$	$R_{(2,1)}=\langle \mathbf{u}\cdot\mathbf{u}' \rangle$		$\left(R_{(p,q+1)}r\right)_{,r}+(p-q-2)L_{(p,q+2)}=\frac{1}{r^{p-q}}\left(r^{p-q+1}L_{(p,q)}\right)_{,r}$		
3	$L_{(3,0)}=\langle u_L^2 u_L' \rangle$	$R_{(3,1)}=\langle u_L \mathbf{u}\cdot\mathbf{u}' \rangle$	$L_{(3,2)}=\langle u^2 u_L' \rangle$			
4	$L_{(4,0)}=\langle u_L^3 u_L' \rangle$	$R_{(4,1)}=\langle u_L^2 \mathbf{u}\cdot\mathbf{u}' \rangle$	$L_{(4,2)}=\langle u^2 u_L u_L' \rangle$	$R_{(4,3)}=\langle u^2 \mathbf{u}\cdot\mathbf{u}' \rangle$		
5	$L_{(5,0)}=\langle u_L^4 u_L' \rangle$	$R_{(5,1)}=\langle u_L^3 \mathbf{u}\cdot\mathbf{u}' \rangle$	$L_{(5,2)}=\langle u^2 u_L^2 u_L' \rangle$	$R_{(5,3)}=\langle u^2 u_L \mathbf{u}\cdot\mathbf{u}' \rangle$	$L_{(5,4)}=\langle u^4 u_L' \rangle$	
6	$L_{(6,0)}=\langle u_L^5 u_L' \rangle$	$R_{(6,1)}=\langle u_L^4 \mathbf{u}\cdot\mathbf{u}' \rangle$	$L_{(6,2)}=\langle u^2 u_L^3 u_L' \rangle$	$R_{(6,3)}=\langle u^2 u_L^2 \mathbf{u}\cdot\mathbf{u}' \rangle$	$L_{(6,4)}=\langle u^4 u_L u_L' \rangle$	$R_{(6,5)}=\langle u^4 \mathbf{u}\cdot\mathbf{u}' \rangle$

For constant divergence flow

For irrotational flow

Kinematic relations (for same order p)

Dynamic relations (for different order p)

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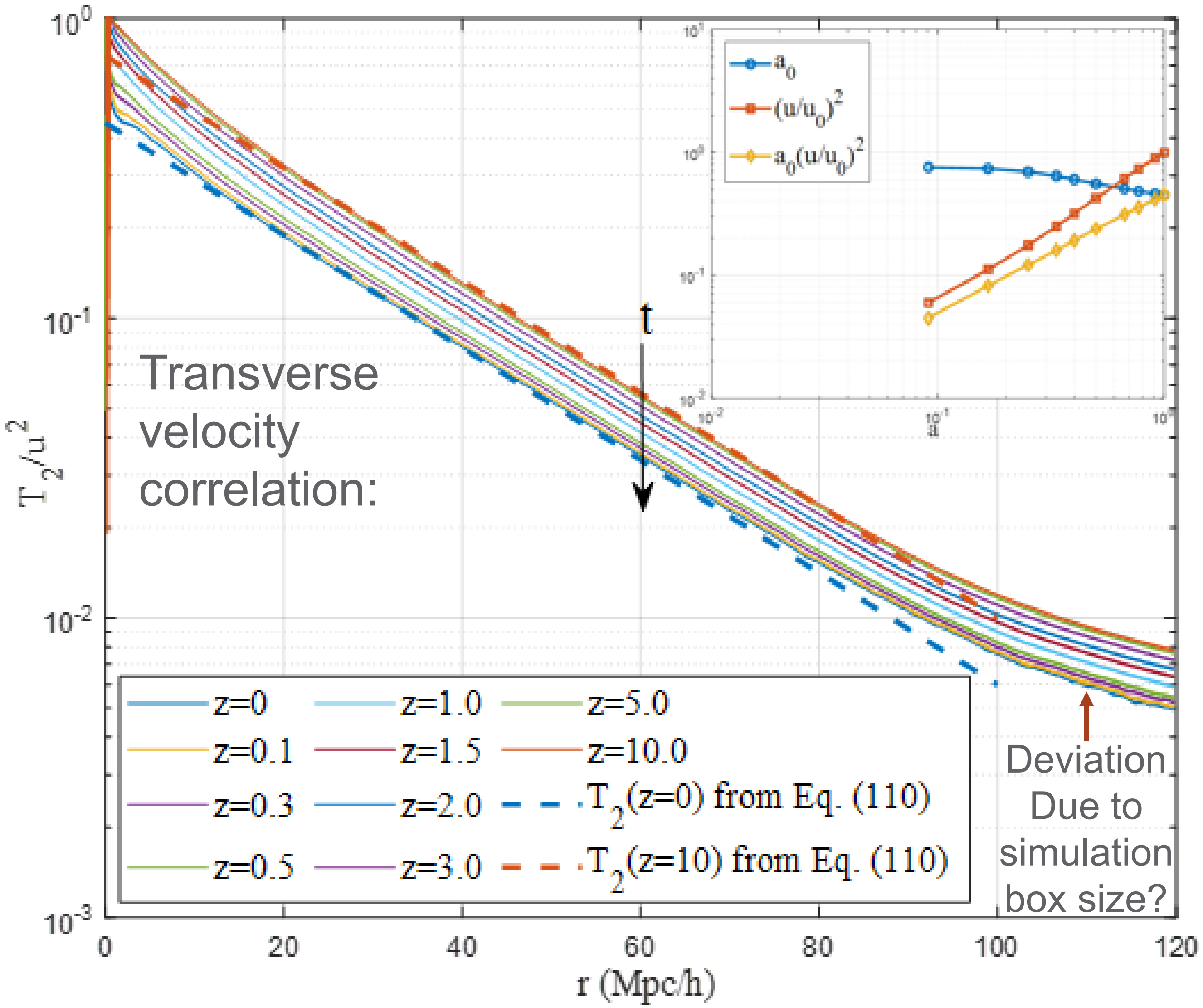
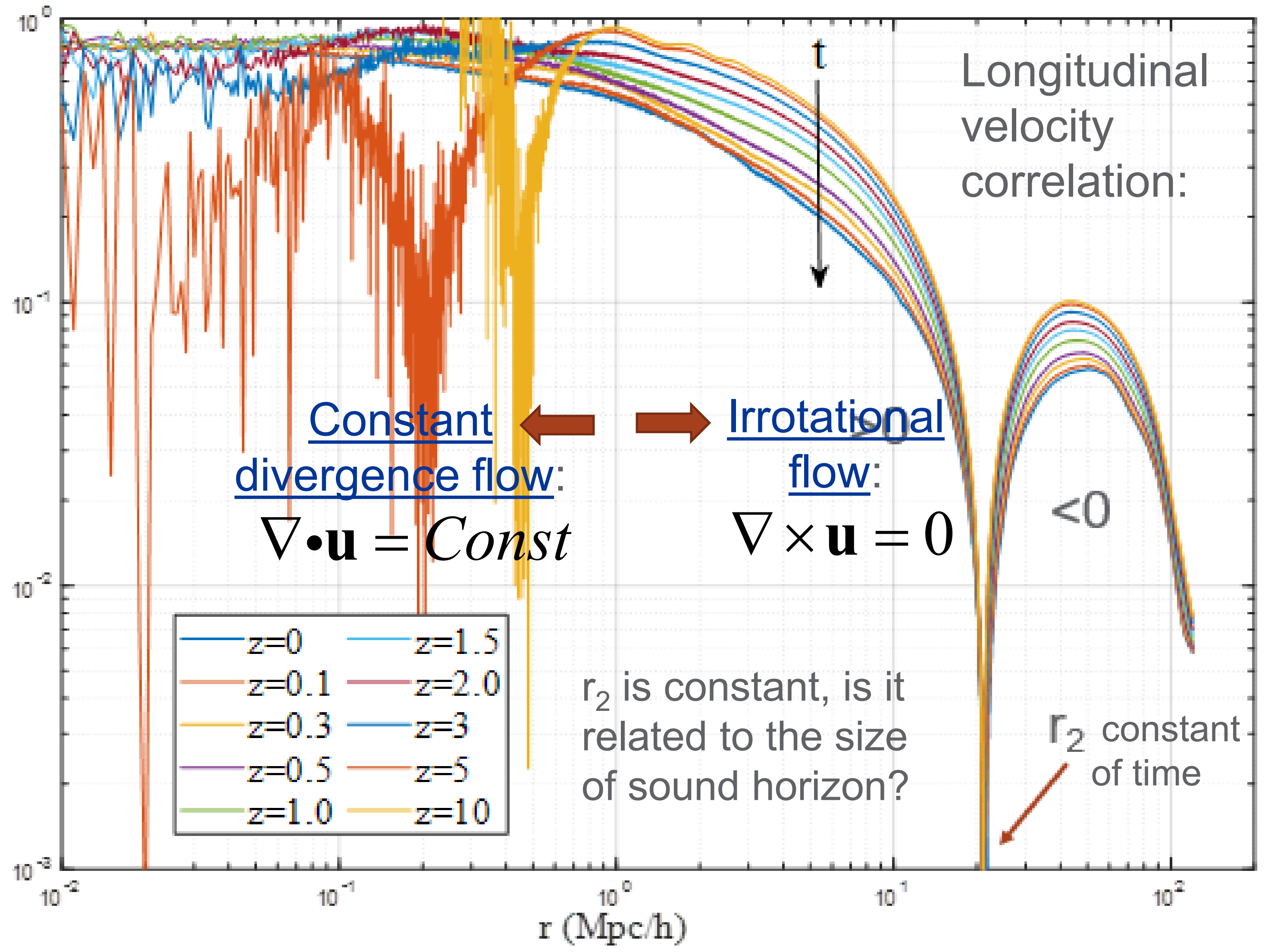
Velocity correlation functions on large scale

On large scale, velocity correlation (exponential): applying kinematic relations for irrotational flow

$$T_2(r,a) = a_0 u^2 \exp(-r/r_2) \propto a \;\Rightarrow\; L_2(r,a) = a_0 u^2 \exp\left(-\frac{r}{r_2}\right)\left(1-\frac{r}{r_2}\right) \;\Rightarrow\; R_2(r,a) = a_0 u^2 \exp\left(-\frac{r}{r_2}\right)\left(3-\frac{r}{r_2}\right)$$



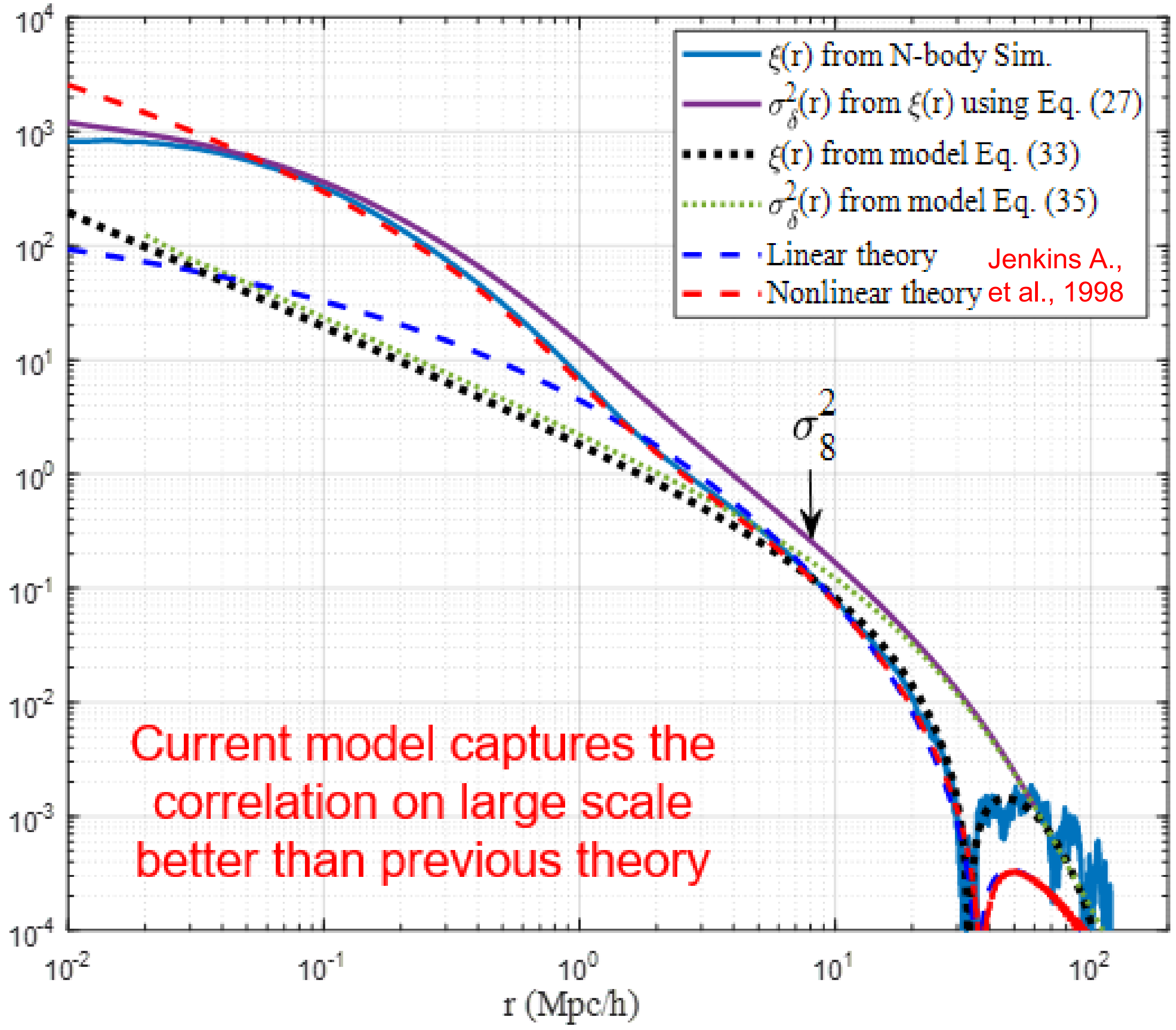


Density correlation function on large scale

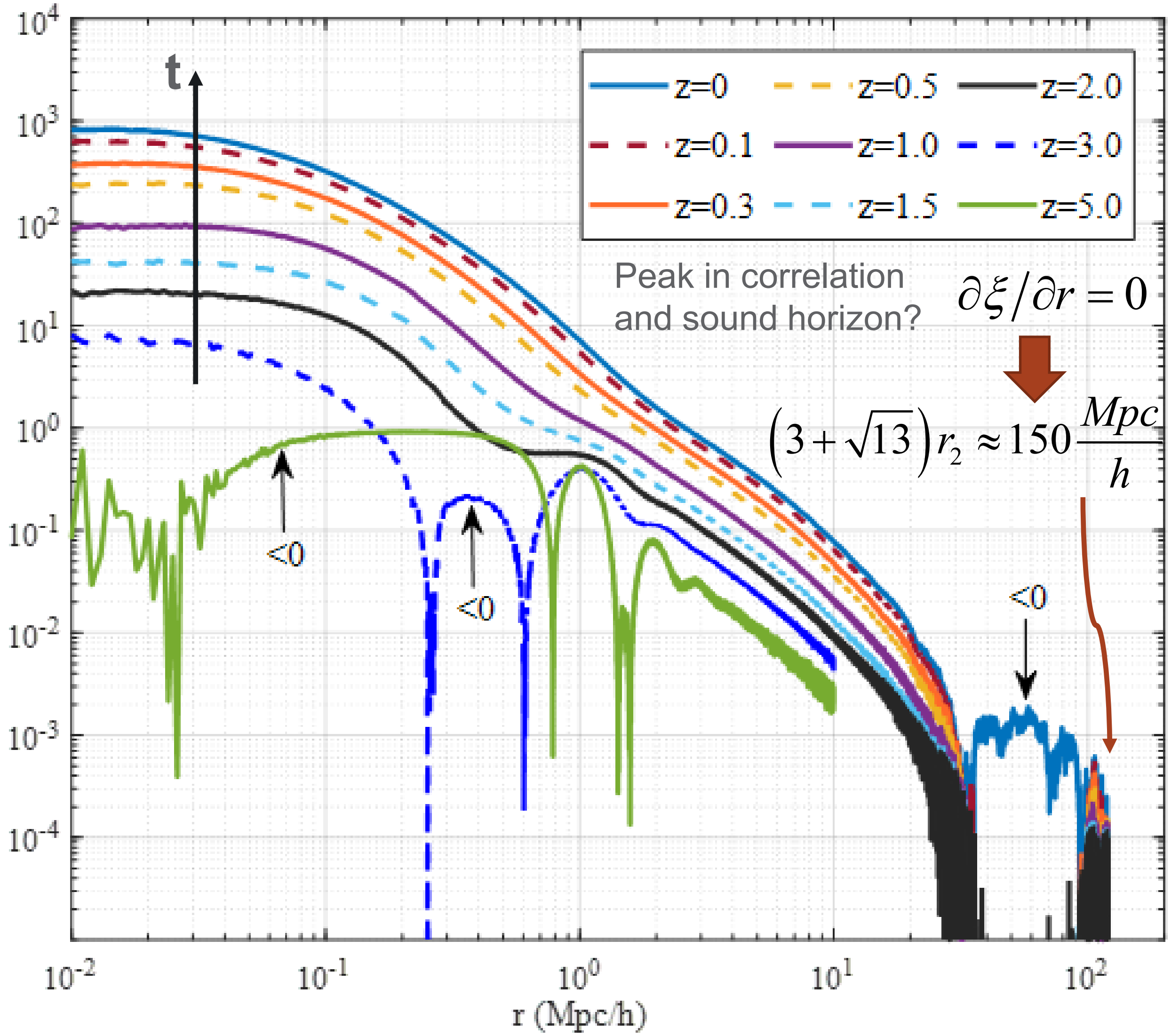
On large scale, density correlation (exponential):

$$R_2(r,a) = a_0 u^2 \exp\left(-\frac{r}{r_2}\right)\left(3-\frac{r}{r_2}\right)$$

$$\xi(r,a) = \frac{1}{\left(aHf(\Omega_m)\right)^2}\cdot\frac{a_0 u^2}{r r_2}\exp\left(-\frac{r}{r_2}\right)\left[\left(\frac{r}{r_2}\right)^2 - 7\left(\frac{r}{r_2}\right)+8\right]$$

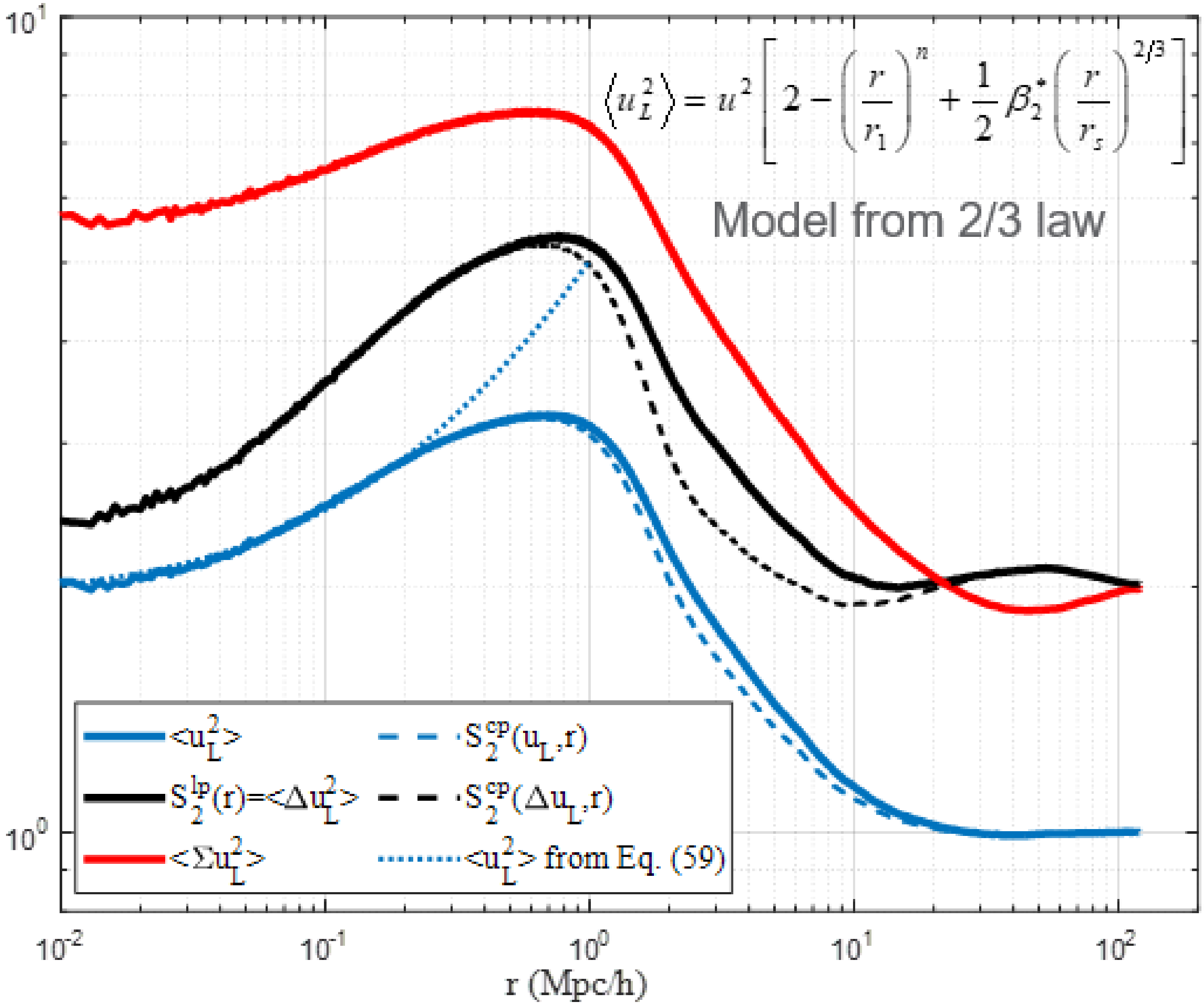


Density correlation function at z=0

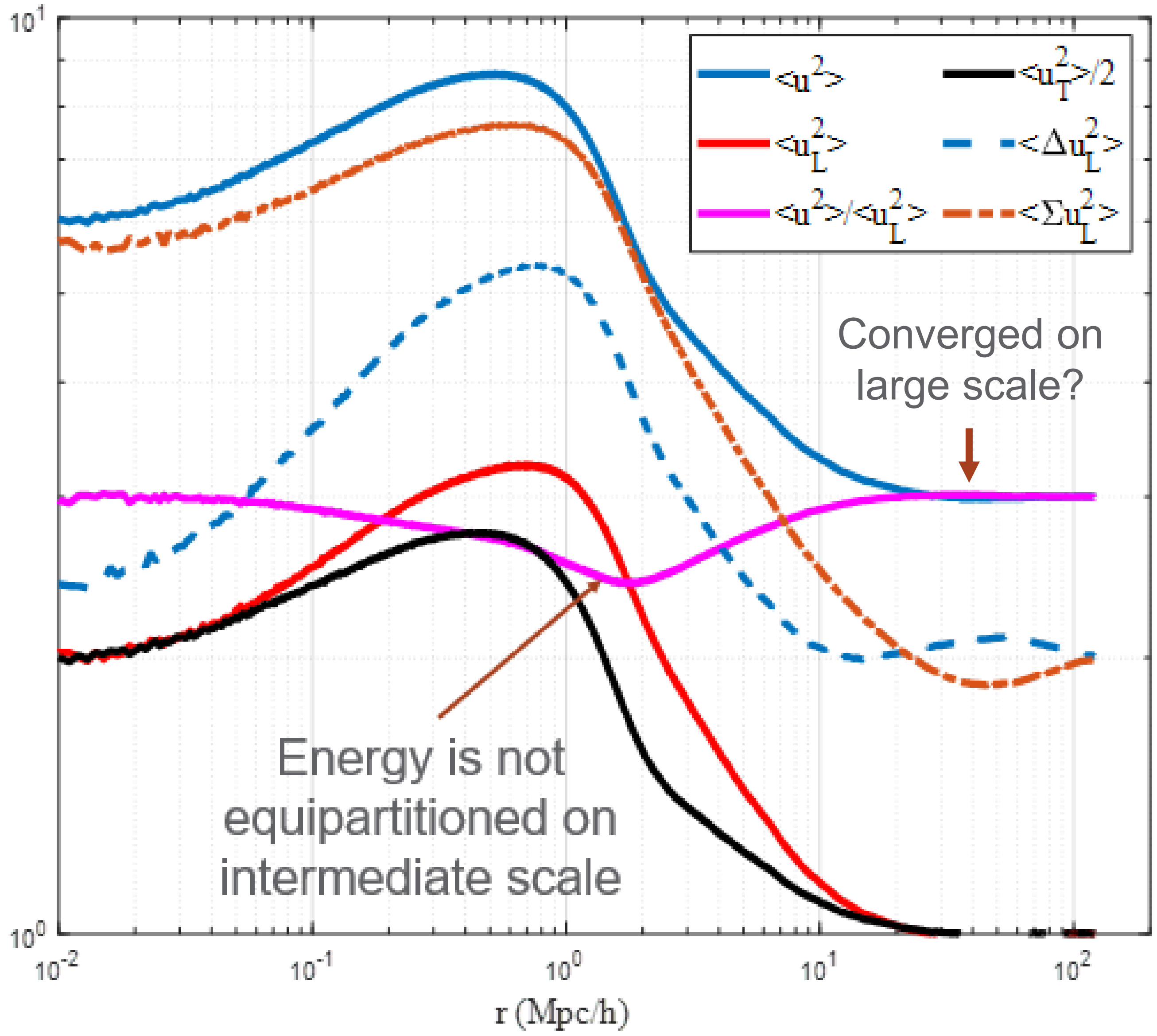


Density correlation function at different z

Second moments of velocity field



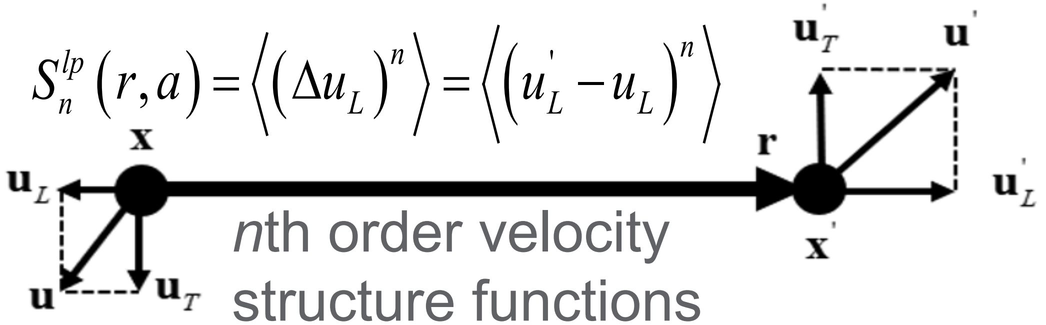
Increase of velocity dispersions with r for $r<r_t$ (pair of particles are more likely from same halos) is mostly due to the increase of velocity dispersion with halo size.



Second moment of velocity (normalized by u^2) varying with scale r at z=0

Two-thirds law and generalized stable clustering (GSCH)

$$S_n^{lp}(r,a) = \left\langle (\Delta u_L)^n \right\rangle = \left\langle (u_L^{'} - u_L)^n \right\rangle$$



*n*th order velocity structure functions

Zeroth order: $S_0^{lp}(r,a) = 1$

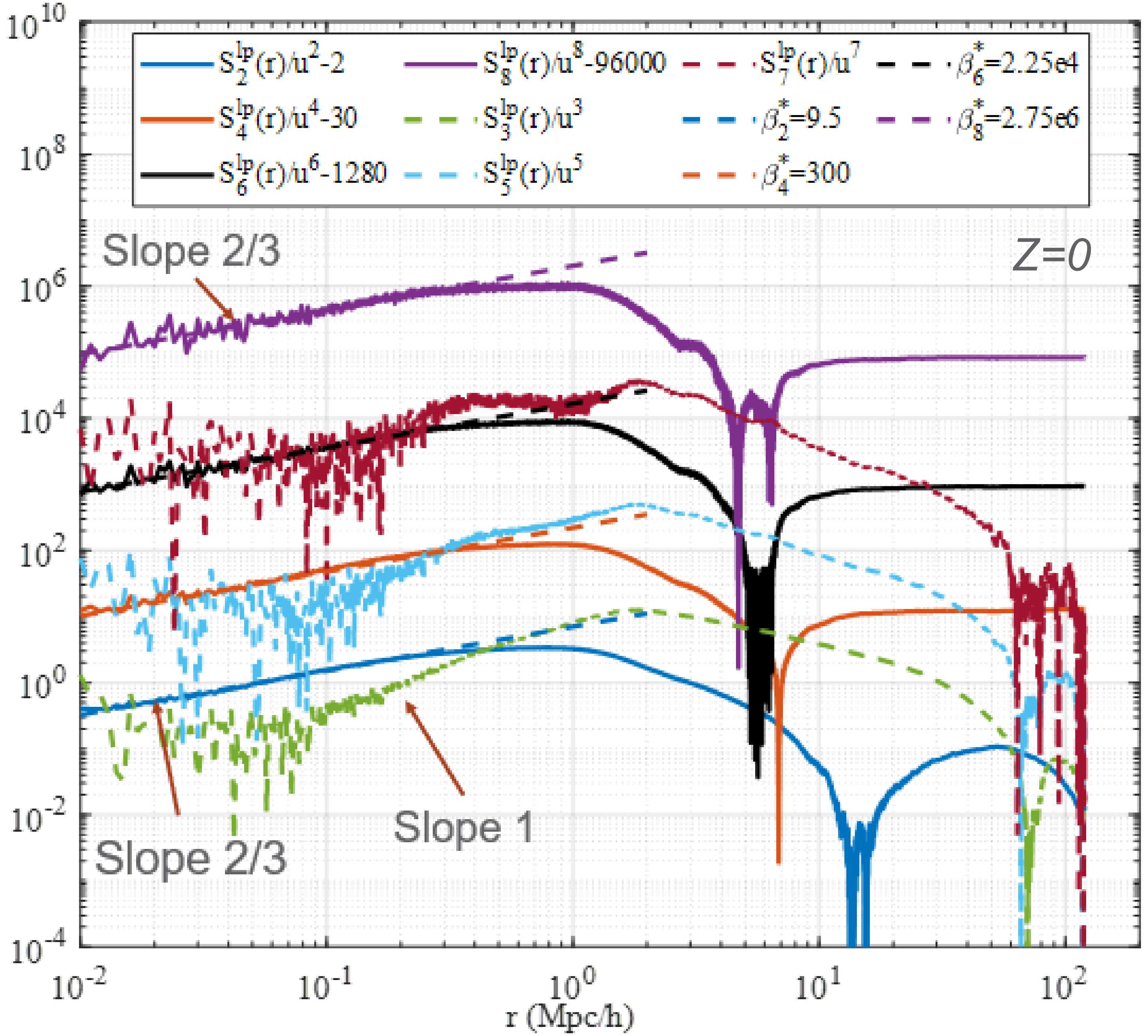
First order: $S_1^{lp}(r,a) = \left\langle \Delta u_L \right\rangle = -Har$ *Stable clustering hypothesis*

All even order reduced structure functions follow two-thirds law:

$$S_{2n}^{lp}(r) = u^{2n}\left[2^n K_{2n}(\Delta u_L, 0) + \beta_{2n}^{*}(r/r_s)^{2/3}\right]$$

All odd order structure functions follow linear law from generalized stable clustering hypothesis

$$S_{2n+1}^{lp}(r) = (2n+1) S_1^{lp}(r) S_{2n}^{lp}(r) \propto r^{1}$$



Maximum entropy distributions in kinetic theory of gases

Review on how to derive maximum entropy distributions (Boltzmann distribution)

Assume the distribution of one-dimensional gas molecule velocity is some unknown function X(v)

Two constraints on X(v): normalization and fixed mean kinetic energy

$$\int_{-\infty}^{\infty} X(v)\,dv = 1 \qquad \text{and} \qquad \int_{-\infty}^{\infty} X(v)\varepsilon(v)\,dv = \frac{3}{2}k_B T = \frac{3}{2}\sigma_0^2$$

Particle energy:

$$\boxed{\varepsilon(v) = 3v^2/2}$$

This is the key to be identified for dark matter flow

Write down the entropy functional with Lagrangian multiplier:

$$S[X(v)] = -\int_{-\infty}^{\infty} X(v)\ln X(v)\,dv + \lambda_1\left(\int_{-\infty}^{\infty} X(v)\,dv - 1\right) + \lambda_2\left(\int_{-\infty}^{\infty} X(v)\varepsilon(v)\,dv - \frac{3}{2}\sigma_0^2\right)$$

Taking the variation of the entropy functional with respect to distribution X:

$$\frac{\delta S(X(v))}{\delta X} = -\ln X(v) - 1 + \lambda_1 + \lambda_2\varepsilon(v) = 0 \;\Rightarrow\; X(v) \propto \exp\left(\lambda_2 v^2\right)$$

Boltzmann distribution

Distribution for particle energy: $E(\varepsilon) = 2\sqrt{\dfrac{\varepsilon}{\pi\sigma_0^2}}\dfrac{1}{\sigma_0^2}e^{-\varepsilon/\sigma_0^2}$

Maxwell-Boltzmann distribution for speed: $Z(v) = \sqrt{\dfrac{2}{\pi}}\dfrac{v^2}{\sigma_0^3}e^{-v^2/2\sigma_0^2}$

Maximum entropy distributions in dark matter

Deriving maximum entropy distributions in dark matter flow (X distribution)

Two constraints on X(v):

$$\int_{-\infty}^{\infty} X(v)\,dv = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} X(v)\varepsilon(v)\,dv = \frac{3}{2}k_B T = \frac{3}{2}\sigma_0^2$$

Particle energy:

$$\varepsilon(v) = -\frac{X(v)v}{\partial X/\partial v}\left(\frac{3}{2}+\frac{3}{n}\right)$$

This is the key

Write down the entropy functional with Lagrangian multiplier:

$$S\left[X(v)\right] = -\int_{-\infty}^{\infty} X(v)\ln X(v)\,dv + \lambda_1\left(\int_{-\infty}^{\infty} X(v)\,dv - 1\right) + \lambda_2\left(\int_{-\infty}^{\infty} X(v)\varepsilon(v)\,dv - \frac{3}{2}\sigma_0^2\right)$$

Taking the variation of the entropy functional with respect to X:

$$\frac{\delta S(X(v))}{\delta X} = -\ln X(v) - 1 + \lambda_1 + \lambda_2\varepsilon(v) = 0 \;\Rightarrow\; X(v) = \frac{1}{2\alpha v_0}\frac{e^{-\sqrt{\alpha^2+(v/v_0)^2}}}{K_1(\alpha)}$$

The X distribution

E distribution for particle energy:

$$E(\varepsilon) = -\frac{2n}{3(n+2)}\frac{e^{-\gamma}\sqrt{\gamma^2-\alpha^2}}{\alpha K_1(\alpha)v_0^2}$$

Z distribution for speed:

$$Z(v) = \frac{1}{\alpha K_1(\alpha)}\cdot\frac{v^2}{v_0^3}\cdot\frac{e^{-\sqrt{\alpha^2+(v/v_0)^2}}}{\sqrt{\alpha^2+(v/v_0)^2}}$$

Maximum entropy distributions in dark matter

Gaussian core for $|v| \ll v_0$

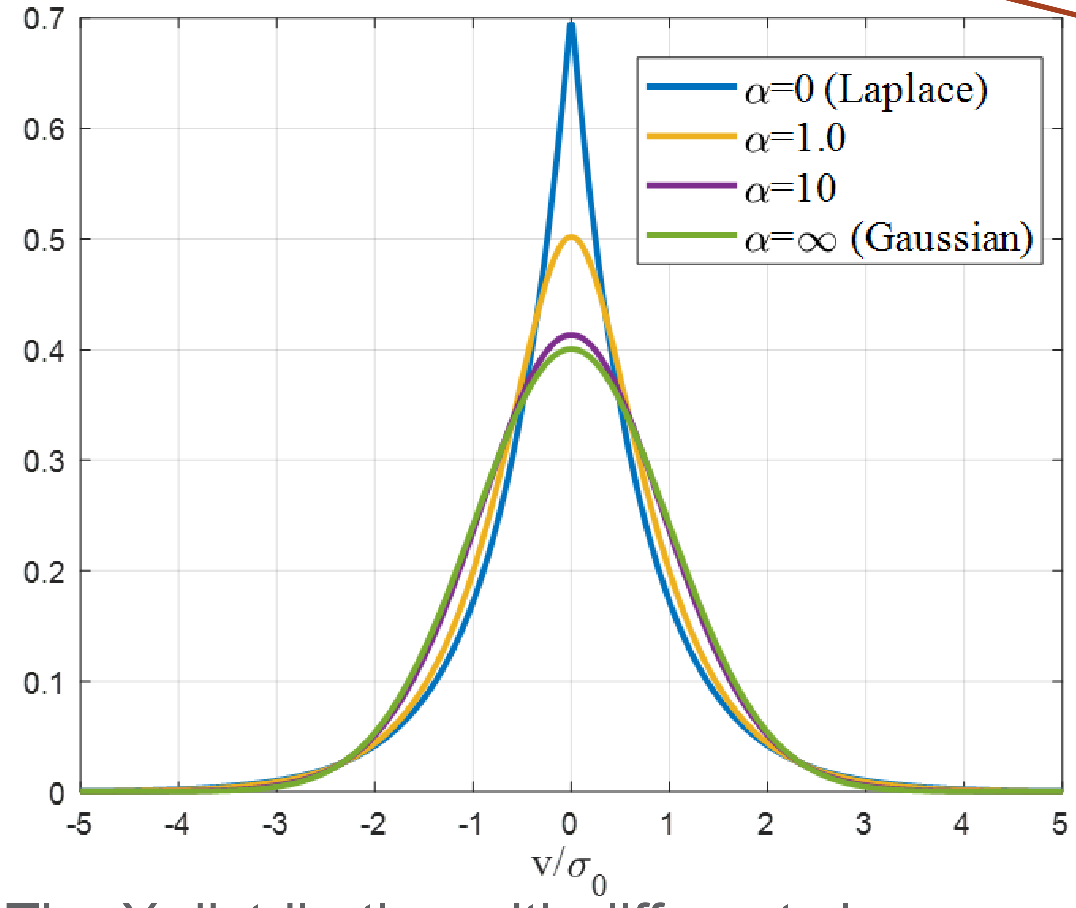
$$X(v) = \frac{e^{-\alpha}}{2\alpha v_0 K_1(\alpha)} \exp\left(-\frac{v^2}{2\alpha v_0^2}\right)$$

Exponential wings for $|v| \gg v_0$

$$X(v) = \frac{1}{2\alpha v_0 K_1(\alpha)} \exp\left(-\frac{v}{v_0}\right)$$

$$X(v) = \frac{1}{2\alpha v_0} \frac{e^{-\sqrt{\alpha^2 + (v/v_0)^2}}}{K_1(\alpha)}$$

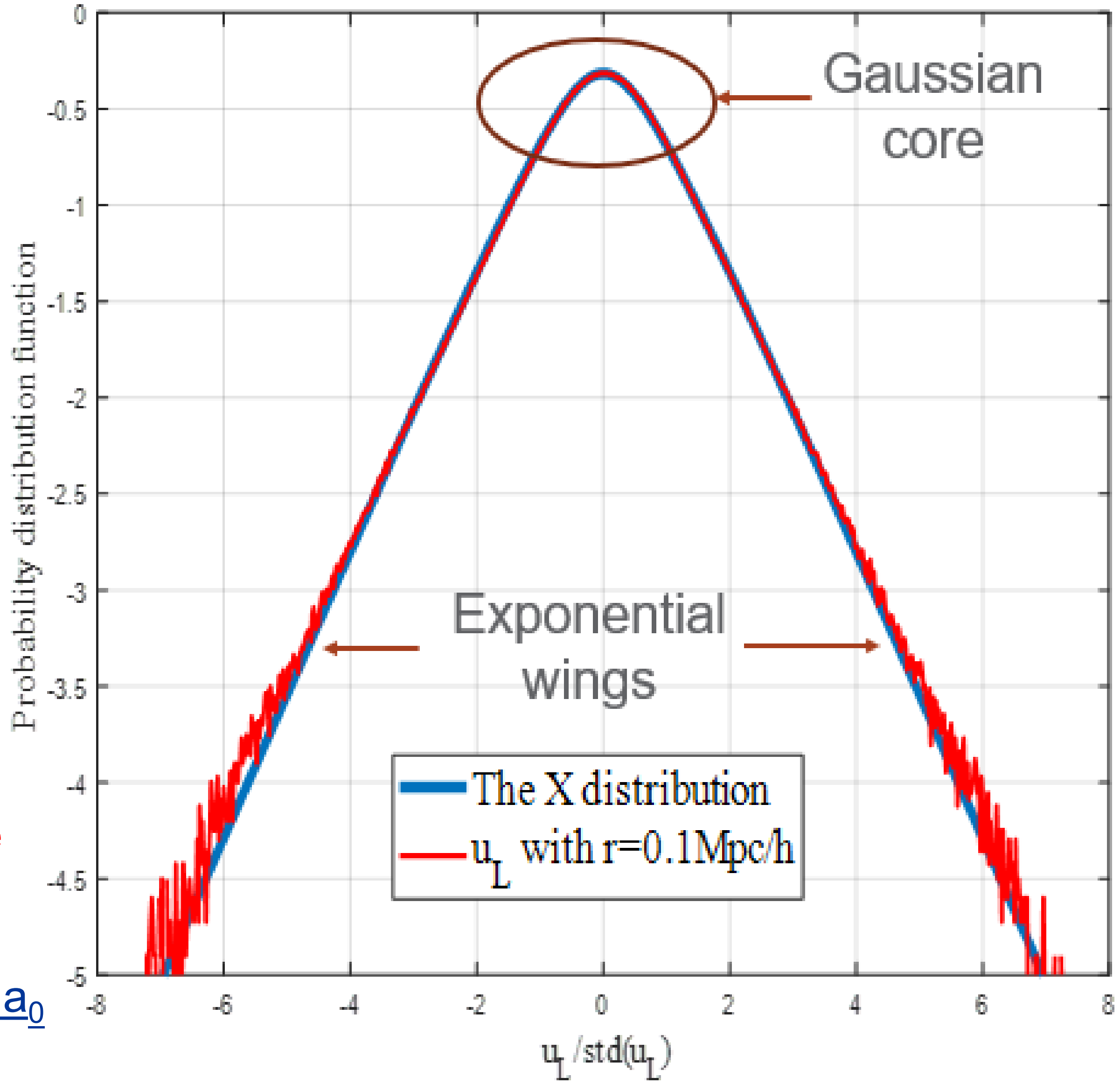
Bessel function



The X distribution with different shape parameter α

X is a two-parameter distribution with shape parameter α and a velocity scale v_0 or an acceleration scale a_0

$$\varepsilon_u \propto a_0 v_0\ ?$$



Comparison with N-body simulation

Particle energy vs. particle velocity in dark matter

$$X(v)=\frac{1}{2\alpha v_0}\frac{e^{-\sqrt{\alpha^2+(v/v_0)^2}}}{K_1(\alpha)} \qquad \varepsilon(v)=-\frac{X(v)v}{\partial X/\partial v}\left(\frac{3}{2}+\frac{3}{n}\right)$$

Particle energy:

$$\varepsilon(v)=\frac{3}{2}\left(1+\frac{2}{n}\right)v_0^2\sqrt{\alpha^2+\left(\frac{v}{v_0}\right)^2}$$

Gaussian core for $|v| \ll v_0$

$$\varepsilon(v)\approx\frac{3}{2}\left(1+\frac{2}{n}\right)\left(\alpha v_0^2+\frac{v^2}{2\alpha}\right)\propto v^2$$

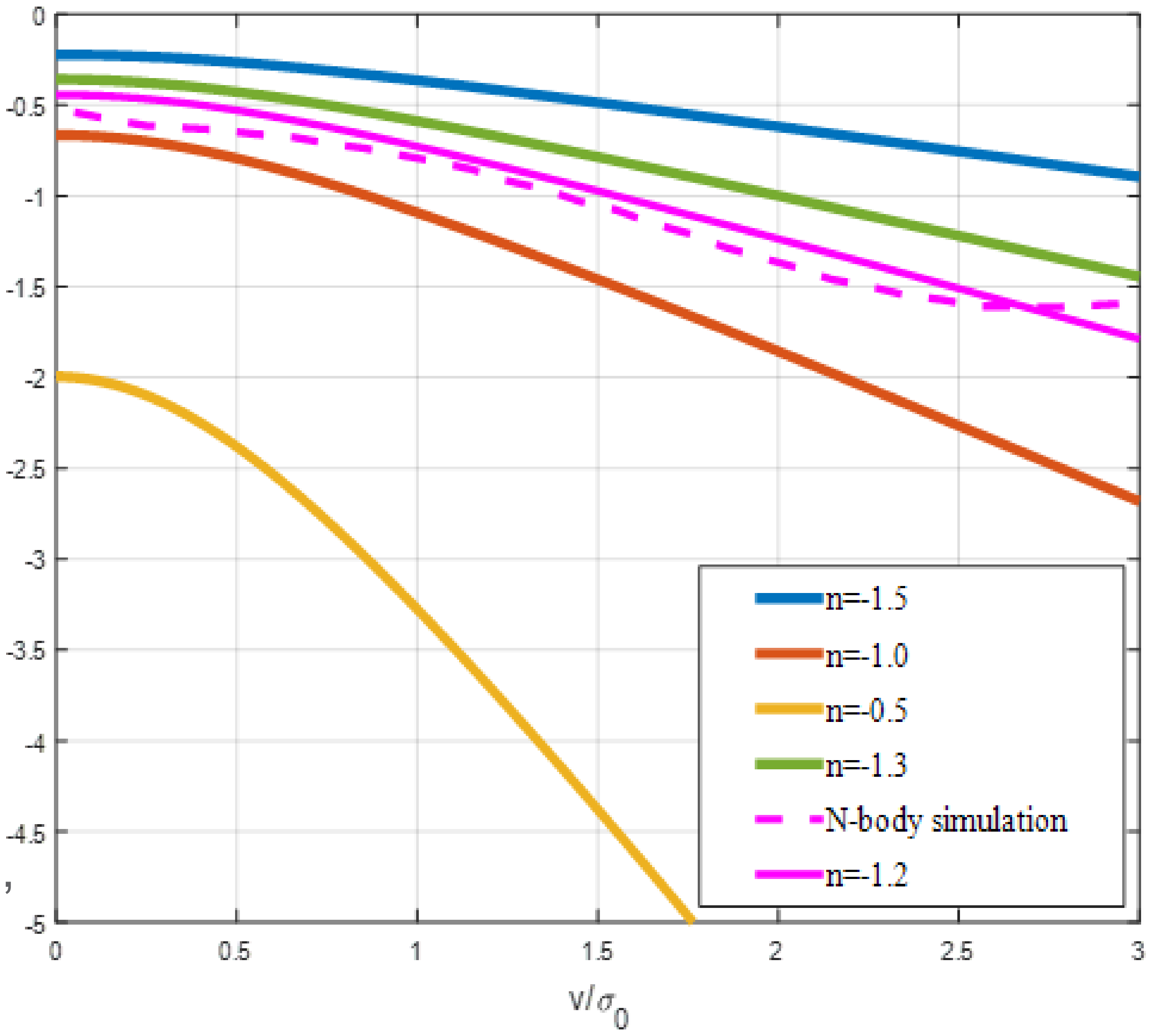
Inner halo, Newtonian behavior

Exponential wings for $|v| \gg v_0$

$$\varepsilon(v)\approx\frac{3}{2}\left(1+\frac{2}{n}\right)v_0 v\propto v$$

Outer region of halo, non-Newtonian behavior

Deep-MOND?



Comparison with N-body simulation for particle energy ε(v)

MOND theory and acceleration fluctuation in DMF

- Empirical Tully-Fisher relation:

Flat rotation speed $v_f \propto M_b^{1/4}$ $\Leftarrow$ observed baryonic mass

- MOND (Milgrom 1983) is an empirical model to reproduce flat rotation curve without dark matter.

$$a_0 \approx 1.2\times 10^{-10}\, m/s^2$$ Critical MOND acceleration

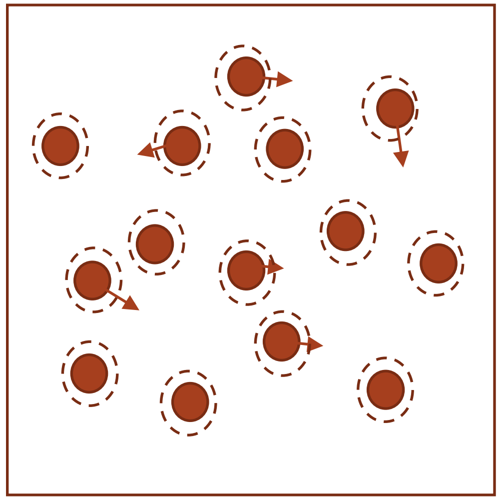
$$F = ma \qquad a \gg a_0$$ Newtonian

$$F = m a^2/a_0 \propto a^2 \qquad a \ll a_0$$ Deep MOND

$$\frac{GMm}{r^2} = m\frac{\left(v_f^2/r\right)^2}{a_0} \Rightarrow v_f = \left(GMa_0\right)^{1/4}$$

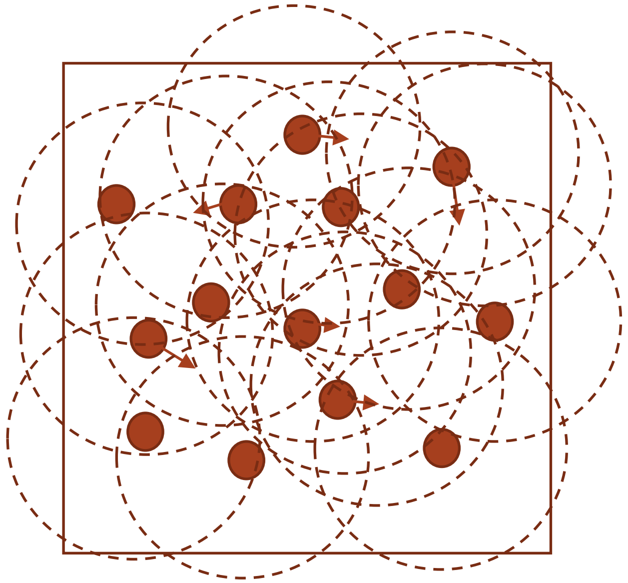
- What is the origin of MOND acceleration?
- What is the origin of deep "MOND"?
- Could MOND be an intrinsic property of dark matter flow in CDM cosmology?

- In kinetic theory of gases, molecules undergo random elastic collisions with a short-range of interaction. Only velocity fluctuation, no fluctuation of acceleration.



Short range: molecule acceleration vanishes

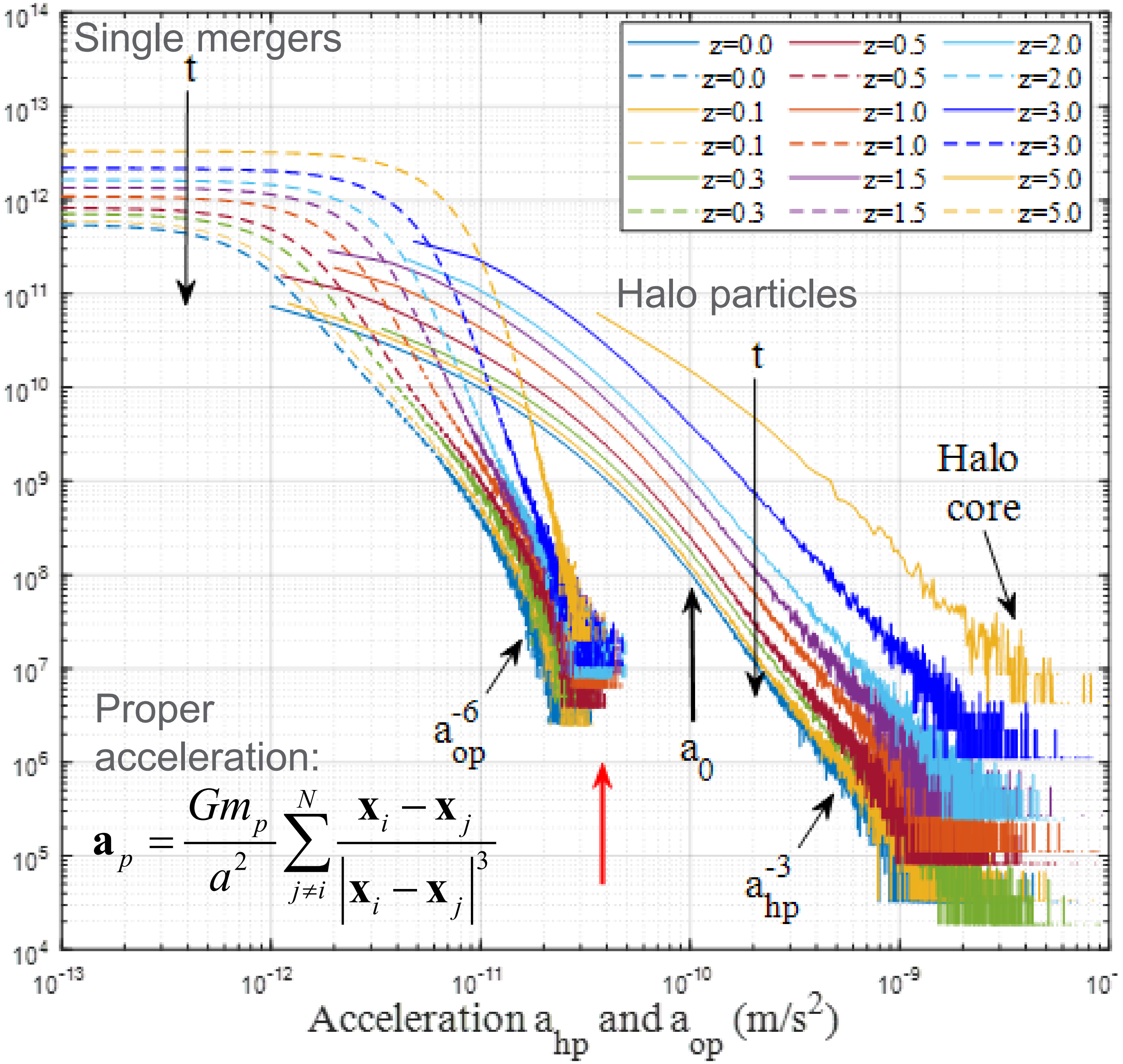
- The long-range gravity in dark matter flow leads to fluctuations in acceleration, in addition to the fluctuation in velocity.



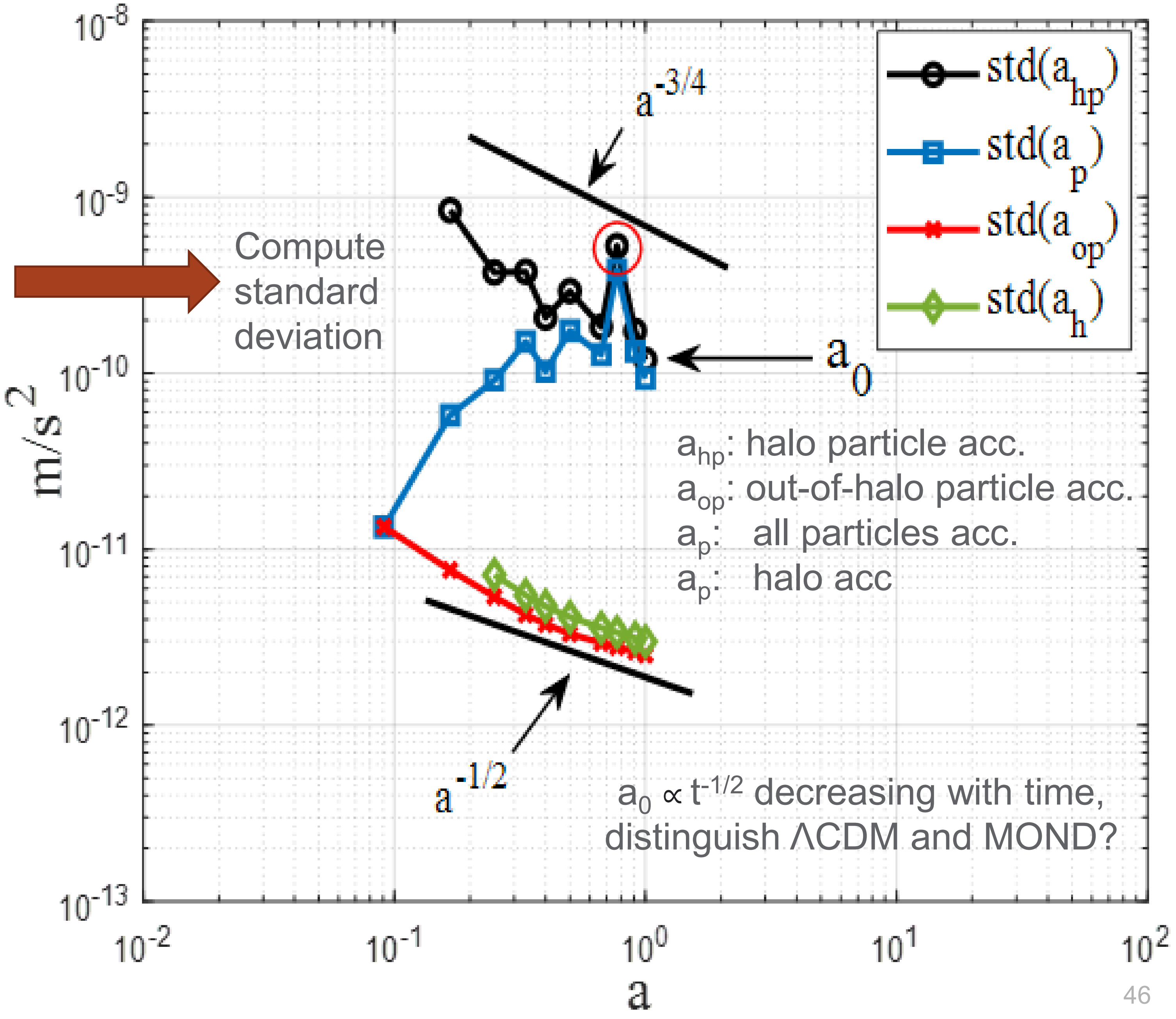
Long range: nonvanishing and fluctuating acceleration

Acceleration distributions in dark matter

Acc fluctuation leads to distribution of acceleration



Time variation of acceleration fluctuation (DM only sim.)



MOND acceleration a_0 from energy cascade

Assume a_0 is the typical acceleration scale of fluctuation,
u is the typical velocity scale of fluctuation, θ_{ur} is the angle of incidence.

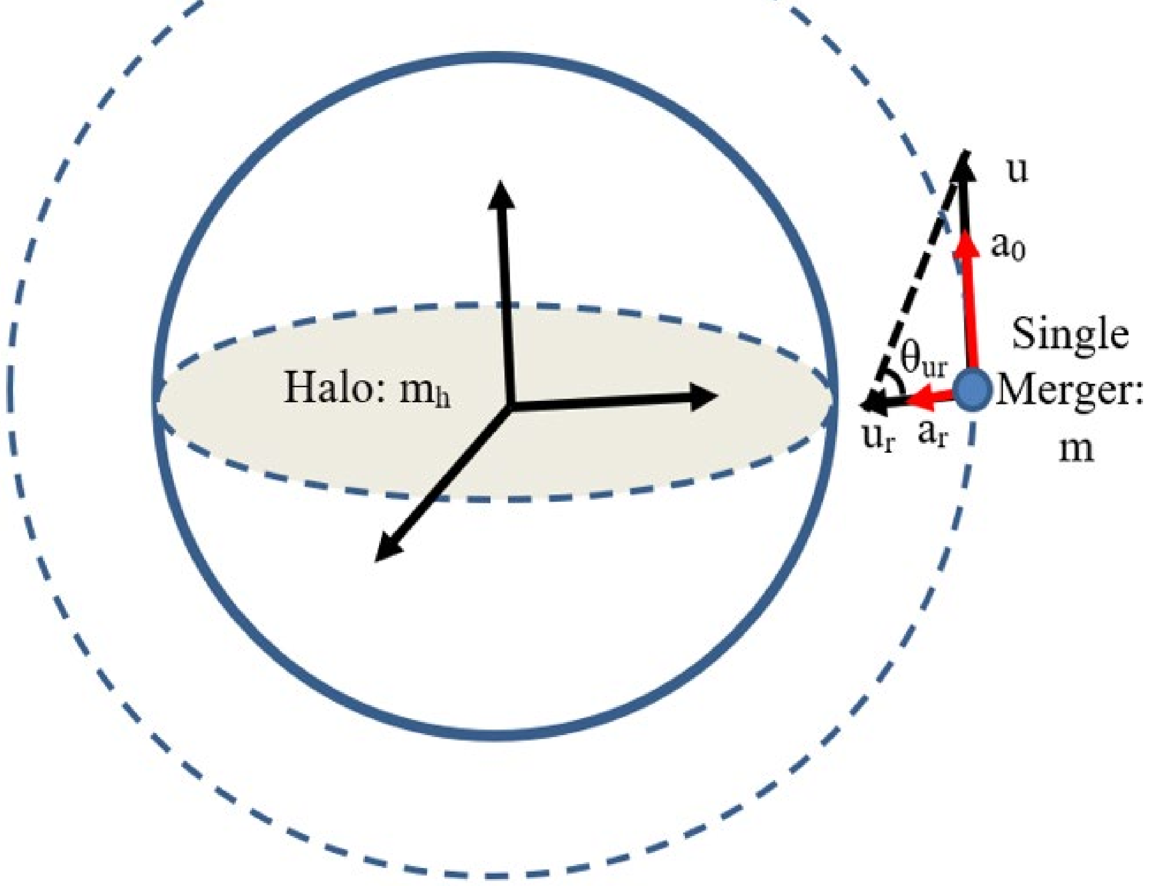
$$\cot(\theta_{ur}) = \frac{1}{3\pi}$$

The rate of energy cascade in terms of a_0, u and θ_{ur} :

$$\varepsilon_u = -a_r u_r = -a_0(a)\cot(\theta_{ur})u(a)\cot(\theta_{ur})$$

$$a_0(a) = -(3\pi)^2 \frac{\varepsilon_u}{u} = \frac{81}{4}\pi^2 H_0 \frac{u_0^2}{u} \propto a^{-3/4} \propto t^{-1/2}$$

Confirmed by simulations, arXiv:2206.04333 arXiv:1712.01654 what about observations?



The rate of energy cascade:

$$\varepsilon_u \approx -\frac{3}{2}\frac{u^2}{t} = -\frac{3}{2}\frac{u_0^2}{t_0} = -\frac{9}{4}H_0 u_0^2 = -4.6\times 10^{-7}\frac{m^2}{s^3}$$

$$a_0(a=1) \approx 200 H_0 u_0 \approx 1.2\times 10^{-10}\, m/s^2$$

In Earth's atmosphere: $\varepsilon_u \approx 10^{-3}\, m^2/s^3$

Potential connection with dark energy??

$$\rho_{vac} = \frac{\Lambda c^2}{8\pi G} = \frac{3\pi}{2G}\left(\frac{(3\pi)^2 \varepsilon_u}{u_0}\right)^2 = \frac{3\pi}{2}\frac{a_0^2 H_0}{GH} \propto \frac{a_0^2}{H}$$

Milgrom coincidence

$$a_0(z=0) \approx c\frac{(\Lambda/3)^{1/2}}{2\pi}$$

- Ideal gas pressure P (N/m^2) ∝ temperature T ∝ velocity fluctuation
- DE density (N/m^2) ∝ a_0^2 ∝ acceleration fluctuation (implies an entropic origin?)

Redshift dependence of acceleration fluctuation a_0

How to compute the angle of incidence?

$$m_h = \frac{4}{3}\pi r_h^3 \Delta_c \bar{\rho} \Rightarrow v_{cir} = \frac{G m_h}{r_h} = H r_h \sqrt{\frac{\Delta_c}{2}} = 3\pi u_r$$

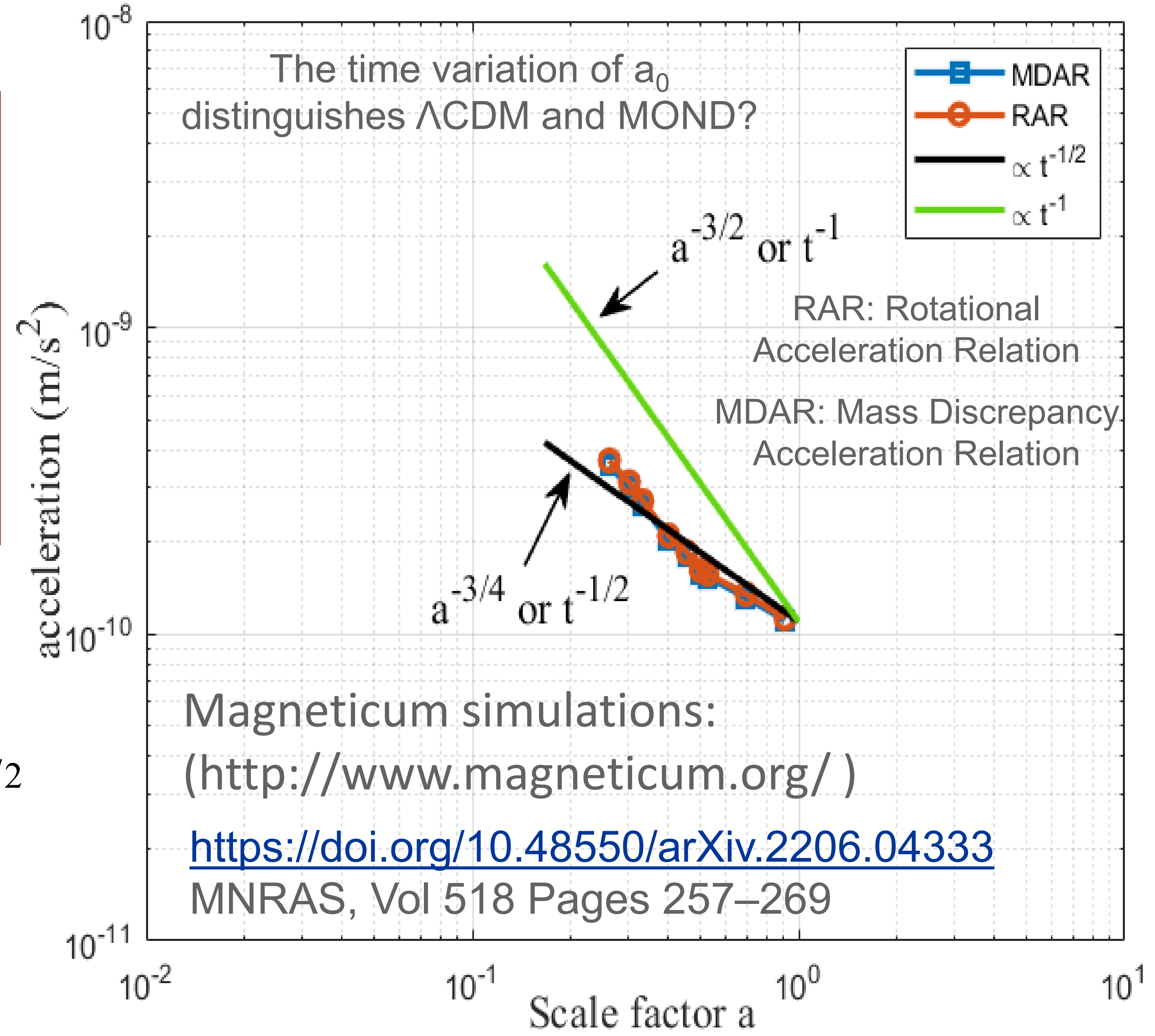
Critical density ratio: $$\Delta_c = 2/(\beta_{s2})^2 = 18\pi^2$$

$$\cot(\theta_{ur}) = \frac{u_r}{v_{cir}} = \beta_{s2} = \frac{1}{3\pi}$$

Finally, our Model predicts:

$$a_0(a) = -\frac{\Delta_c}{2} \cdot \frac{\varepsilon_u}{u} = -(3\pi)^2 \frac{\varepsilon_u}{u} \propto a^{-3/4} \propto t^{-1/2}$$

Agree with hydrodynamic simulations



Magneticum simulations: (http://www.magneticum.org/)

https://doi.org/10.48550/arXiv.2206.04333

MNRAS, Vol 518 Pages 257–269

The origin of deep MOND behavior?

- Fluctuation of acceleration introduces a scale of acceleration a_0
- Deep MOND for baryonic particles with acceleration $a_p \ll a_0$
- Consider baryonic mass in a one-dimensional dark matter fluid with a velocity fluctuation v_0 and acceleration fluctuation a_0 (Similar to Brownian motion)

$$\frac{1}{2}\frac{dv_p^2}{dt} = v_p\frac{dv_p}{dt} = a_p v_p = a_0 v_0 = -\varepsilon_u$$

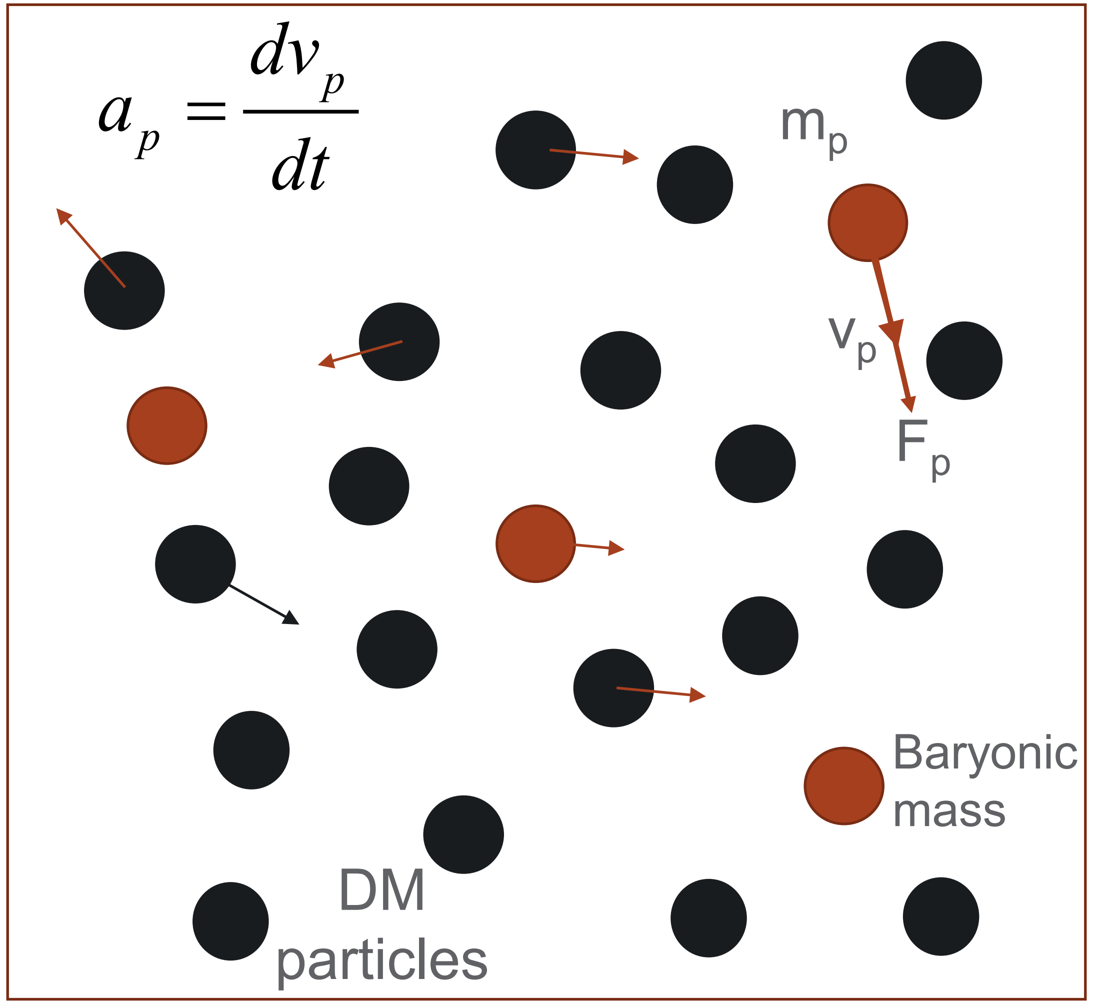
Constant rate of Energy cascade

$$\varepsilon_K(v) = v_0 v_p$$

Maximum entropy distribution: particle kinetic energy ε_k is proportional to velocity when $a_p \ll a_0$ (deep-MOND)

Power (Joule/second) of baryonic mass:

$$F_p v_p = m_p\frac{d\varepsilon_K}{dt} \quad\Rightarrow\quad F_p = m_p\frac{v_0}{v_p}a_p = m_p\frac{a_p^2}{a_0} \propto a_p^2$$



Baryonic mass immersed in DM fluid subject to external force F_p (two miscible phases)

Energy cascade for baryonic-to-halo mass relation

- Total galaxy baryonic mass = stellar mass + cold gas.
- Stellar-to-halo mass relation (SHMR)
  - halo abundance matching

Goals:

- Baryonic-to-halo mass ration (BHMR>SHMR)
- The average mass fraction of baryons in all halos?
- The fraction of total baryons residing in all galaxies?

- Baryonic Tully-Fisher (BTFR) for flat rotation speed:

$$v_f^4 = Gm_b a_0$$

observed baryonic mass

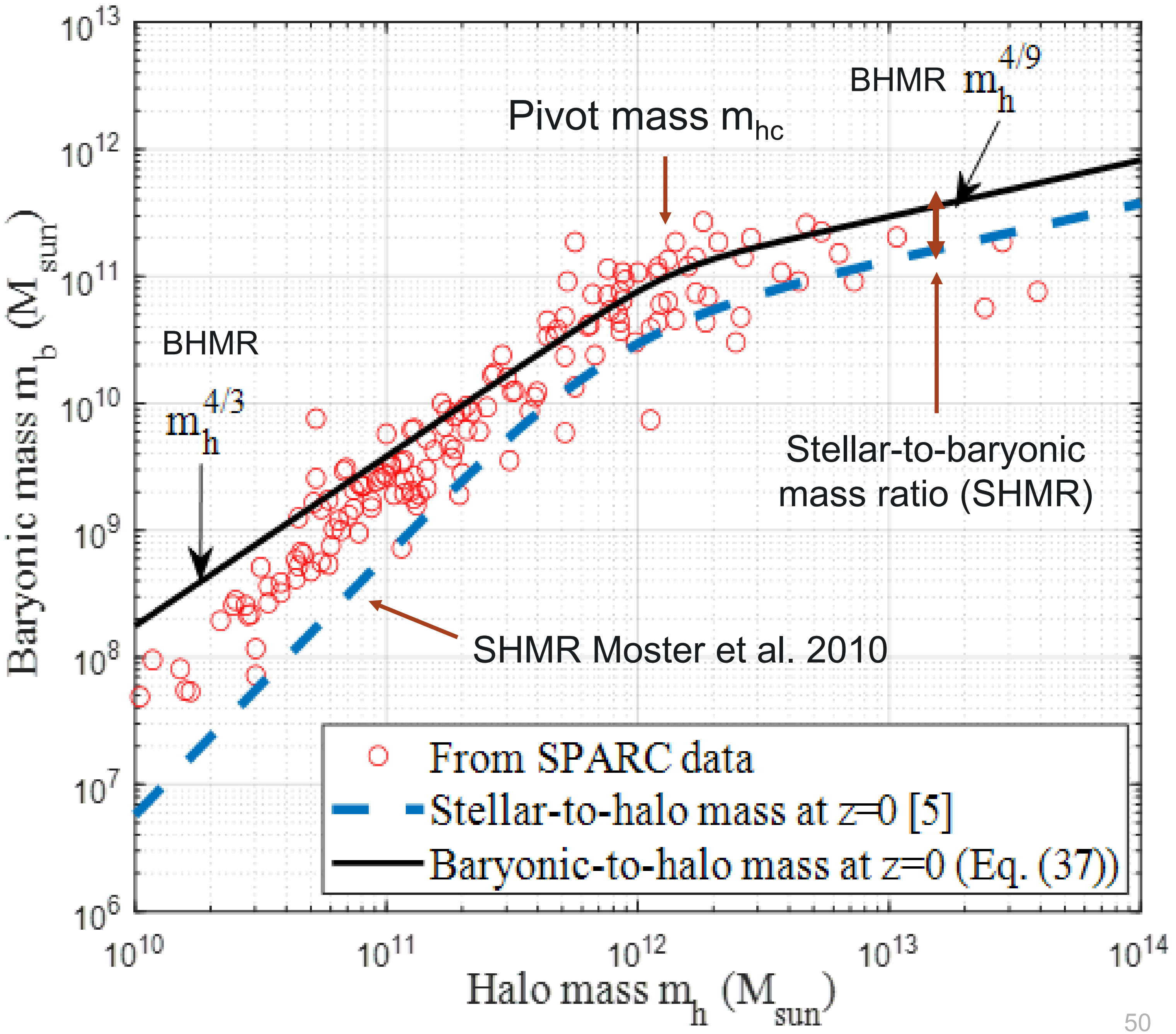
- Halo mass m_h can be related to the halo virial radius r_h through constant density ratio Δ_c

$$m_h = \frac{4}{3}\pi\left(r_h\right)^3 \Delta_c \bar{\rho}_0\left(a\right)$$

- The BHMR (between m_b and m_h) can be obtained only if the relation between v_f and r_h is known.

Relate to energy cascade in baryonic flow? see 2/3 law

$$\varepsilon_u \propto v_f^3 / r_h$$



Energy cascade for the flow of baryonic mass

Baryonic Tully-Fisher relation (BTFR):

$$v_f^4 = Gm_b a_0$$

Halo mass and halo size relation:

$$m_h = \frac{4}{3}\pi r_h^3 \Delta_c \bar{\rho}_0 a^{-3}$$

Rate of energy cascade

$$\varepsilon_u = -\beta_f \frac{u^2}{r_h/v_f} a^q$$

Small halos <m_L: Baryonic mass in equilibrium with DM, i.e. same kinetic energy as DM particles u^2

DM Circular velocity

$$v_{cir} = \frac{4}{9}\sqrt{\frac{\Delta_c}{2}}\beta_f v_f a^q \propto v_f$$

DM halo size

$$r_h = \frac{4}{9}\beta_f v_f H^{-1} a^q \propto v_f$$

Flat rotation speed

$$v_f = \frac{9}{4\beta_f}\left(\frac{2}{\Delta_c}\right)^{\frac{1}{3}} \left(Gm_h H\right)^{1/3} a^{-q} \propto \left(m_h\right)^{1/3}$$

Baryonic Tully-Fisher relation (BTFR):

$$v_f^4 = Gm_b a_0$$

Halo mass and halo size relation:

$$m_h = \frac{4}{3}\pi r_h^3 \Delta_c \bar{\rho}_0 a^{-3}$$

$$\varepsilon_u = -\alpha_f \frac{v_f^2}{r_h/v_f} a^p$$

Turnaround time

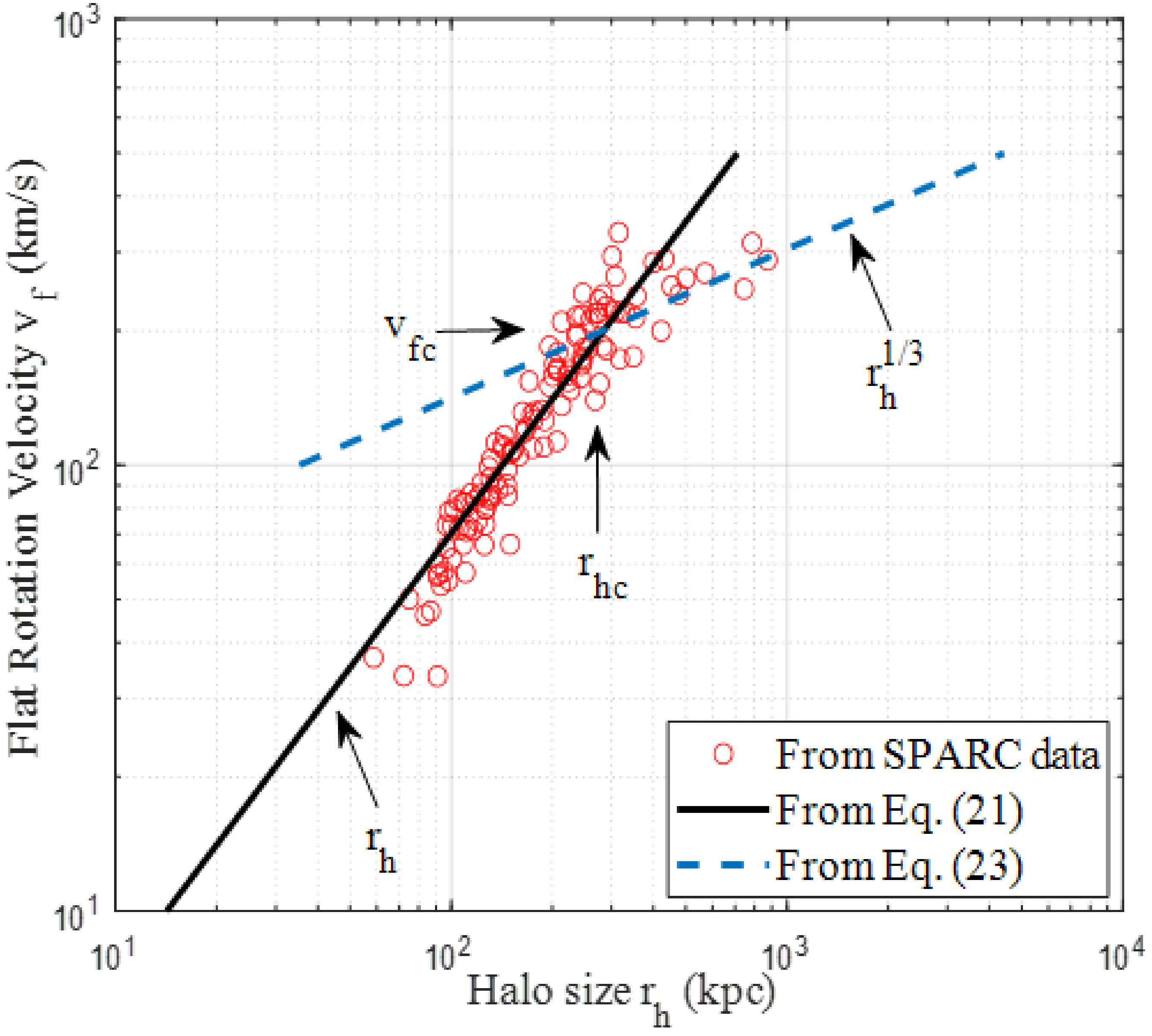
Large halos >m_L: Baryonic mass and DM are two miscible phases sharing the same rate of cascade.

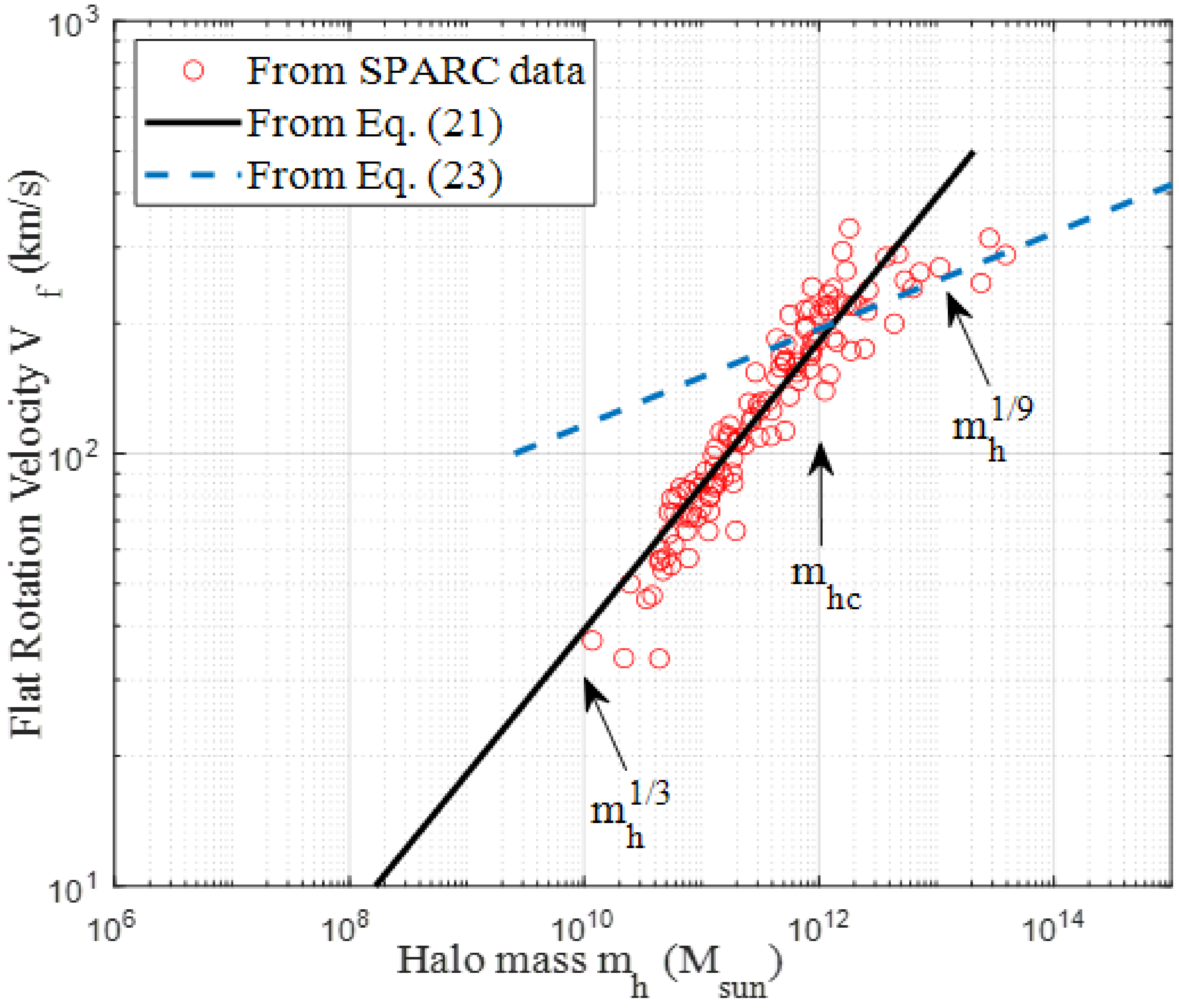
$$v_{cir} = \frac{4}{9}\sqrt{\frac{\Delta_c}{2}}\alpha_f \frac{v_f^3}{u^2} a^p \propto v_f^3$$

$$r_h = \frac{4}{9}\alpha_f \frac{v_f^3}{Hu^2} a^p \propto v_f^3$$

$$v_f = \left(\frac{3}{2\sqrt{\alpha_f}}\right)^{\frac{2}{3}} \left(\frac{2}{\Delta_c}\right)^{\frac{1}{9}} \left(Gm_h H\right)^{1/9} u^{2/3} a^{-p/3} \propto \left(m_h\right)^{1/9}$$

Model prediction and validation by SPARC data I





Model prediction and validation by SPARC data II

Baryonic mass in small halos < pivot mass m_{hc}:

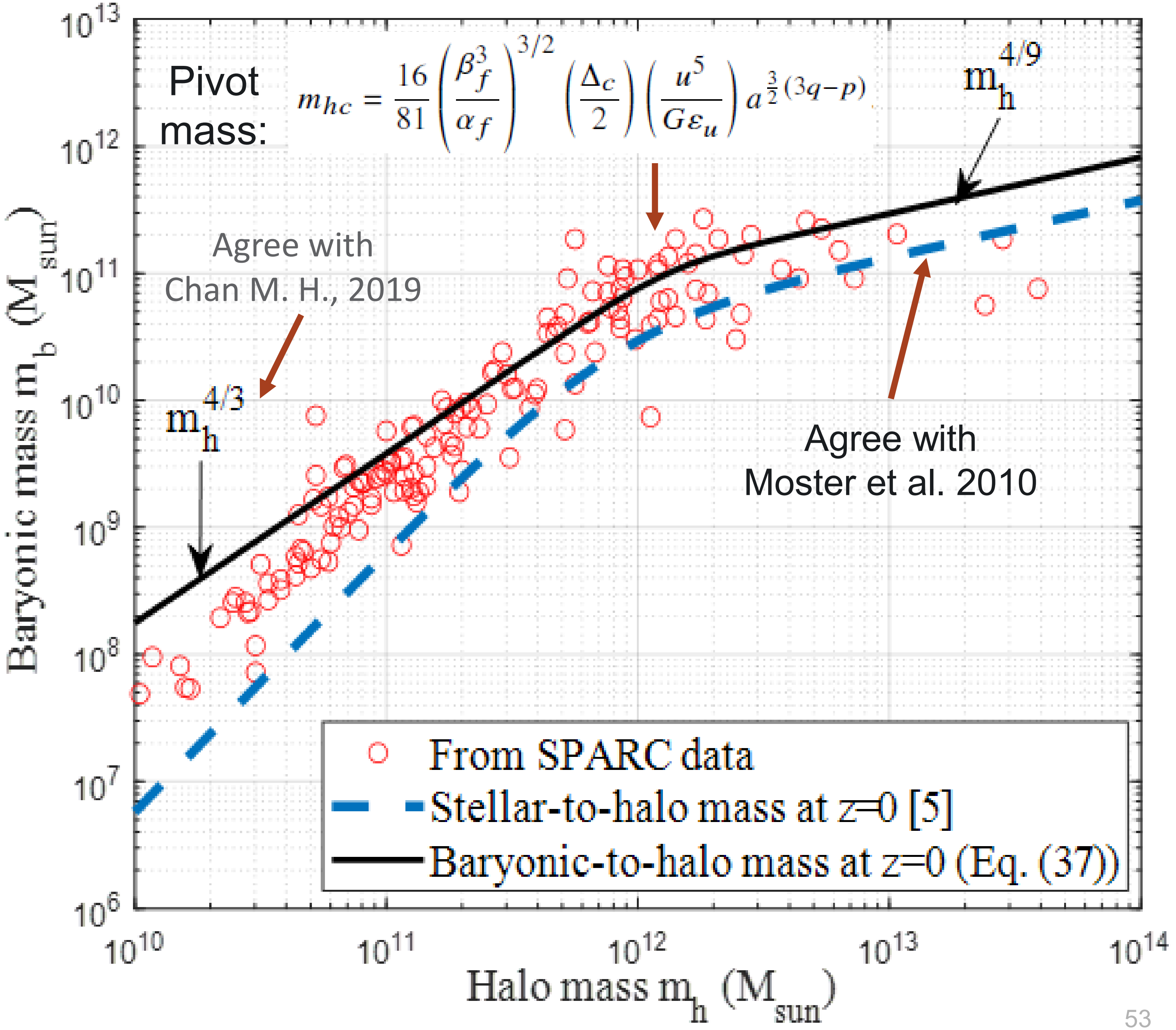
$$m_b = \left(M_{c1}\right)^{-1/3}\left(m_h\right)^{4/3}$$

Baryonic mass in large halos < pivot mass m_{hc}:

$$m_b = \left(M_{c2}\right)^{5/9}\left(m_h\right)^{4/9}$$

Model incorporate two limits:

$$\frac{m_b}{m_h} = 2^{\frac{1}{m}} A(z)\left[\left(\frac{m_h}{m_{hc}(z)}\right)^{-\frac{m}{3}} + \left(\frac{m_h}{m_{hc}(z)}\right)^{\frac{5m}{9}}\right]^{-\frac{1}{m}}$$



- Dash line: the stellar-to-halo mass ratio (SHMR) obtained from halo abundance matching (required to match the stellar mass function)
- The 4/9 scaling law for both SHMR and BHMR

Redshift evolution of baryonic-halo-mass ratio

Overall cosmic baryonic-to-DM mass ratio (including both halos and out-of-halo) is ~18.8% in ΛCDM model:

Baryonic-to-DM mass ratio in out-of-halos

Average Baryonic-to-halo mass ratio in all halos

$$A_{boh}(z)=\frac{0.188-A_{dh}(z)A_{bh}(z)}{1-A_{dh}(z)}$$

Cosmic ratio

Fraction of DM mass in halos

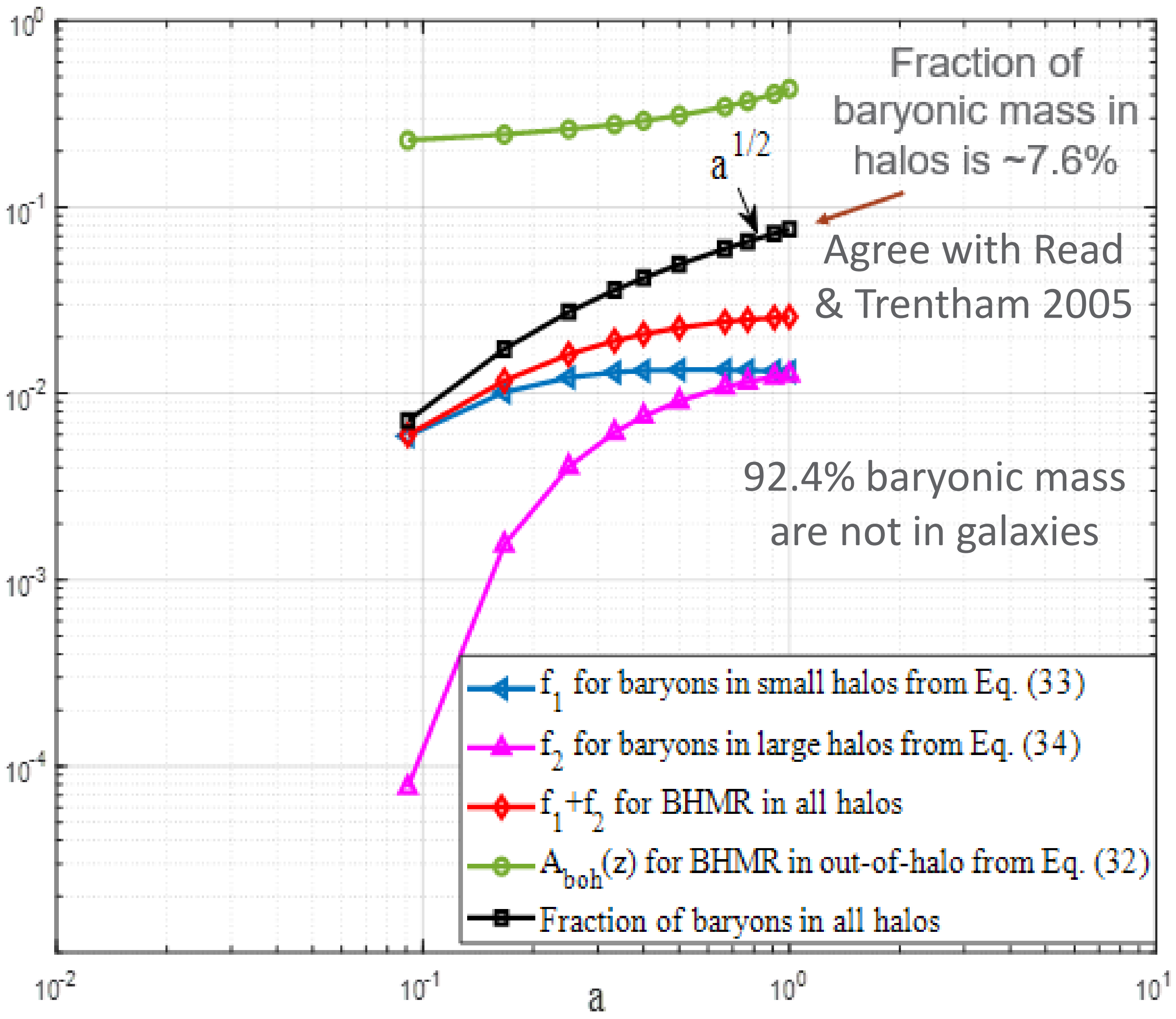
Use double-λ mass function to compute:

$$f_1=\int_0^{\nu_c} f_{D\lambda}(\nu)\left(M_{c1}\right)^{-1/3}\left(\nu^{3/2}m_h^*\right)^{1/3}d\nu$$

The baryonic-to-halo mass ratio in small halos

$$f_2=\int_{\nu_c}^{\infty} f_{D\lambda}(\nu)\left(M_{c2}\right)^{5/9}\left(\nu^{3/2}m_h^*\right)^{-5/9}d\nu$$

The baryonic-to-halo mass ratio in large halos



Redshift evolution of BHMR

Energy cascade for SMBH-galaxy evolution

- Strong correlations between supermassive black holes (SMBHs) and host galaxies suggest a co-evolution.
  - M_B–σ_b relation (BH mass vs. velocity dispersion)
  - M_B–M_b relation (BH mass vs. bulge mass)
  - M_B–L_b relation (BH mass vs. bulge luminosity)

(Ferrarese et al. 2005)	$\frac{M_B}{10^8 M_\odot} = 1.66\left(\frac{\sigma}{200 km/s}\right)^{4.86}$
(Marconi et al. 2003)	$M_B \approx 0.002 M_b$
Virial theorem	$M_b \approx 3 r_b \sigma_b^2 / G$

- Proposed mechanisms for BH-galaxy co-evolution
  - AGN Feedback
  - Statistical origin
  - Effect of energy cascade?

$M_B \propto \sigma_b^5$	$M_B \propto M_b$	$M_b \propto r_b \sigma_b^2$
M_B–σ_b correlation	M_B–M_b correlation	Virial theorem

$$\Rightarrow \frac{\sigma_b^3}{r_b} = Const$$

Bulge dispersion

Bulge size

why?

$$\varepsilon_b = \sigma_b^3 / r_b \approx 10^{-4} m^2/s^3$$

For comparison:

M31 bulge: $6\times10^{-5}\, m^2/s^3$
Average local galaxies: $10^{-4}\, m^2/s^3$
Sun: mass-to-light ratio 5122 kg/W or $2\times10^{-4}\, m^2/s^3$
Cascade in dark matter: $4.6\times10^{-7}\, m^2/s^3$

- Two-thirds law: $\sigma_b^2 \propto (\varepsilon_b r_b)^{2/3} \Rightarrow \varepsilon_b = \sigma_b^3 / r_b \Rightarrow$ The rate of energy cascade in bulge

- Does energy cascade exist in SMBH-bulge system?
- How energy cascade impacts SMBH-galaxy coevolution?
- Can cascade induced accretion exceed Eddington limit?

Energy cascade in galaxy bulge

Dynamics on large scale does not feel the dissipation of baryons. Flow is self-gravitating collisionless with the same scaling laws on scale r:

Mass:	$m_r = \alpha_r \varepsilon_b^{2/3} G^{-1} r^{5/3}$	5/3 law
Density:	$\rho_r = \beta_r \varepsilon_b^{2/3} G^{-1} r^{-4/3}$	-4/3 law
Kinetic energy:	$v_r^2 = \left(\varepsilon_b r\right)^{2/3}$	2/3 law
Cascade pressure:	$P_r = \rho_r v_r^2 \propto \varepsilon_b^{4/3} G^{-1} r^{-2/3}$	Due to random motion
Cascade Force:	$F_r = 4\pi r^2 P_r \propto \varepsilon_b^{4/3} G^{-1} r^{4/3} \propto v_r^4 / G$	

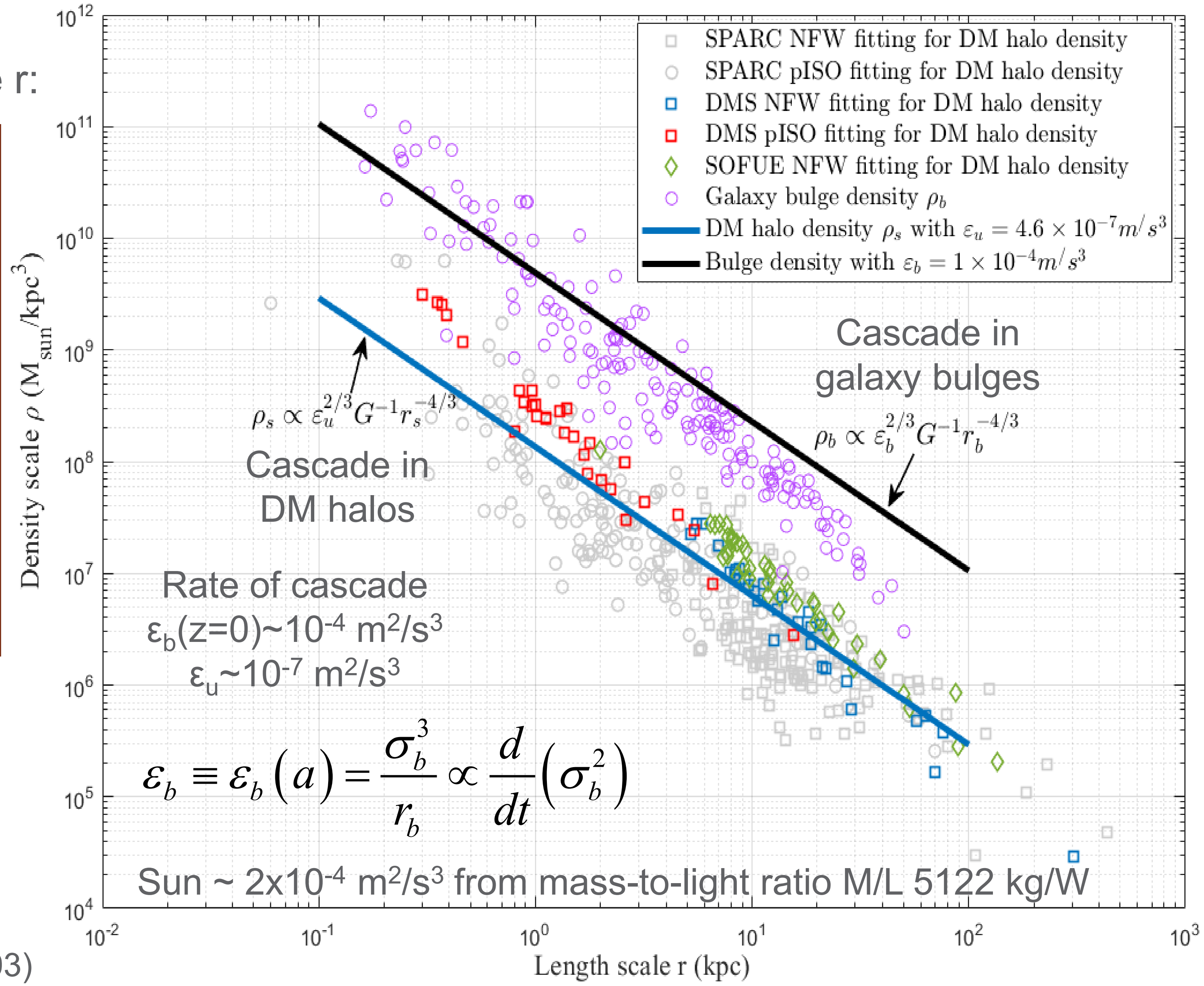
Predicted galaxy mass-size relation: $r \propto m_r^{3/5}$

Observed mass-size relation (ETG only, why?):

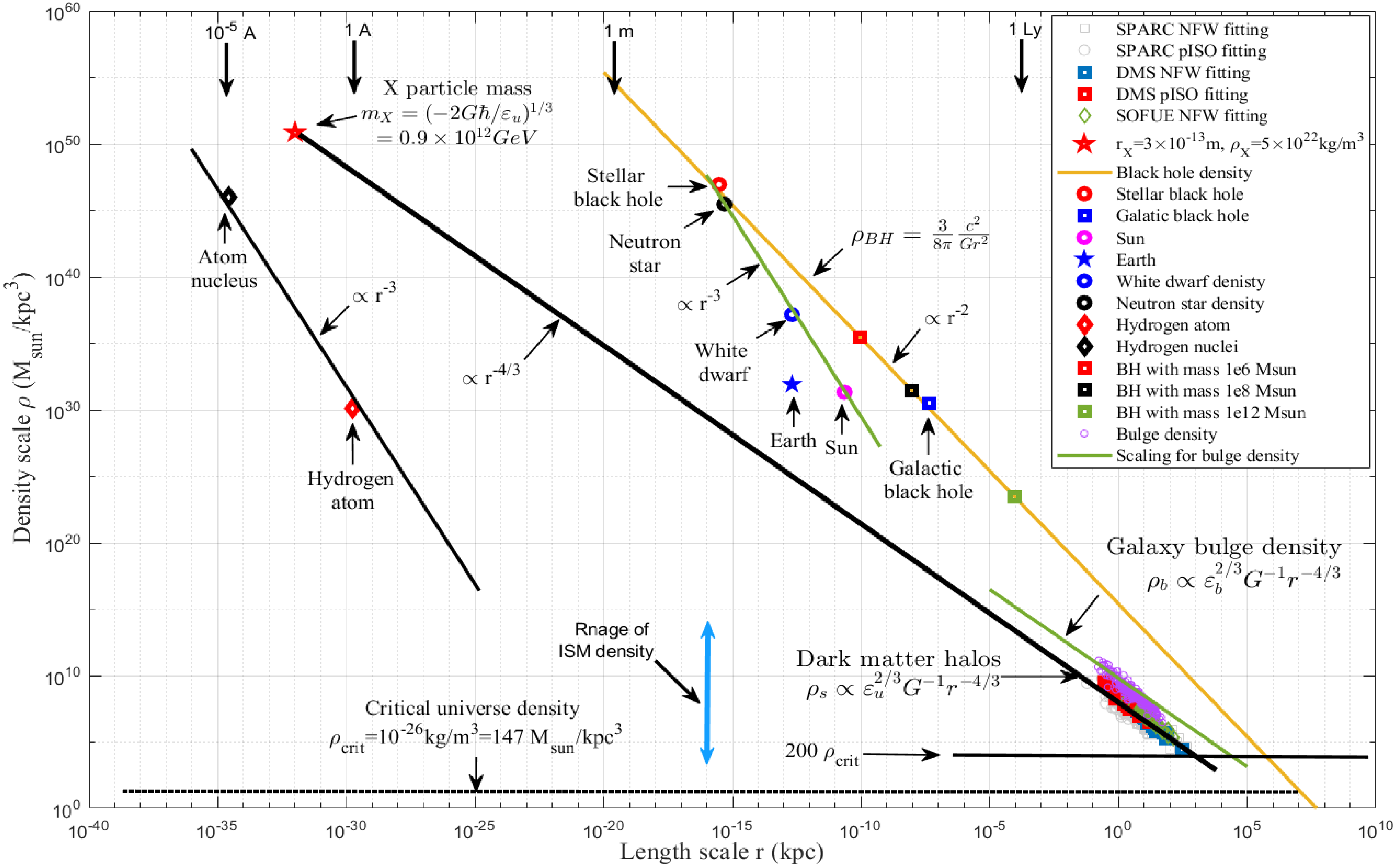
$r \propto m_r^{[0.5\ \ 0.6]}$ (Huertas-Company et al. 2013)

$r \propto m_r^{0.6}$ (Mowla et al. 2019a)

$r \propto m_r^{0.55}$ (Shen et al. 2003)



Astronomical density variation on length scales



Dynamics on the bulge scale and time-variation of ε_b

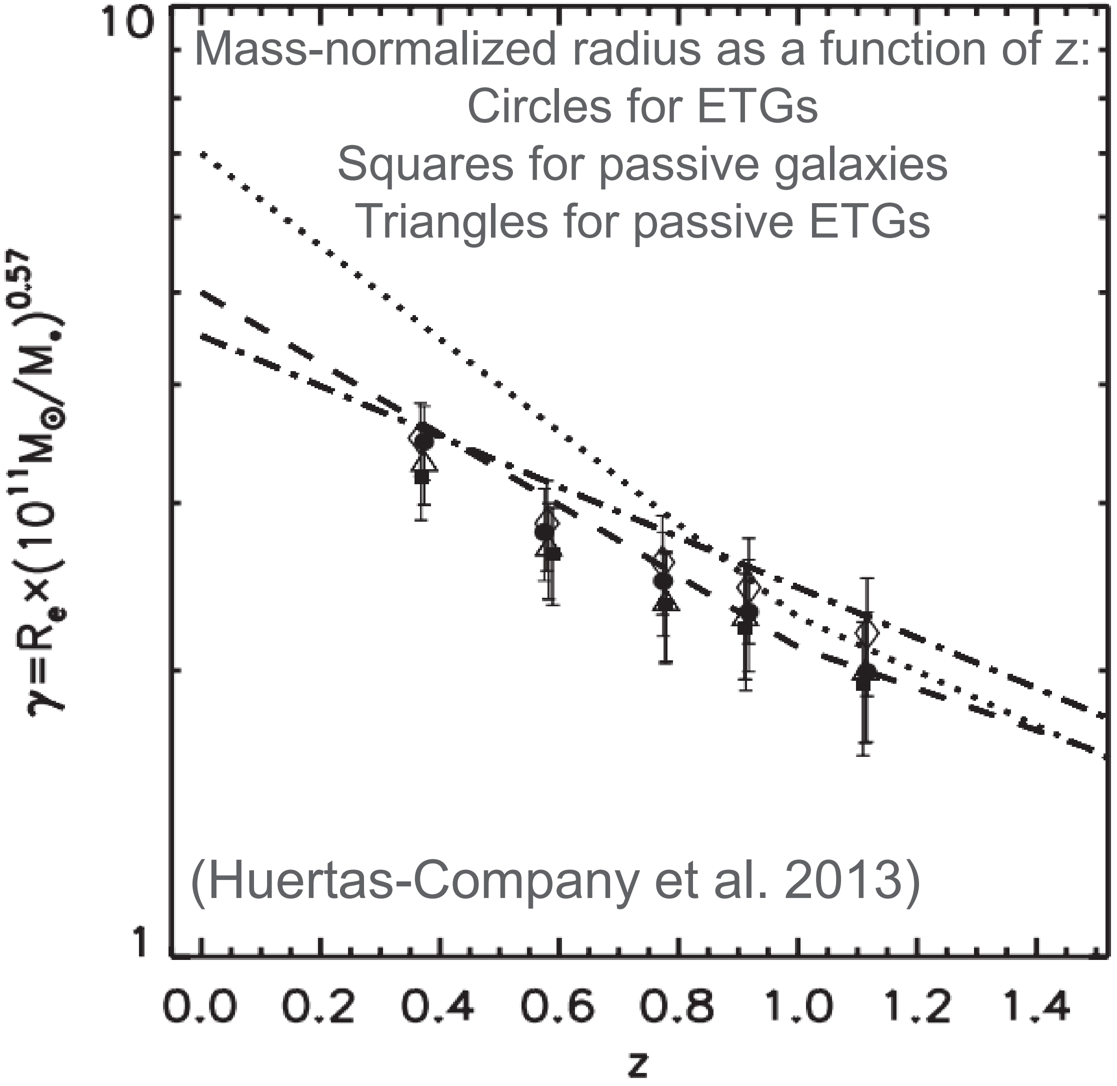
$$\varepsilon_b = \frac{\sigma_b^3}{r_b} \propto \frac{d}{dt}\left(\sigma_b^2\right) \quad \sigma_b^2 r_b^n = Const \quad \sigma_b^2 \propto GM_b/r_b$$

$$r_b \propto a^{\frac{3}{2+n}}, \quad \sigma_b \propto a^{-\frac{3n}{4+2n}}, \quad M_b \propto a^{\frac{3-3n}{2+n}}$$

$$\rho_b \propto a^{-3}, \quad \varepsilon_b \propto a^{-\frac{6+9n}{4+2n}}, \quad r_M \propto a^{\frac{6+9n}{10+5n}}$$

From the observed evolution of galaxy mass-size relation

r_M: size with fixed bulge mass at different z



r_M: the size of bulge with a fixed mass M_b at different z

$r_M \propto a^{1.01}$ (Huertas-Company et al. 2013)

$r_M \propto a^{1.05}$ (Yang et al. 2020)

$r_M \propto a^{0.95}$ (Mowla et al. 2019b)

$n = 1$

$$r_b \propto a \quad \sigma_b \propto a^{-1/2} \quad M_b \propto a^0$$

$$\rho_b \propto a^{-3} \quad \varepsilon_b \propto a^{-5/2} \quad r_M \propto a$$

Key quantities and length scales for SMBH-Bulge

Six physical quantities:

Bulge mass M_b	Rate of energy cascade ε_b
BH mass M_B	Gravitational constant G
BH luminosity L_B	Light speed c

Five length scales:

Bulge scale: $r_b = (1/\alpha_r)^{3/5} M_b^{3/5} G^{3/5} \varepsilon_b^{-2/5}$

BH sphere of influence: $r_B = (1/\alpha_r)^{3/5} M_B^{3/5} G^{3/5} \varepsilon_b^{-2/5}$

Schwarzschild Radius: $r_s = 2GM_B/c^2$

Radiation scale: $r_p = \left(\frac{GL_B}{3\alpha_r^2 \gamma_r c}\right)^{\frac{3}{4}} \varepsilon_b^{-1}$

Dissipation scale: $r_x = \left(\frac{\nu_B^3}{\varepsilon_b}\right)^{\frac{1}{4}} = \left(\frac{8z_r^3 G^3 M_B^3}{c^3 \varepsilon_b}\right)^{\frac{1}{4}}$

Equivalent BH kinematic viscosity: $\nu_B = z_r c r_s = 2z_r G M_B / c$

Scaling laws:

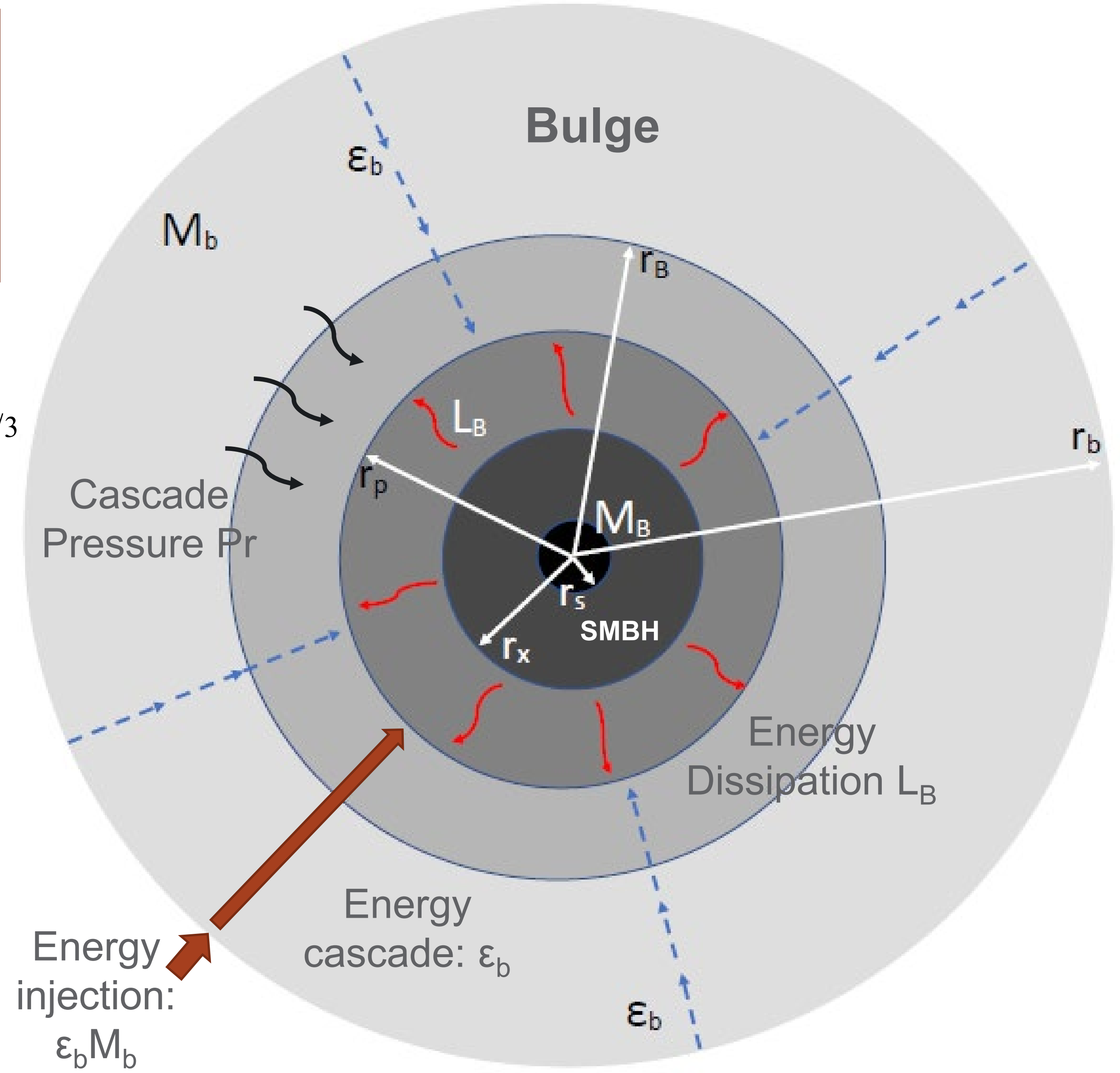
$$m_r = \alpha_r \varepsilon_b^{2/3} G^{-1} r^{5/3}$$

Radiation pressure

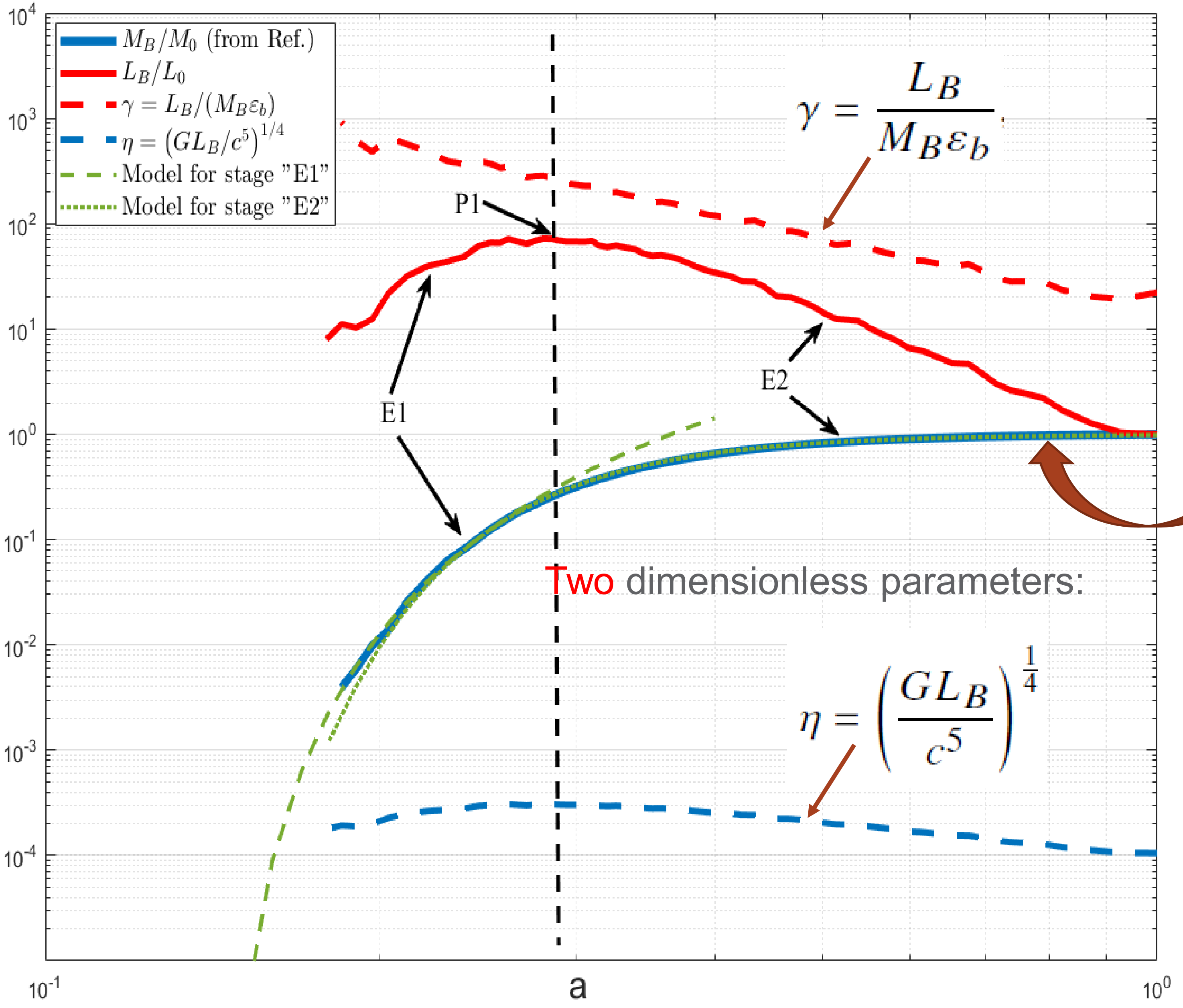
$$p_{rad}(r) = \frac{L_B}{4\pi r^2}$$

$$p_r(r) = \frac{\varepsilon_b^{4/3}}{Gr^{2/3}}$$

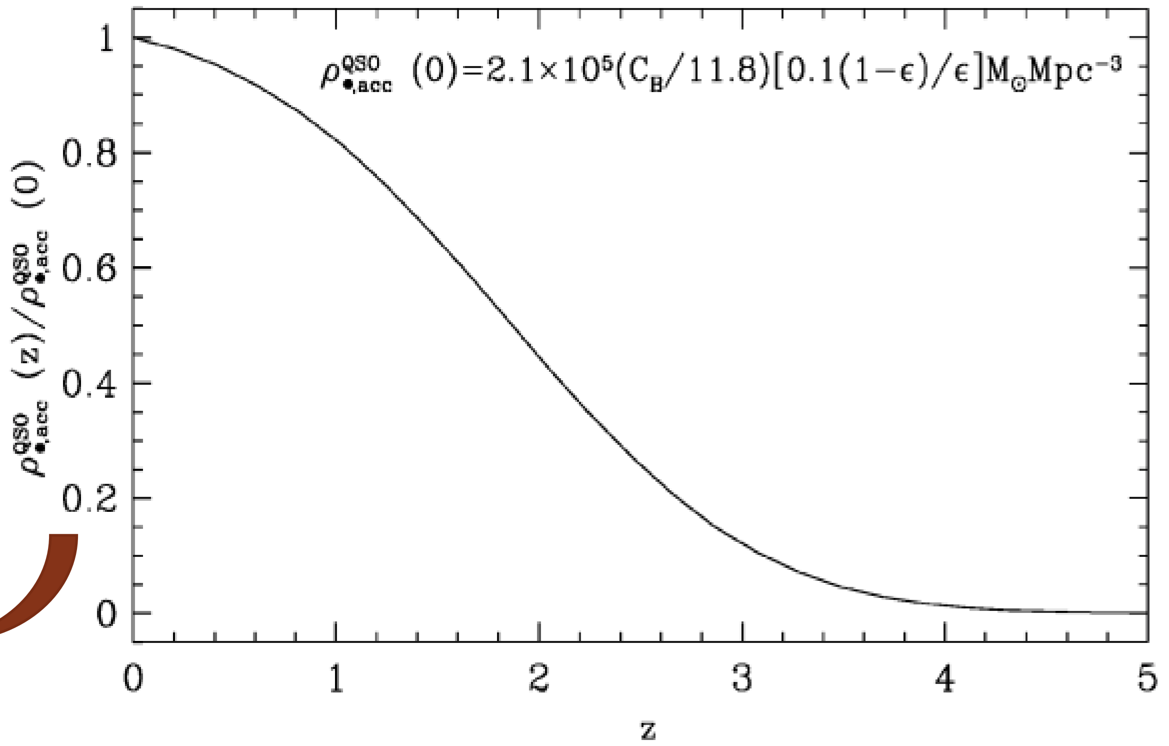
Cascade pressure



SMBH evolution from quasar luminosity function



Time evolution of BH mass M_B, Luminosity L_B, dimensionless γ and η

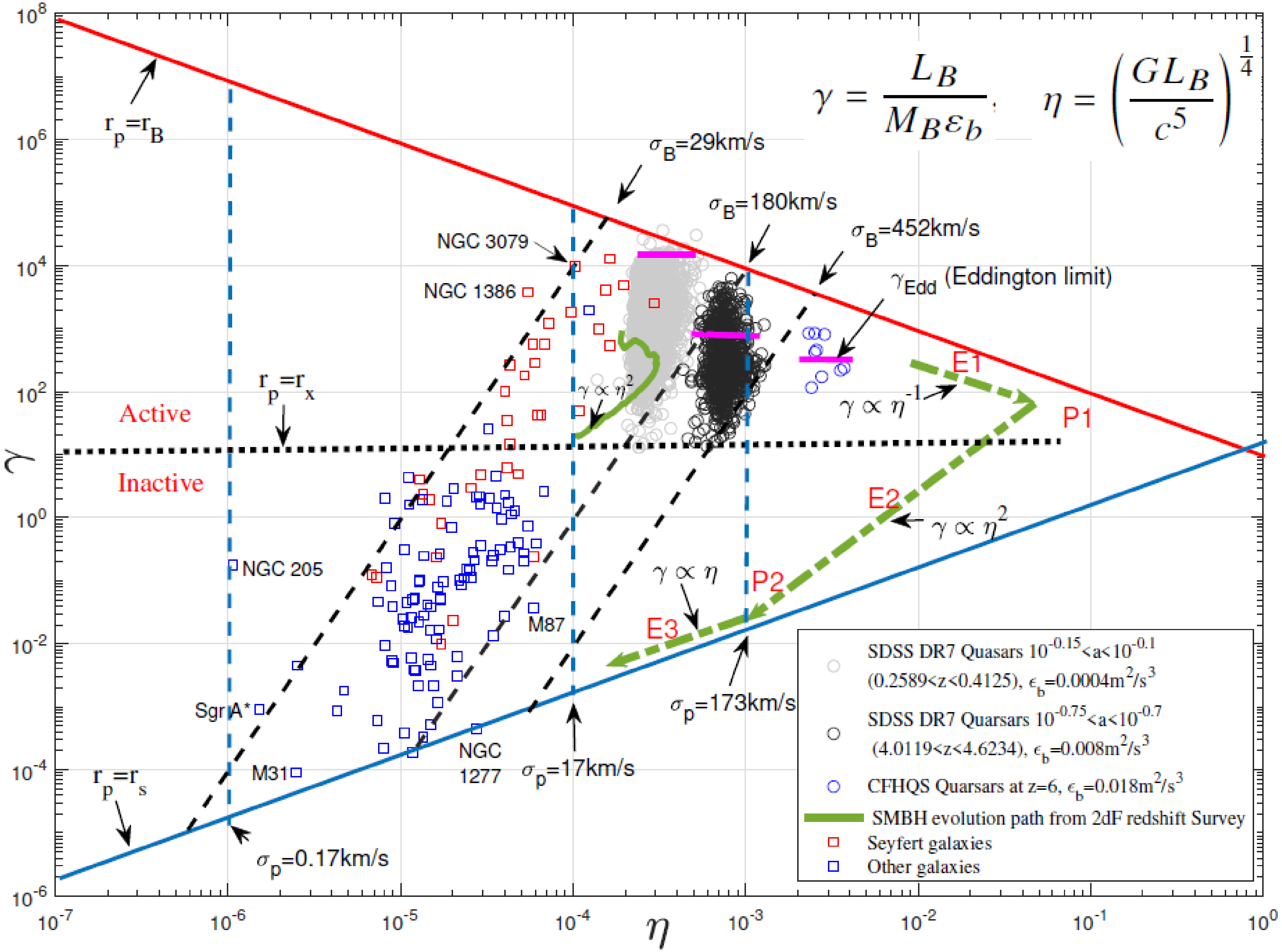


Evolution of co-moving BH mass density from Quasar luminosity function from 2dF Redshift Survey (Yu & Tremaine 2002)

Luminosity is converted from mass evolution :

$$\frac{L_B}{M_0} = \frac{\dot{M}_B}{M_0}\frac{\epsilon c^2}{1-\epsilon} = \frac{\partial(M_B/M_0)}{\partial a}H_0 a^{-1/2}\frac{\epsilon c^2}{1-\epsilon}$$

The SMBH distribution and evolution in γ -- η plane



Data sources:

1) Survey of local galaxies from literature (squares) Multiple sources

2) Quasars from Sloan Digital Sky Survey DR7 (gray and black circles) Schneider et.al 2010, Shen et al. 2011.

3) High redshift quasars from Canada–France High-z Quasar Survey (blue circles) Willott et.al 2010

4) BH evolution from the luminosity function from 2dF Redshift Survey (solid green) Yu & Tremaine et.al 2002

Any other potential sources?

Galaxy bulge and SMBH data

Velocity scales

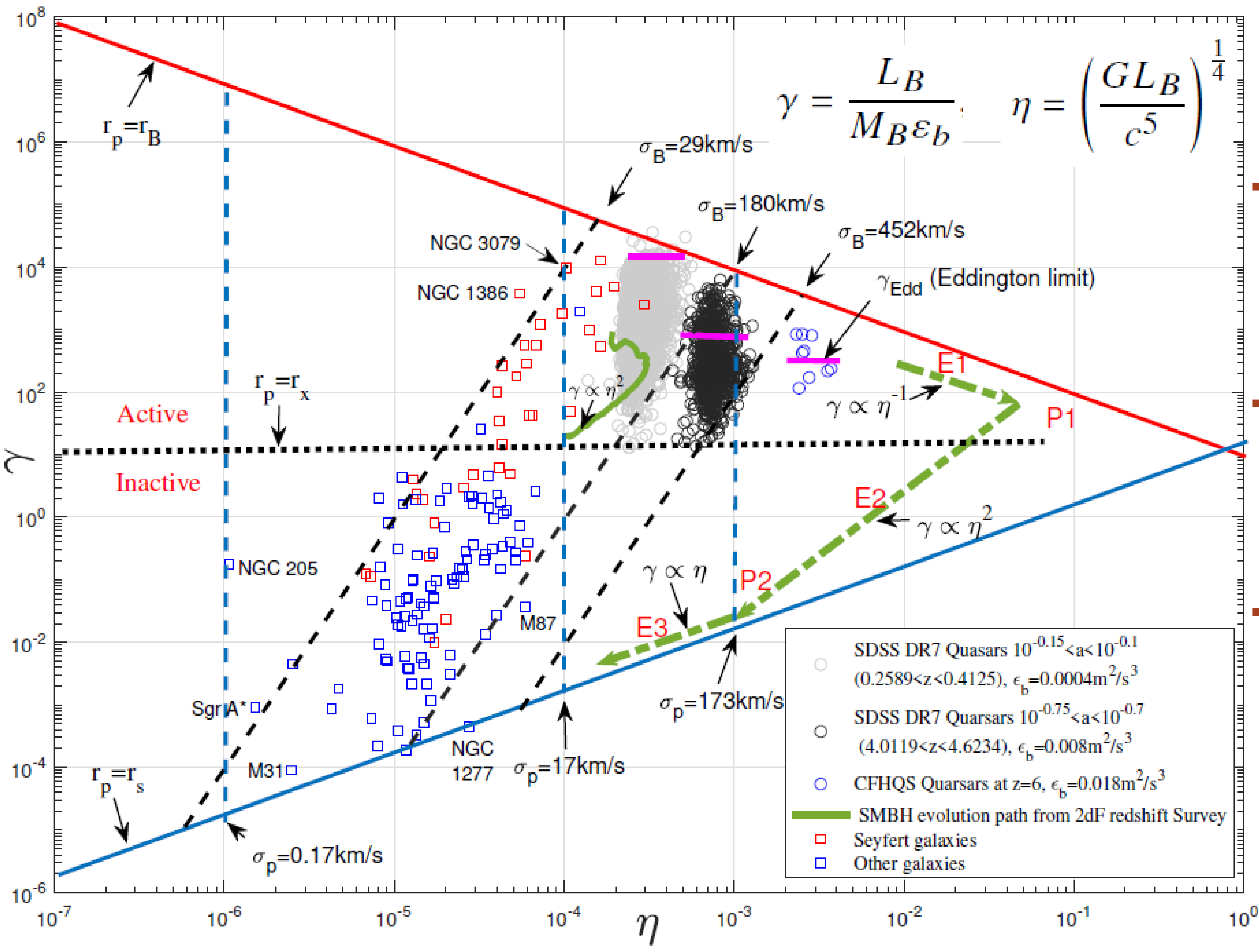
Length scales

Rate of cascade

Table A1. Samples of SMBHs and their host galaxies

Galaxy Name	Type	M_B ($M_\odot$)	Ref.	L_B (erg/s)	Ref.	σ_b (km/s)	Ref.	σ_B (km/s)	σ_P (km/s)	M_b ($M_\odot$)	Ref.	r_b (kpc)	r_B (kpc)	r_P (kpc)	r_X (kpc)	r_S (kpc)	ε_b (m^2/s^3)
Cygnus A	Seyfert	2.7E+09	5	2.7E+45	2	270.0	1	67.1	38.2	1.6E+12	1	31.6	4.8E-01	9.0E-02	3.1E-03	2.6E-07	2.0E-05
A1836-BCG		3.9E+09	1	3.3E+42	5	288.0	1	89.6	7.2	7.6E+11	1	13.2	4.0E-01	2.0E-04	3.1E-03	3.7E-07	5.9E-05
Circinus	Seyfert	1.1E+06	5	4.8E+42	2	158.0	1	29.2	7.9	3.0E+09	1	0.2	1.1E-03	2.1E-05	3.7E-06	1.1E-10	7.4E-04
IC 1262				3.6E+43	63	232.5	63		13.0	9.3E+11	63	24.7		4.4E-03			1.6E-05
IC 1459		2.5E+09	5	1.3E+42	3a	340.0	1	99.4	5.6	6.6E+11	1	8.2	2.1E-01	3.8E-05	1.8E-03	2.4E-07	1.5E-04
IC 1633				8.3E+42	63	356.6	63		9.0	2.4E+12	63	27.0		4.4E-04			5.4E-05
IC 2560	Seyfert	5.0E+06	5	1.2E+42	5	137.0	1	22.7	5.6	2.3E+10	1	1.8	8.0E-03	1.2E-04	2.3E-05	4.8E-10	4.7E-05
IC 4296		1.3E+09	5	1.6E+42	3a	322.0	1	69.3	6.0	1.6E+12	1	22.2	2.2E-01	1.4E-04	1.4E-03	1.2E-07	4.9E-05
IC 5267				6.2E+40	63	167.7	63		2.6	1.5E+11	63	7.6		3.0E-05			2.0E-05
IC 5358				1.1E+44	63	214.2	63		17.2	1.6E+12	63	50.2		2.6E-02			6.3E-06
Sgr A*		4.1E+06	1	1.9E+36	3a	105.0	1	19.3	0.2	1.1E+10	1	1.4	9.0E-03	9.6E-09	2.2E-05	3.9E-10	2.6E-05
NGC193		2.5E+08	59	1.6E+41	59	187.0	59	70.5	3.4	1.9E+10	59	0.8	4.1E-02	4.4E-06	2.7E-04	2.4E-08	2.7E-04
NGC 205		3.8E+04	5	4.8E+35	58	35.0	13	5.1	0.1	3.3E+08	13	0.4	1.2E-03	2.5E-08	1.1E-06	3.7E-12	3.6E-06
NGC 221		2.5E+06	5	1.5E+37	3a	75.0	1	21.0	0.3	8.0E+08	1	0.2	4.5E-03	1.7E-08	1.2E-05	2.4E-10	6.7E-05
NGC 224		1.4E+08	5	1.4E+37	3a	160.0	1	45.4	0.3	4.4E+10	1	2.5	5.7E-02	2.1E-08	2.7E-04	1.4E-08	5.4E-05
NGC 315	BCG	1.7E+09	3	7.6E+42	3a	341.0	11	81.6	8.8	1.2E+12	11	14.9	2.0E-01	2.6E-04	1.5E-03	1.6E-07	8.6E-05
NGC 326				1.3E+42	63	231.9	63		5.7	1.4E+12	63	38.3		5.6E-04			1.1E-05
NGC 383		5.8E+08	59	9.5E+41	59	240.0	59	55.4	5.2	5.0E+11	59	12.5	1.5E-01	1.3E-04	8.5E-04	5.5E-08	3.6E-05
NGC 499				8.9E+42	63	253.3	63		9.2	5.1E+11	63	11.5		5.4E-04			4.6E-05
NGC 507	BCG	1.6E+09	3	7.3E+41	3a	331.0	12	78.1	4.9	1.3E+12	12	16.6	2.2E-01	5.4E-05	1.6E-03	1.6E-07	7.1E-05
NGC 524		8.7E+08	5	1.8E+40	5	235.0	1	67.1	1.9	2.6E+11	1	6.8	1.6E-01	3.8E-06	1.0E-03	8.3E-08	6.2E-05
NGC 533				1.3E+43	63	271.2	63		10.1	1.1E+12	63	22.4		1.2E-03			2.9E-05
NGC 541		3.9E+08	59	4.3E+41	59	191.0	59	48.5	4.3	2.1E+11	59	8.3	1.4E-01	9.4E-05	6.8E-04	3.7E-08	2.7E-05
NGC 708				3.0E+43	63	222.2	63		12.5	7.6E+11	63	22.0		3.9E-03			1.6E-05
NGC 720				6.5E+41	63	235.6	63		4.8	2.5E+11	63	6.4		5.3E-05			6.6E-05
NGC 741				5.2E+42	63	286.0	63		8.0	1.0E+12	63	17.6		3.9E-04			4.3E-05
NGC 821		1.7E+08	5	4.4E+39	2	209.0	1	49.2	1.4	1.3E+11	1	4.3	5.6E-02	1.2E-06	2.8E-04	1.6E-08	6.9E-05
NGC 1023		4.1E+07	5	1.0E+40	2	205.0	1	41.5	1.7	6.9E+10	1	2.4	2.0E-02	1.3E-06	8.7E-05	4.0E-09	1.2E-04
NGC 1052	BCG	1.7E+08	59	3.5E+40	59	191.0	59	53.8	2.3	5.6E+10	59	2.2	4.9E-02	3.8E-06	2.7E-04	1.7E-08	1.0E-04
NGC 1068	Seyfert	8.4E+06	5	2.5E+44	19a	151.0	1	30.2	21.2	1.5E+10	1	0.9	7.6E-03	2.6E-03	2.6E-05	8.1E-10	1.2E-04
NGC 1194	Seyfert	7.1E+07	5	5.5E+44	19a	148.0	1	42.8	25.7	2.0E+10	1	1.3	3.2E-02	6.9E-03	1.4E-04	6.8E-09	8.0E-05

The SMBH distribution in γ -- η plane



The upper limit (red): $r_p = r_B \Rightarrow \gamma\eta = 12 \Rightarrow L_B \propto \varepsilon_b^{4/5} M_B^{4/5} G^{-1/5} c$

The lower limit (blue): $r_p = r_s \Rightarrow \gamma = 24\eta \Rightarrow L_B \propto \varepsilon_b^{4/3} M_B^{4/3} G^{1/3} c^{-5/3}$

The boundary of active and inactive (black): $r_p = r_x \Rightarrow \gamma = 18.6 \Rightarrow L_B \sim \varepsilon_b M_B$

$$\frac{\sigma_p}{c} = \left(\frac{\gamma_r}{3}\right)^{\frac{1}{4}} \eta \Rightarrow L_B/c \propto \sigma_p^4/G$$

Radiation force | Cascade force

$$\frac{\sigma_B}{c} = \frac{(\alpha_r \gamma_r)^{1/2}}{\alpha_r^{1/5}} \left(\frac{\eta^4}{\gamma}\right)^{\frac{1}{5}} \Rightarrow \boxed{\frac{M_B}{10^8 M_\odot} = 0.99 \left(\frac{\sigma_b}{200 km/s}\right)^5}$$

$M_B - \sigma$ relation

The three-stage SMBH evolution in γ -- η plane

Co-evolution stage (E1): parallel to the upper limit

$$r_p \propto r_B \Rightarrow \gamma\eta = Const \Rightarrow L_B \propto \varepsilon_b^{4/5} M_B^{4/5} G^{-1/5} c$$

$$\boxed{L_B \sim \varepsilon_b M_b \gg \varepsilon_b M_B} \quad \frac{dM_B}{dt} = L_B \frac{1-\epsilon}{\epsilon c^2} \quad \Downarrow \quad \boxed{\varepsilon_b = a^{-m} \quad \text{m=5/2}}$$

$$M_B = \left[\alpha_r^{-3/2}\gamma_r^{-5/2}\xi_r^{-5/3}\right]\frac{\sigma_p^5}{\varepsilon_b G} \Leftarrow M_B = M_{\infty 1}\left[1-\left(\frac{a}{a_1}\right)^{-\frac{4}{5}m+\frac{3}{2}}\right]^5$$

Transitional stage (E2):

$$L_B \propto M_B\, \varepsilon_b^2 / \varepsilon_b^* \text{ or } \gamma \propto \eta^2$$

$$\boxed{L_B \sim \varepsilon_b M_B} \quad \Downarrow$$

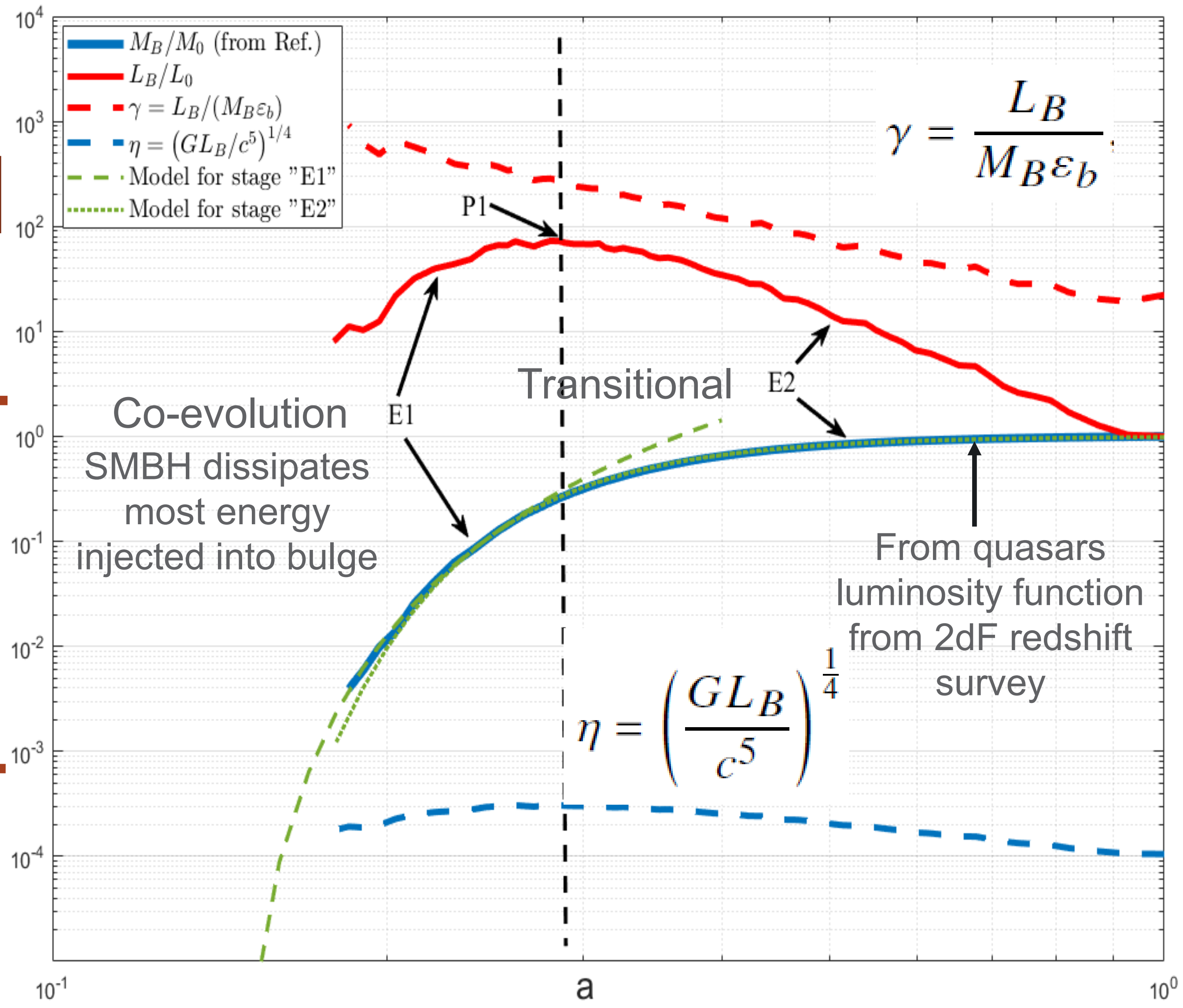
$$M_B = \left(\frac{3}{\gamma_r\gamma^*}\right)\left(\frac{\varepsilon_b}{\varepsilon_b^*}\right)^{1-p}\frac{\sigma_p^4 c}{\varepsilon_b G} \Leftarrow M_B = M_{\infty 2}\exp\left[-\left(\frac{a}{a_2}\right)^{-mp+\frac{3}{2}}\right]$$

Dormant stage (E3): parallel to the lower limit

$$r_p = r_s \Rightarrow \gamma \propto \eta \Rightarrow L_B \propto \varepsilon_b^{4/3} M_B^{4/3} G^{1/3} c^{-5/3}$$

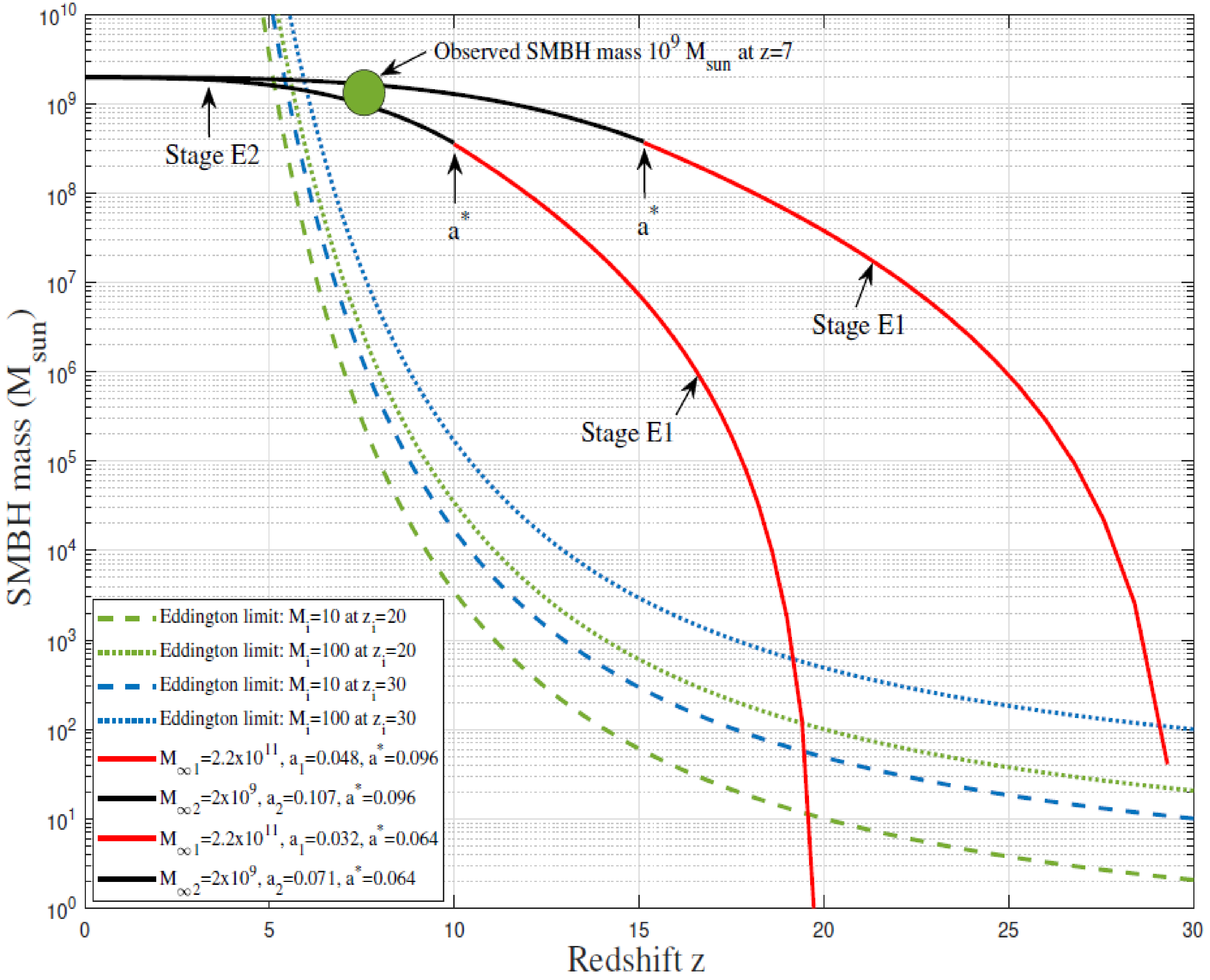
$$\boxed{L_B \ll \varepsilon_b M_B} \quad \Downarrow$$

$$M_B = \left(\frac{\alpha_r^{-3/2}\gamma_r^{-3/2}}{2}\right)\frac{\sigma_p^3 c^2}{\varepsilon_b G} \Leftarrow M_B = M_{\infty 3}\left[1+\left(\frac{a}{a_3}\right)^{-\frac{4}{3}m+\frac{3}{2}}\right]^{-3}$$



Three-stage SMBH evolution

Cascade induced accretion vs. Eddington accretion



Eddington accretion: $$M_B = M_i \exp\left(\frac{t-t_i}{t_{sal}}\right)$$

Radiation force balances the weight of static gas:

$$\frac{L_{Edd}}{4\pi c r^2} = \frac{G M_B m_p}{r^2 \sigma_T} \quad \text{or} \quad \frac{L_{Edd}}{c} \approx M_B \times \left(2.1\times10^{-8}\frac{m}{s^2}\right)$$

Alternatively, radiation force must balance the cascade force: $M_B \propto \sigma_p^5 / \varepsilon_b G$ (in stage E1)

$$\frac{L_B}{c} \propto \frac{\sigma_p^4}{G} \propto M_B \times \left(\frac{\varepsilon_b}{\sigma_p}\right) \gg \frac{L_{Edd}}{c} \quad \varepsilon_b \propto a^{-5/2}$$

Cascade induced accretion (first stage E1):

$$M_B = M_{\infty 1}\left[1-\left(\frac{a}{a_1}\right)^{-\frac{4}{5}m+\frac{3}{2}}\right]^5 \qquad a_1 = 1/(1+z_i)$$

$$M_B = M_{\infty 2}\exp\left[-\left(\frac{a}{a_2}\right)^{-mp+\frac{3}{2}}\right] \quad \text{(second stage E2)}$$

In early universe, cascade accretion >> Eddington?

Potential flaws in this argument?

Conclusions, keywords, and hyperlinks

- Cascade is ubiquitous in our universe
- Inverse mass cascade with a scale-independent rate ε_m (kg/s)
  - Random walk of halos in mass space (diffusion) ➡ Double-λ halo mass function
  - Random walk of DM particles ➡ Double-γ halo density profile
  - Halo mass function and density profile share the same origin and similar functional form.
  - No critical density ratio δc or spherical/ellipsoidal collapse model required
- Energy cascade with a constant rate ε_u (m^2/s^3)
  - 2/3 law for kinetic energy $v_r^2 \propto (\varepsilon_u r)^{2/3}$
  - 5/3 law for enclosed mass, $m_r \propto \varepsilon_u^{2/3} G^{-1} r^{5/3}$
  - -4/3 law for halo density, $\rho_r \propto \varepsilon_u^{2/3} G^{-1} r^{-4/3}$
  - The fundamental origin of cascade on the smallest scale (uncertainty principle)?

In propagation range, all quantity by ε_u, G, and r

- The smallest scale dependent on the nature of dark matter:
  - Collisionless dark matter: $r_\eta \propto (\varepsilon_u G h)^{1/3}$ ➡ DM particle mass & properties ⬅ All quantity by ε_u, G, and *h*
  - Self-interacting dark matter: $r_\eta \propto \varepsilon_u^2 G^{-3} (\sigma/m)^3$ ➡ the smallest structure ⬅ All quantity by ε_u, G, and σ/m
- The largest scale determined by u_0, ε_u, and G ➡ the largest halo & its properties ⬅ All quantity by ε_u, G, u_0, a

- Velocity/density correlation/moment functions
- The maximum entropy distributions in dark matter
- Energy cascade for the origin or MOND acceleration
- Energy cascade for the baryonic-to-halo mass relation
- Energy cascade for SMBH-galaxy co-evolution

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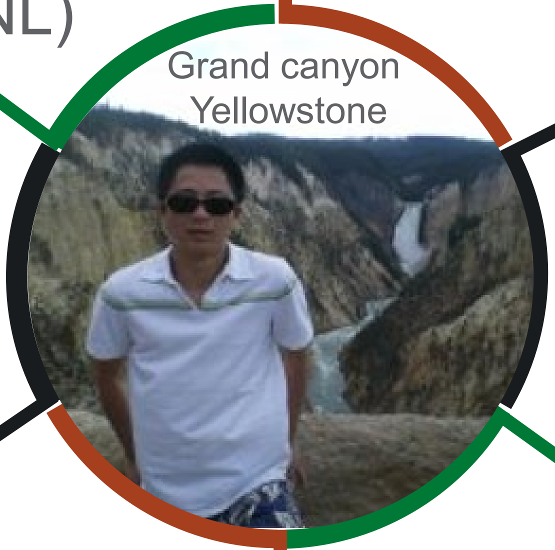

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