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Cascade Theory for Turbulence, Dark Matter and SMBH Evolution

Dec 2022

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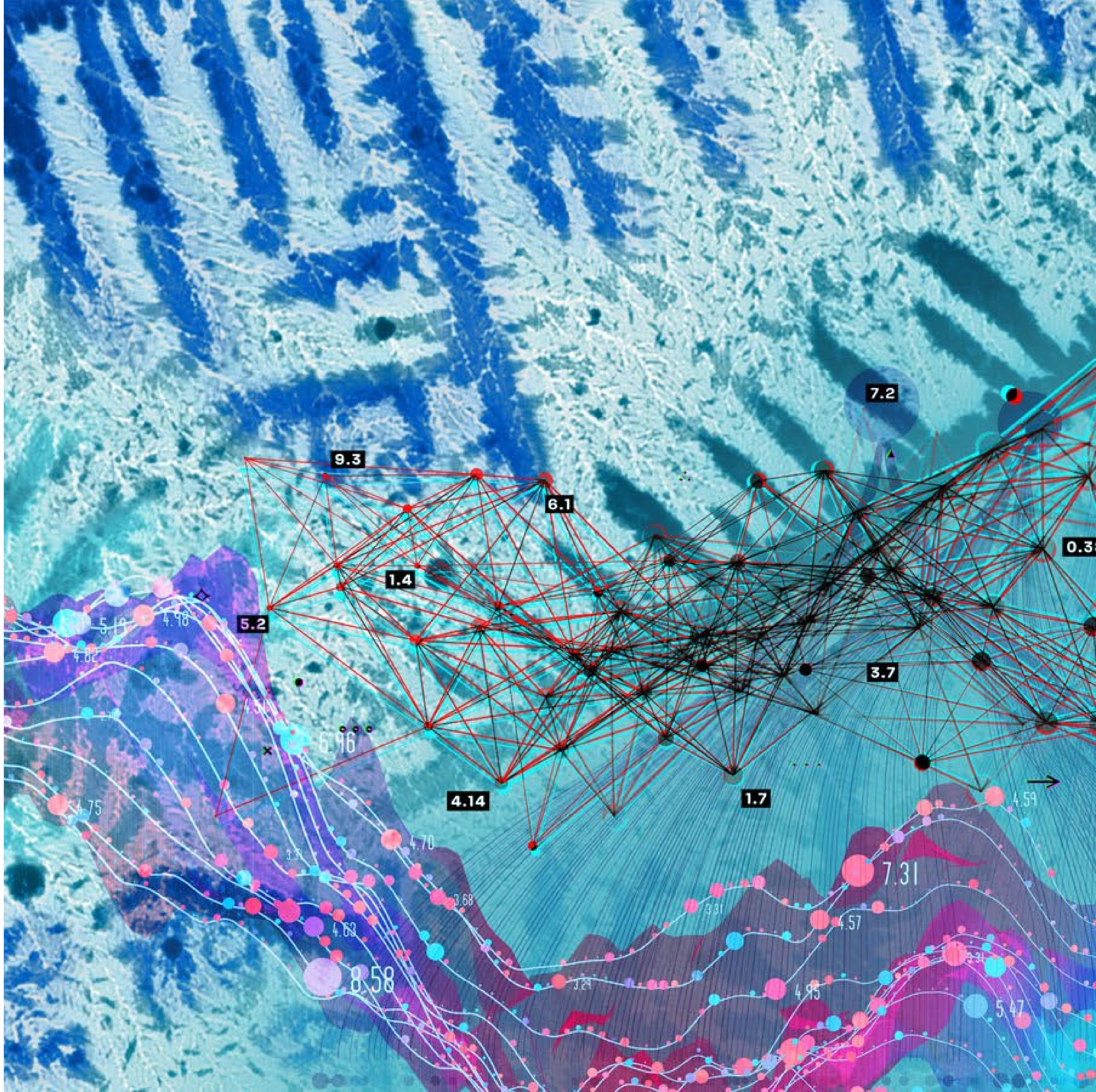


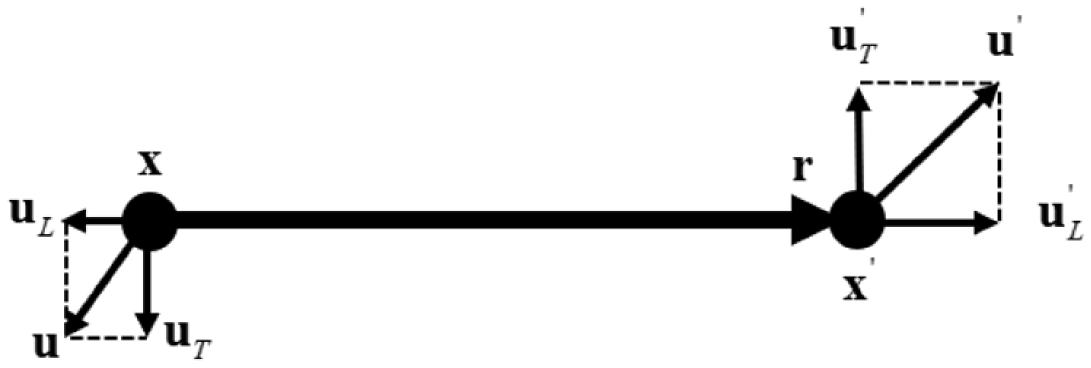
Outline

- Introduction
- Turbulence vs. the flow of dark matter: similarities and differences?
- Inverse mass cascade in dark matter flow
  - Random walk of halos in mass space and halo mass function
  - Random walk of dark matter in real space and halo density profile
- Energy cascade in dark matter flow
  - Universal scaling laws from N-body simulations and rotation curves
  - Dark matter properties from energy cascade
  - Uncertainty principle for energy cascade?
  - Extending to self-interacting dark matter
- Velocity/density correlation/moment functions
- Maximum entropy distributions for dark matter
- Energy cascade for the origin of MOND acceleration
- Energy cascade for the baryonic-to-halo mass relation
- Energy cascade for SMBH-bulge coevolution

Relevant datasets are available at:

"A comparative study of dark matter flow & hydrodynamic turbulence and its applications"
http://dx.doi.org/10.5281/zenodo.6569901

Correlation/moment functions from N-body sim.



Velocity correlation:

Longitudinal:	$L_2(r) = \langle u_L u'_L \rangle$	$\langle u_L^2 \rangle$
Transverse:	$T_2(r) = \langle \mathbf{u}_T \cdot \mathbf{u}'_T \rangle / 2$	$\langle u_T^2 \rangle$
Total:	$R_2(r) = \langle \mathbf{u} \cdot \mathbf{u}' \rangle = L_2 + 2T_2$	$\langle u^2 \rangle$

Density correlation: $\xi(r) = \langle \delta \cdot \delta' \rangle$ — 2nd moment

For incompressible or constant divergence flow (small scale):

$$T_2 = \frac{1}{2r}\left(r^2 L_2\right)_{,r} \qquad R_2 = \frac{1}{r^2}\left(r^3 L_2\right)_{,r}$$

$$L_2(r) = \int_0^\infty E_u(k) \frac{2 j_1(kr)}{kr} dk$$

$$T_2(r) = \int_0^\infty E_u(k) \left(j_0(kr) - \frac{j_1(kr)}{kr} \right) dk$$

*n*th order spherical Bessel function of the first kind: $j_n(kr)$

Kinematic relations

Relations to power spectrum function

Relations to density correlation function

For irrotational flow on large scale:

$$R_2 = \frac{1}{r^2}\left(r^3 T_2\right)_{,r} \qquad L_2 = \left(r T_2\right)_{,r}$$

$$L_2(r) = 2\int_0^\infty E_u(k) \left(j_0(kr) - 2\frac{j_1(kr)}{kr} \right) dk$$

$$T_2(r) = \int_0^\infty E_u(k) \frac{2 j_1(kr)}{kr} dk$$

$$\xi(r,a) = \langle \delta(\mathbf{x}) \cdot \delta(\mathbf{x}') \rangle = \frac{\langle \theta(\mathbf{x}) \cdot \theta(\mathbf{x}') \rangle}{(aHf(\Omega_m))^2}$$

$$= -\frac{1}{(aHf(\Omega_m))^2} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial R_2}{\partial r} \right) \right]$$

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Kinematic and dynamics relations for vel. correlation

Table 2. The velocity correlation functions of different order

p	$q=0$	$q=1$	$q=2$	$q=3$	$q=4$	$q=5$
1	$L_{(1,0)}=\langle u_L' \rangle$			$(p-q-1)R_{(p,q+1)}=\frac{1}{r^{p-q}}\left(r^{p-q+1}L_{(p,q)}\right)_{,r}$		
2	$L_{(2,0)}=\langle u_L u_L' \rangle$	$R_{(2,1)}=\langle \mathbf{u}\cdot\mathbf{u}' \rangle$		$\left(R_{(p,q+1)}r\right)_{,r}+(p-q-2)L_{(p,q+2)}=\frac{1}{r^{p-q}}\left(r^{p-q+1}L_{(p,q)}\right)_{,r}$		
3	$L_{(3,0)}=\langle u_L^2 u_L' \rangle$	$R_{(3,1)}=\langle u_L \mathbf{u}\cdot\mathbf{u}' \rangle$	$L_{(3,2)}=\langle u^2 u_L' \rangle$			
4	$L_{(4,0)}=\langle u_L^3 u_L' \rangle$	$R_{(4,1)}=\langle u_L^2 \mathbf{u}\cdot\mathbf{u}' \rangle$	$L_{(4,2)}=\langle u^2 u_L u_L' \rangle$	$R_{(4,3)}=\langle u^2 \mathbf{u}\cdot\mathbf{u}' \rangle$		
5	$L_{(5,0)}=\langle u_L^4 u_L' \rangle$	$R_{(5,1)}=\langle u_L^3 \mathbf{u}\cdot\mathbf{u}' \rangle$	$L_{(5,2)}=\langle u^2 u_L^2 u_L' \rangle$	$R_{(5,3)}=\langle u^2 u_L \mathbf{u}\cdot\mathbf{u}' \rangle$	$L_{(5,4)}=\langle u^4 u_L' \rangle$	
6	$L_{(6,0)}=\langle u_L^5 u_L' \rangle$	$R_{(6,1)}=\langle u_L^4 \mathbf{u}\cdot\mathbf{u}' \rangle$	$L_{(6,2)}=\langle u^2 u_L^3 u_L' \rangle$	$R_{(6,3)}=\langle u^2 u_L^2 \mathbf{u}\cdot\mathbf{u}' \rangle$	$L_{(6,4)}=\langle u^4 u_L u_L' \rangle$	$R_{(6,5)}=\langle u^4 \mathbf{u}\cdot\mathbf{u}' \rangle$

For constant divergence flow

For irrotational flow

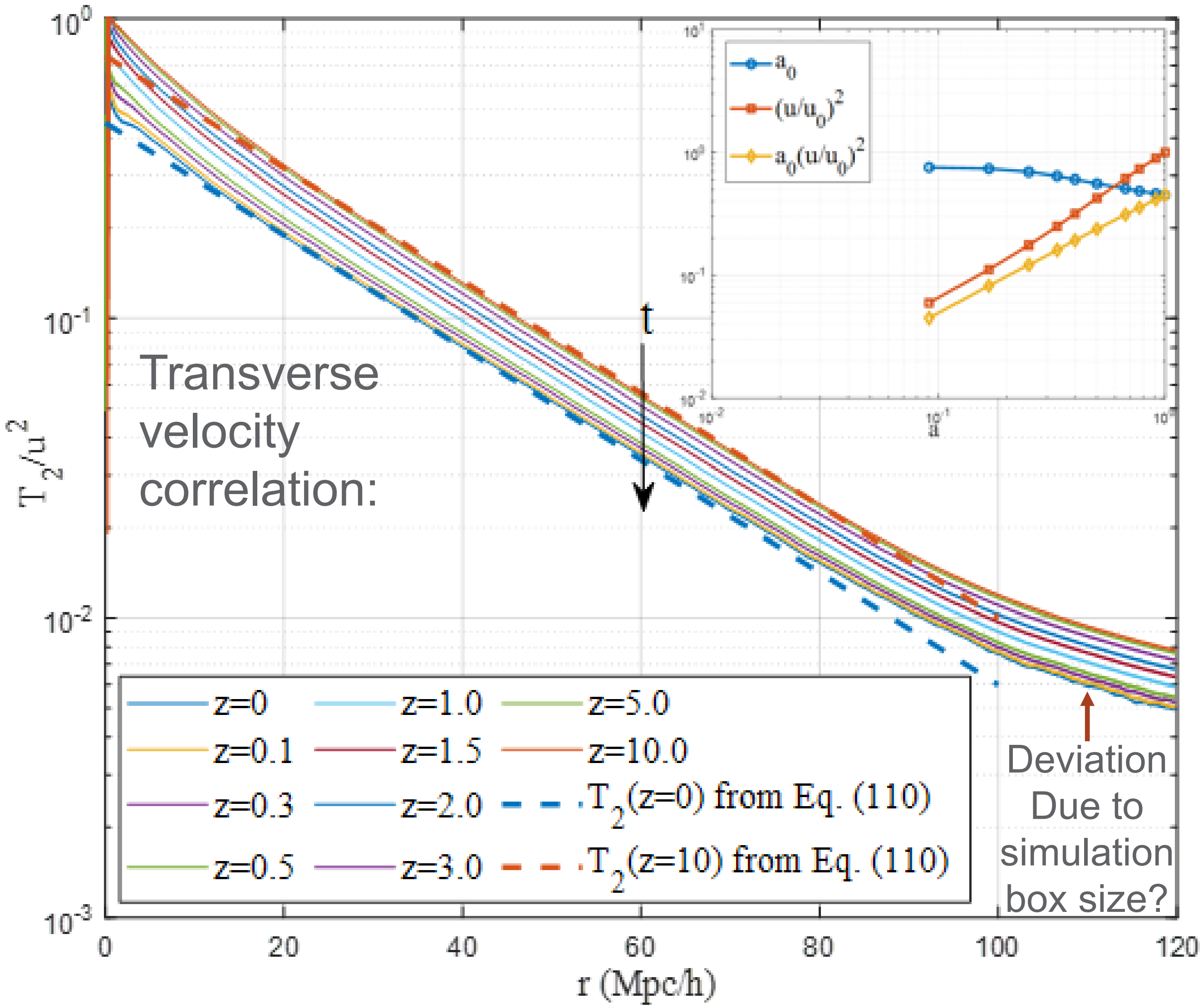
Kinematic relations (for same order p)

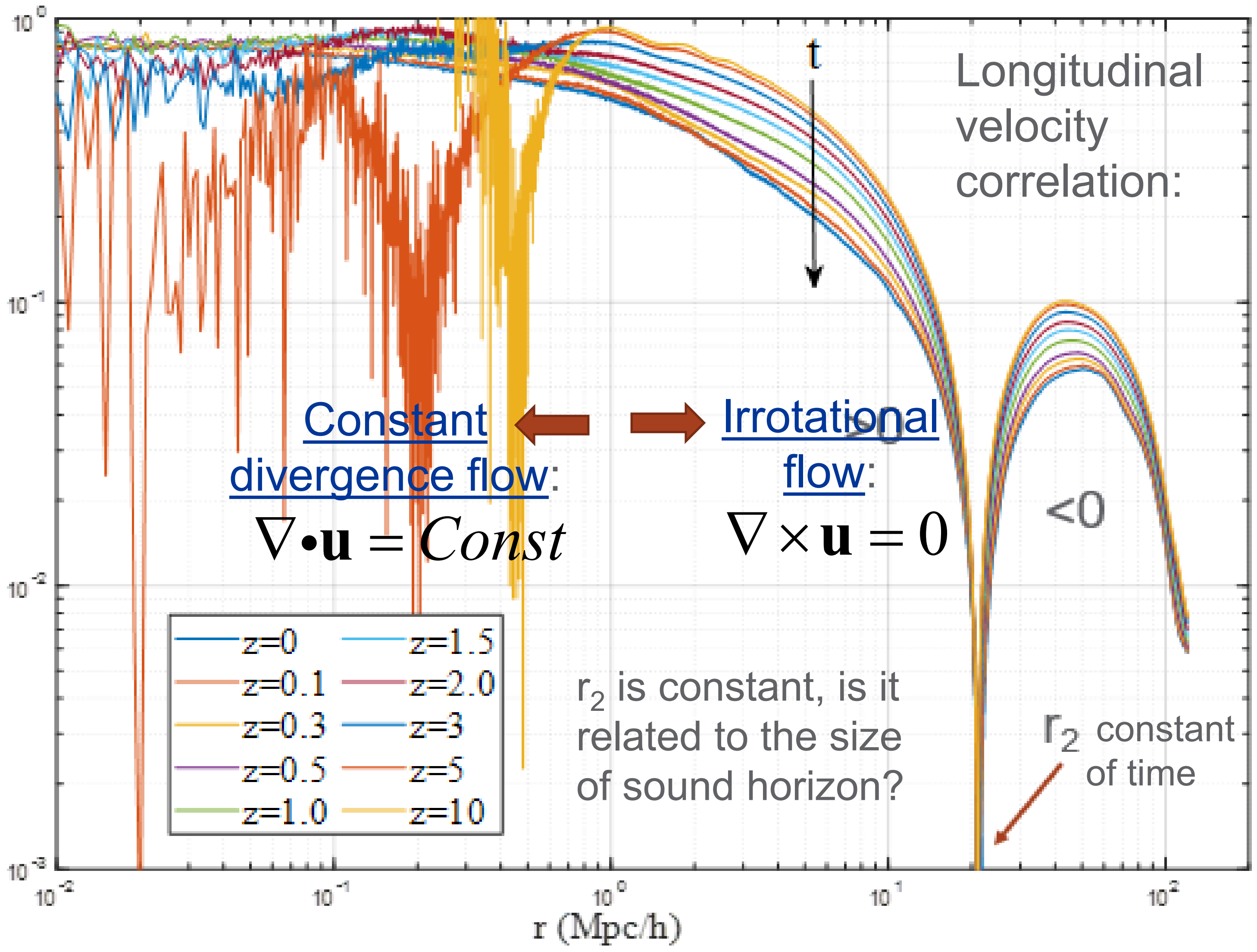
Dynamic relations (for different order p)

Velocity correlation functions on large scale

On large scale, velocity correlation (exponential): applying kinematic relations for irrotational flow

$$T_2(r,a) = a_0 u^2 \exp(-r/r_2) \propto a \;\Rightarrow\; L_2(r,a) = a_0 u^2 \exp\left(-\frac{r}{r_2}\right)\left(1-\frac{r}{r_2}\right) \;\Rightarrow\; R_2(r,a) = a_0 u^2 \exp\left(-\frac{r}{r_2}\right)\left(3-\frac{r}{r_2}\right)$$



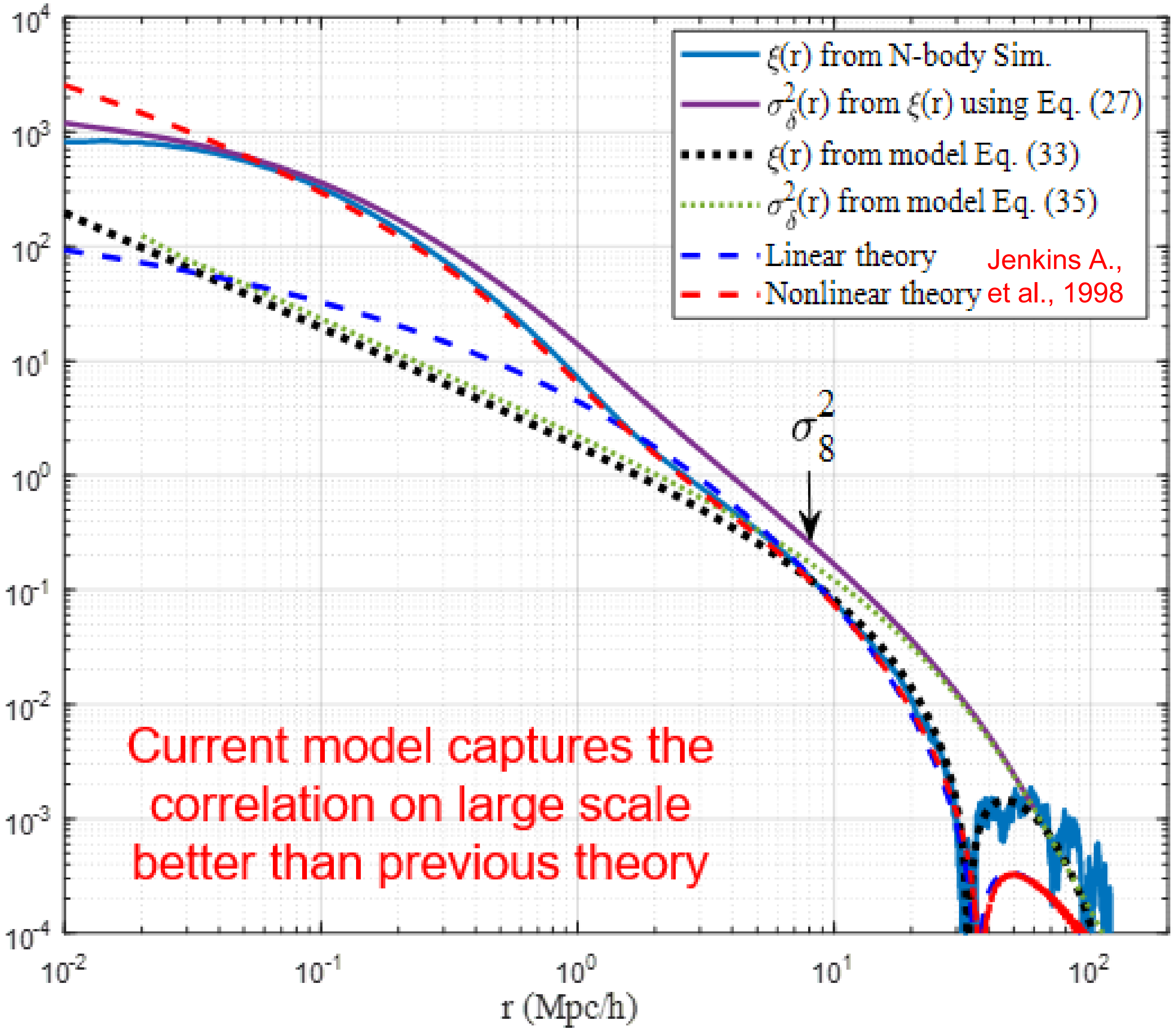


Density correlation function on large scale

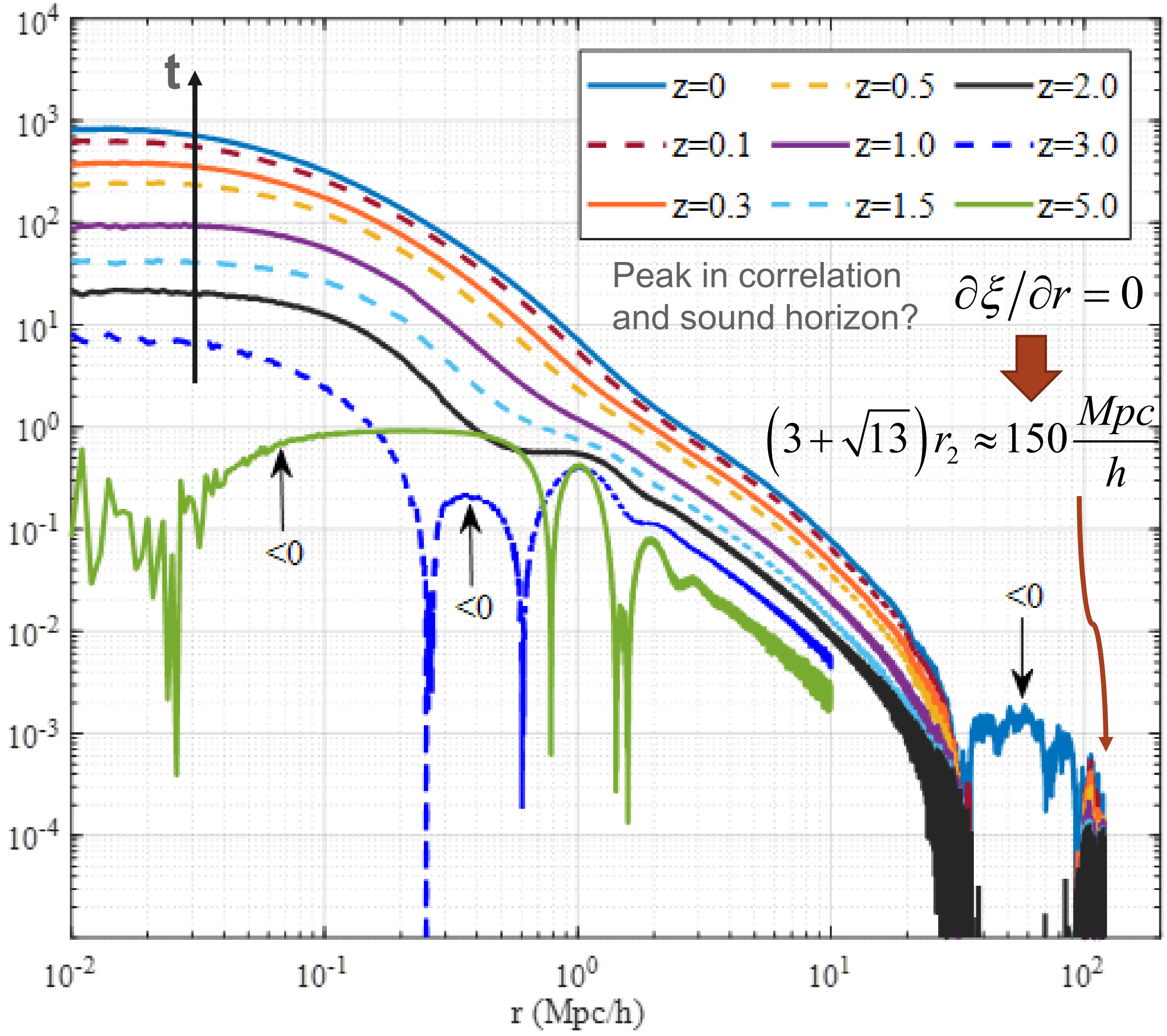
On large scale, density correlation (exponential):

$$R_2(r,a) = a_0 u^2 \exp\left(-\frac{r}{r_2}\right)\left(3 - \frac{r}{r_2}\right)$$

$$\xi(r,a) = \frac{1}{\left(aHf(\Omega_m)\right)^2} \cdot \frac{a_0 u^2}{r r_2} \exp\left(-\frac{r}{r_2}\right)\left[\left(\frac{r}{r_2}\right)^2 - 7\left(\frac{r}{r_2}\right) + 8\right]$$



Density correlation function at z=0



Density correlation function at different z

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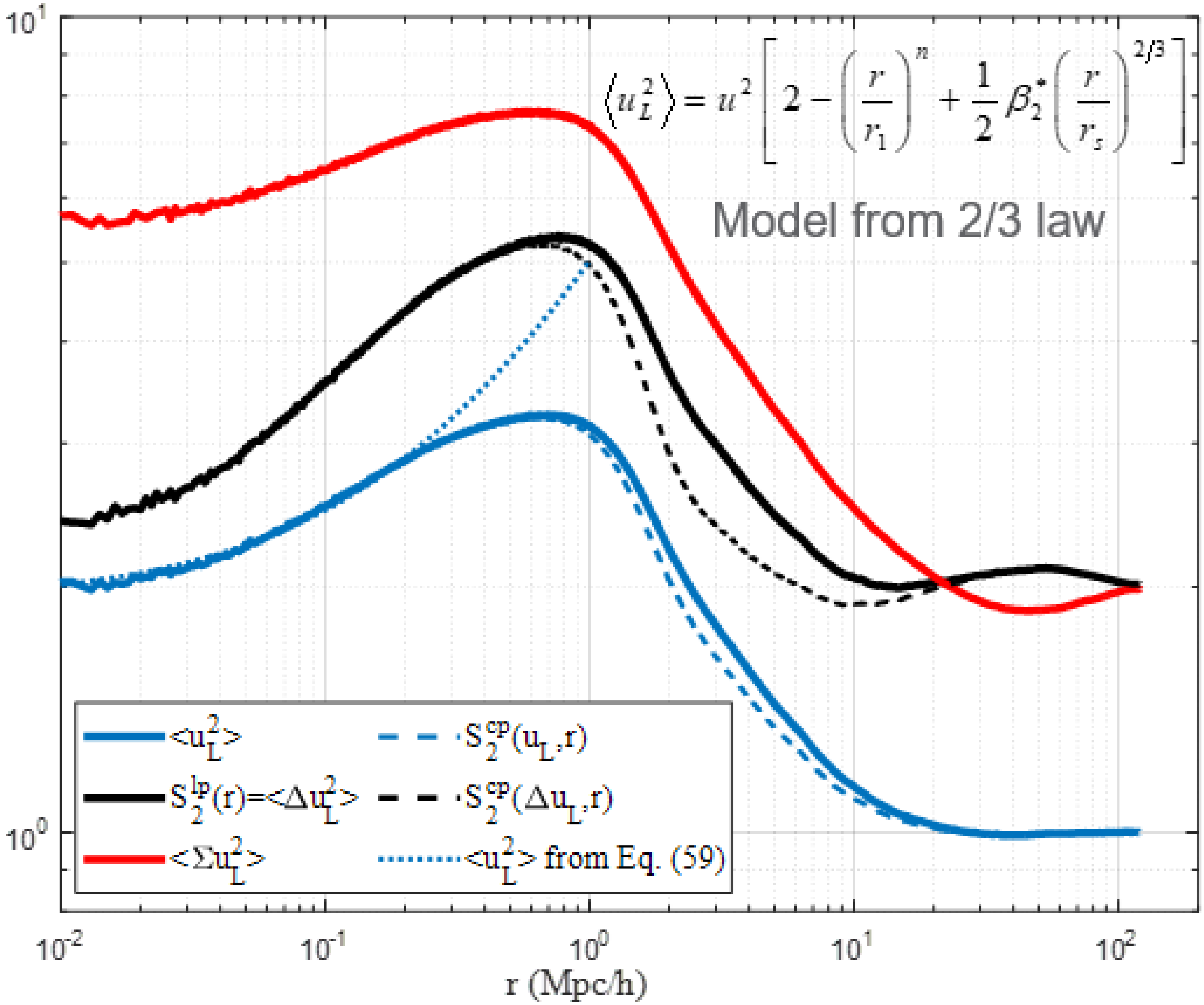
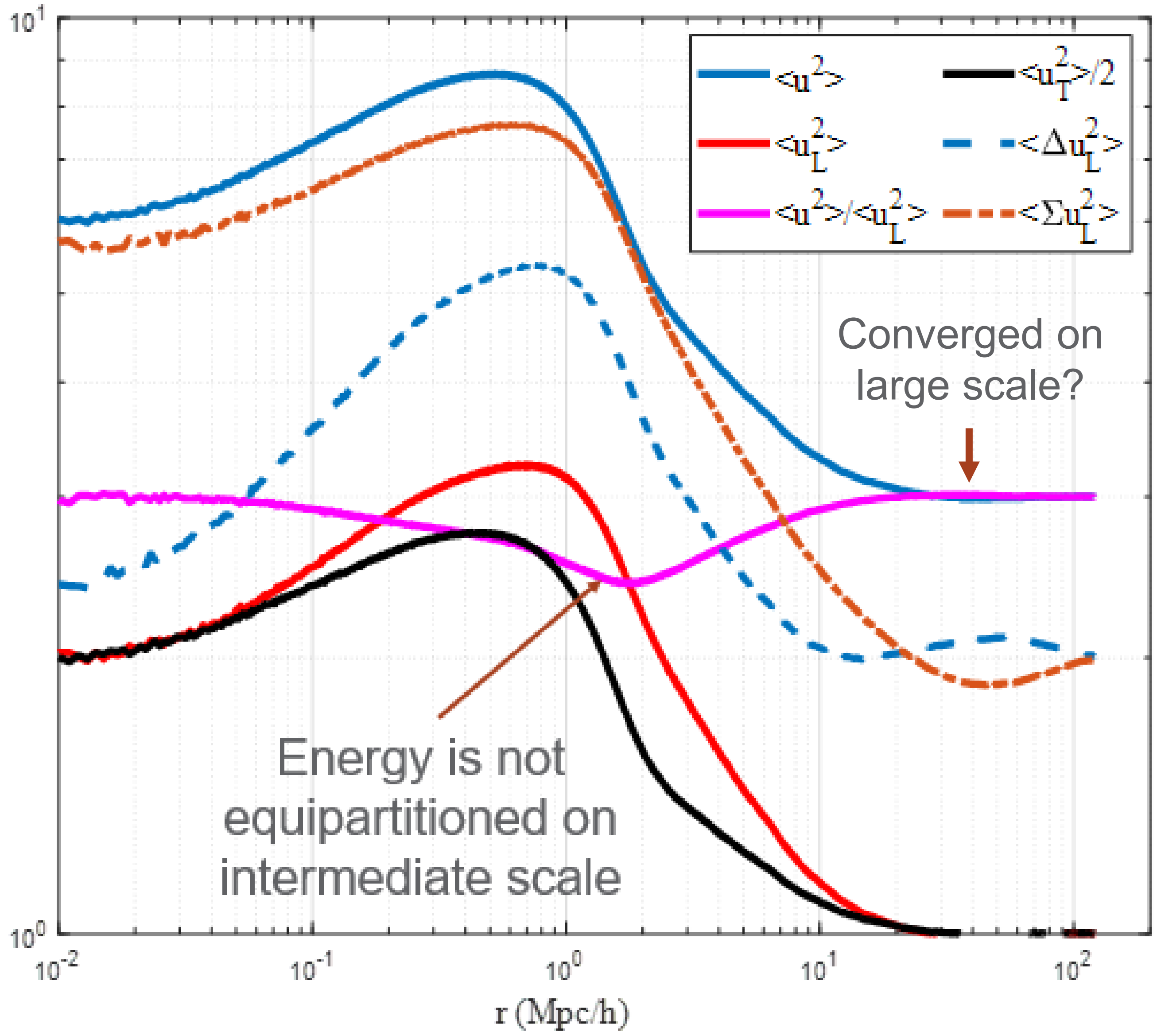
Second moments of velocity field



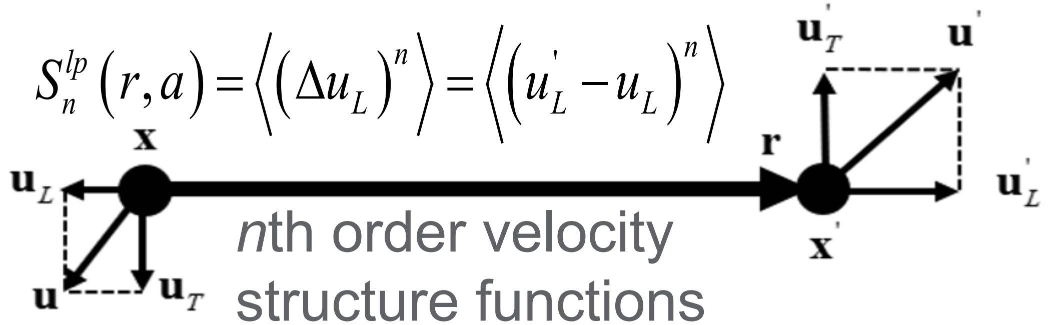
Increase of velocity dispersions with r for $r<r_t$ (pair of particles are more likely from same halos) is mostly due to the increase of velocity dispersion with halo size.



Second moment of velocity (normalized by u^2) varying with scale r at z=0

Two-thirds law and generalized stable clustering (GSCH)

$$S_n^{lp}(r,a) = \left\langle (\Delta u_L)^n \right\rangle = \left\langle (u_L' - u_L)^n \right\rangle$$



*n*th order velocity structure functions

Zeroth order: $S_0^{lp}(r,a) = 1$

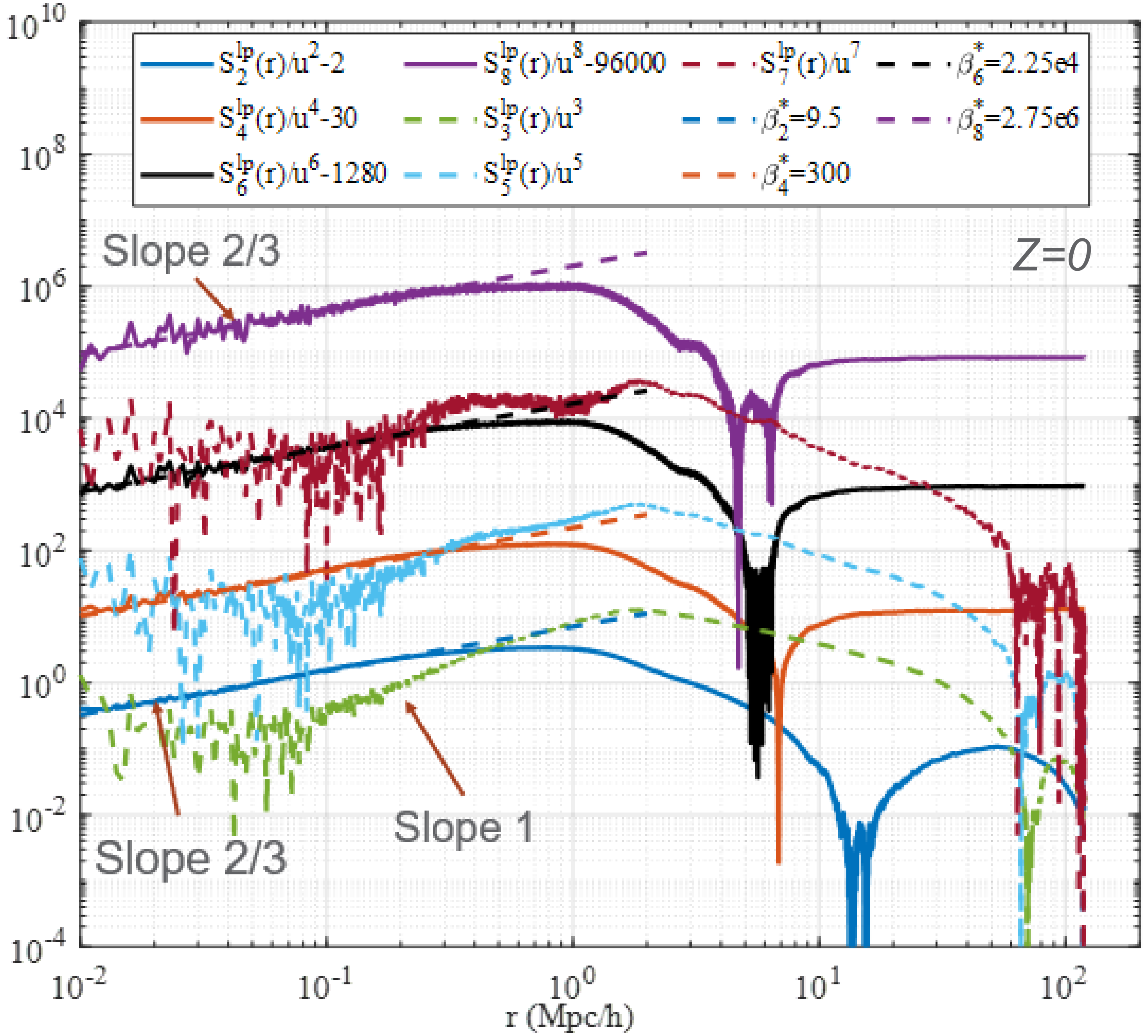
First order: $S_1^{lp}(r,a) = \langle \Delta u_L \rangle = -Har$ *Stable clustering hypothesis*

All even order reduced structure functions follow two-thirds law:

$$S_{2n}^{lp}(r) = u^{2n}\left[2^n K_{2n}(\Delta u_L, 0) + \beta_{2n}^*(r/r_s)^{2/3}\right]$$

All odd order structure functions follow linear law from generalized stable clustering hypothesis

$$S_{2n+1}^{lp}(r) = (2n+1) S_1^{lp}(r) S_{2n}^{lp}(r) \propto r^1$$



Maximum entropy distributions in kinetic theory of gases

Review on how to derive maximum entropy distributions (Boltzmann distribution)

Assume the distribution of one-dimensional gas molecule velocity is some unknown function X(v)

Two constraints on X(v): normalization and fixed mean kinetic energy

$$\int_{-\infty}^{\infty} X(v)\,dv = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} X(v)\varepsilon(v)\,dv = \frac{3}{2}k_B T = \frac{3}{2}\sigma_0^2$$

Particle energy: $\varepsilon(v) = 3v^2/2$

This is the key to be identified for dark matter flow

Write down the entropy functional with Lagrangian multiplier:

$$S[X(v)] = -\int_{-\infty}^{\infty} X(v)\ln X(v)\,dv + \lambda_1\left(\int_{-\infty}^{\infty} X(v)\,dv - 1\right) + \lambda_2\left(\int_{-\infty}^{\infty} X(v)\varepsilon(v)\,dv - \frac{3}{2}\sigma_0^2\right)$$

Taking the variation of the entropy functional with respect to distribution X:

$$\frac{\delta S(X(v))}{\delta X} = -\ln X(v) - 1 + \lambda_1 + \lambda_2\varepsilon(v) = 0 \;\Rightarrow\; X(v) \propto \exp\left(\lambda_2 v^2\right)$$

Boltzmann distribution

Distribution for particle energy: $E(\varepsilon) = 2\sqrt{\frac{\varepsilon}{\pi\sigma_0^2}}\frac{1}{\sigma_0^2}e^{-\varepsilon/\sigma_0^2}$

Maxwell-Boltzmann distribution for speed: $Z(v) = \sqrt{\frac{2}{\pi}}\frac{v^2}{\sigma_0^3}e^{-v^2/2\sigma_0^2}$

Maximum entropy distributions in dark matter

Deriving maximum entropy distributions in dark matter flow (X distribution)

Two constraints on X(v):

$$\int_{-\infty}^{\infty} X(v)\,dv = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} X(v)\,\varepsilon(v)\,dv = \frac{3}{2}k_B T = \frac{3}{2}\sigma_0^2$$

Particle energy:

$$\varepsilon(v) = -\frac{X(v)\,v}{\partial X/\partial v}\left(\frac{3}{2}+\frac{3}{n}\right)$$

This is the key

Write down the entropy functional with Lagrangian multiplier:

$$S[X(v)] = -\int_{-\infty}^{\infty} X(v)\ln X(v)\,dv + \lambda_1\left(\int_{-\infty}^{\infty} X(v)\,dv - 1\right) + \lambda_2\left(\int_{-\infty}^{\infty} X(v)\,\varepsilon(v)\,dv - \frac{3}{2}\sigma_0^2\right)$$

Taking the variation of the entropy functional with respect to X:

$$\frac{\delta S(X(v))}{\delta X} = -\ln X(v) - 1 + \lambda_1 + \lambda_2\varepsilon(v) = 0 \;\Rightarrow\; X(v) = \frac{1}{2\alpha v_0}\frac{e^{-\sqrt{\alpha^2+(v/v_0)^2}}}{K_1(\alpha)}$$

The X distribution

E distribution for particle energy:

$$E(\varepsilon) = -\frac{2n}{3(n+2)}\frac{e^{-\gamma}\sqrt{\gamma^2-\alpha^2}}{\alpha K_1(\alpha)v_0^2}$$

Z distribution for speed:

$$Z(v) = \frac{1}{\alpha K_1(\alpha)}\cdot\frac{v^2}{v_0^3}\cdot\frac{e^{-\sqrt{\alpha^2+(v/v_0)^2}}}{\sqrt{\alpha^2+(v/v_0)^2}}$$

Maximum entropy distributions in dark matter

Gaussian core for $|v| \ll v_0$

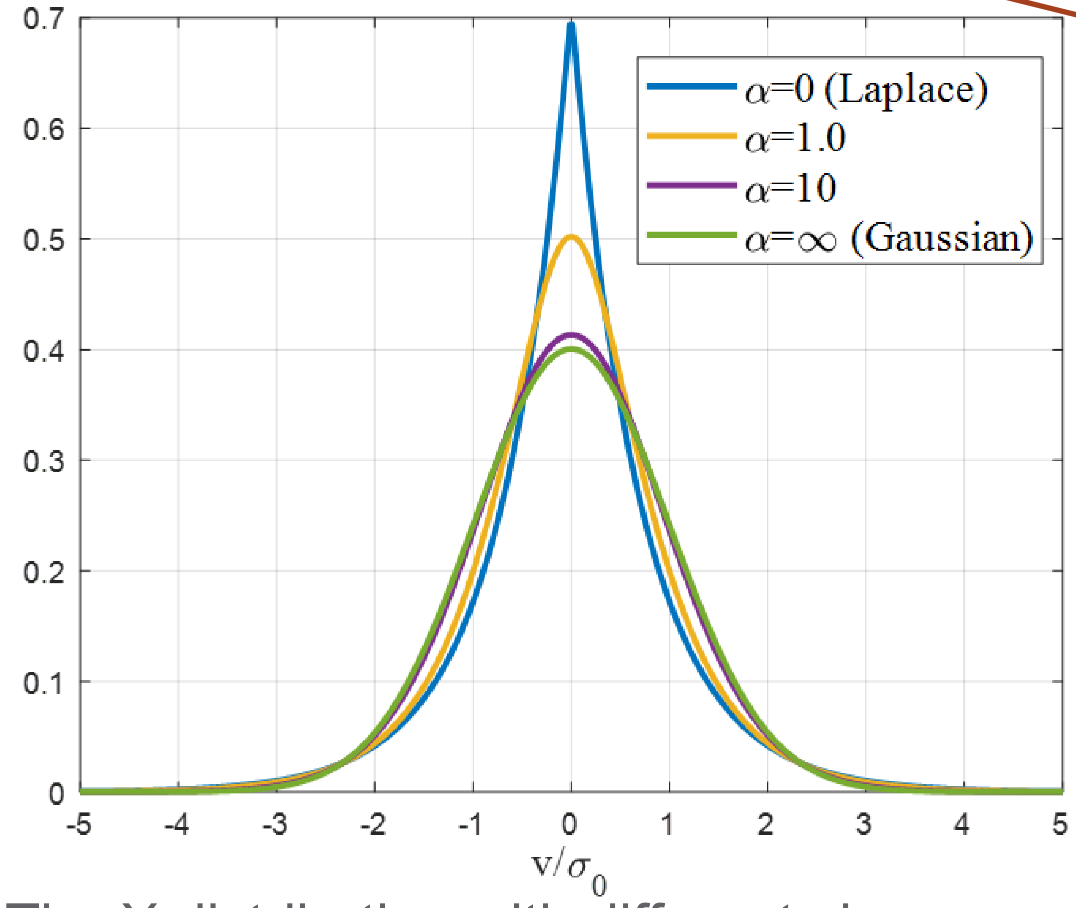
$$X(v) = \frac{e^{-\alpha}}{2\alpha v_0 K_1(\alpha)} \exp\left(-\frac{v^2}{2\alpha v_0^2}\right)$$

Exponential wings for $|v| \gg v_0$

$$X(v) = \frac{1}{2\alpha v_0 K_1(\alpha)} \exp\left(-\frac{v}{v_0}\right)$$

$$X(v) = \frac{1}{2\alpha v_0} \frac{e^{-\sqrt{\alpha^2 + (v/v_0)^2}}}{K_1(\alpha)}$$

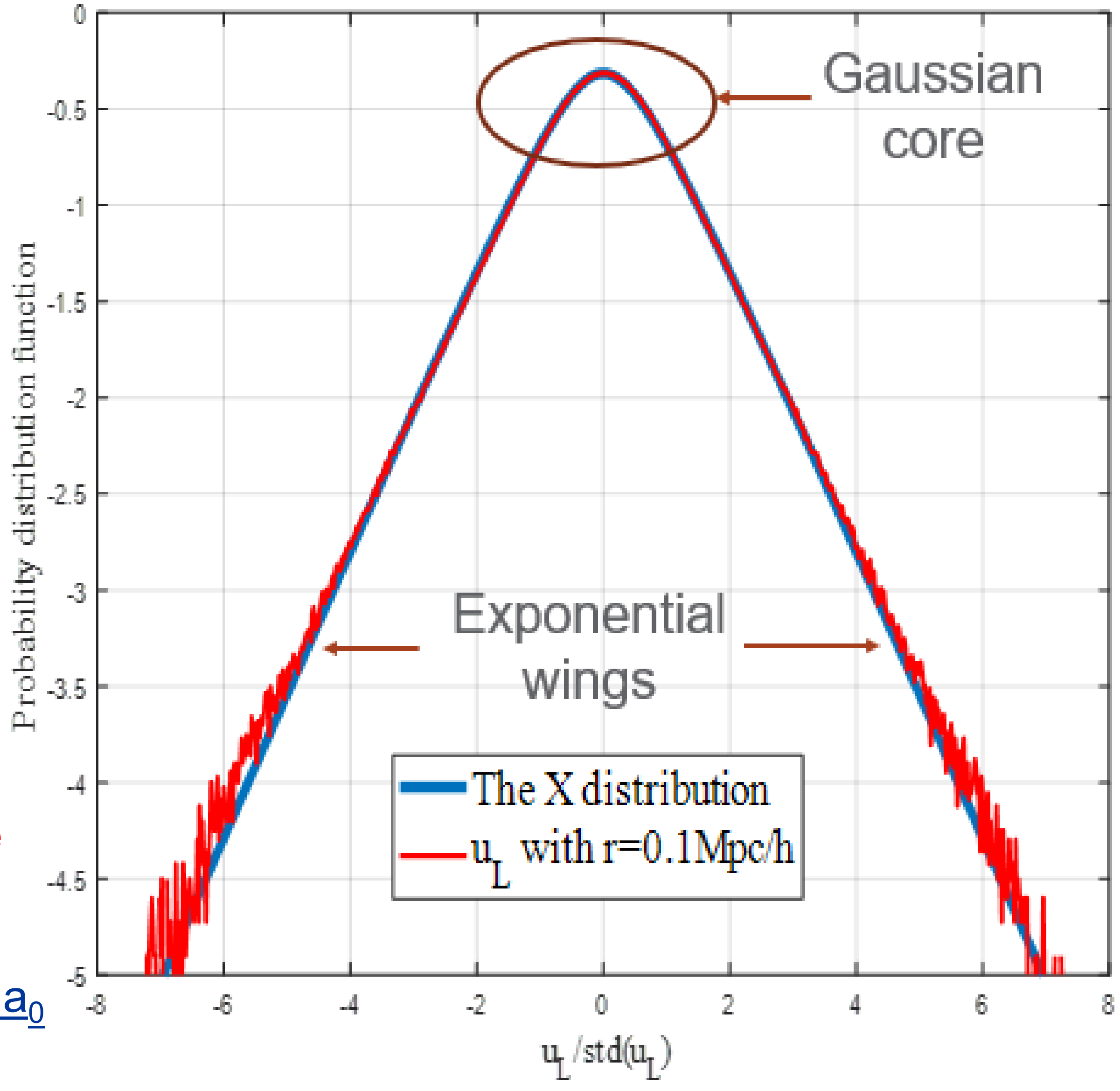
Bessel function



The X distribution with different shape parameter α

X is a two-parameter distribution with shape parameter α and a velocity scale v_0
or an acceleration scale a_0

$$\varepsilon_u \propto a_0 v_0 \,?$$



Comparison with N-body simulation

Particle energy vs. particle velocity in dark matter

$$X(v) = \frac{1}{2\alpha v_0} \frac{e^{-\sqrt{\alpha^2 + (v/v_0)^2}}}{K_1(\alpha)} \qquad \varepsilon(v) = -\frac{X(v)v}{\partial X/\partial v}\left(\frac{3}{2} + \frac{3}{n}\right)$$

Particle energy:

$$\varepsilon(v) = \frac{3}{2}\left(1 + \frac{2}{n}\right) v_0^2 \sqrt{\alpha^2 + \left(\frac{v}{v_0}\right)^2}$$

Gaussian core for $|v| \ll v_0$

$$\varepsilon(v) \approx \frac{3}{2}\left(1 + \frac{2}{n}\right)\left(\alpha v_0^2 + \frac{v^2}{2\alpha}\right) \propto v^2$$

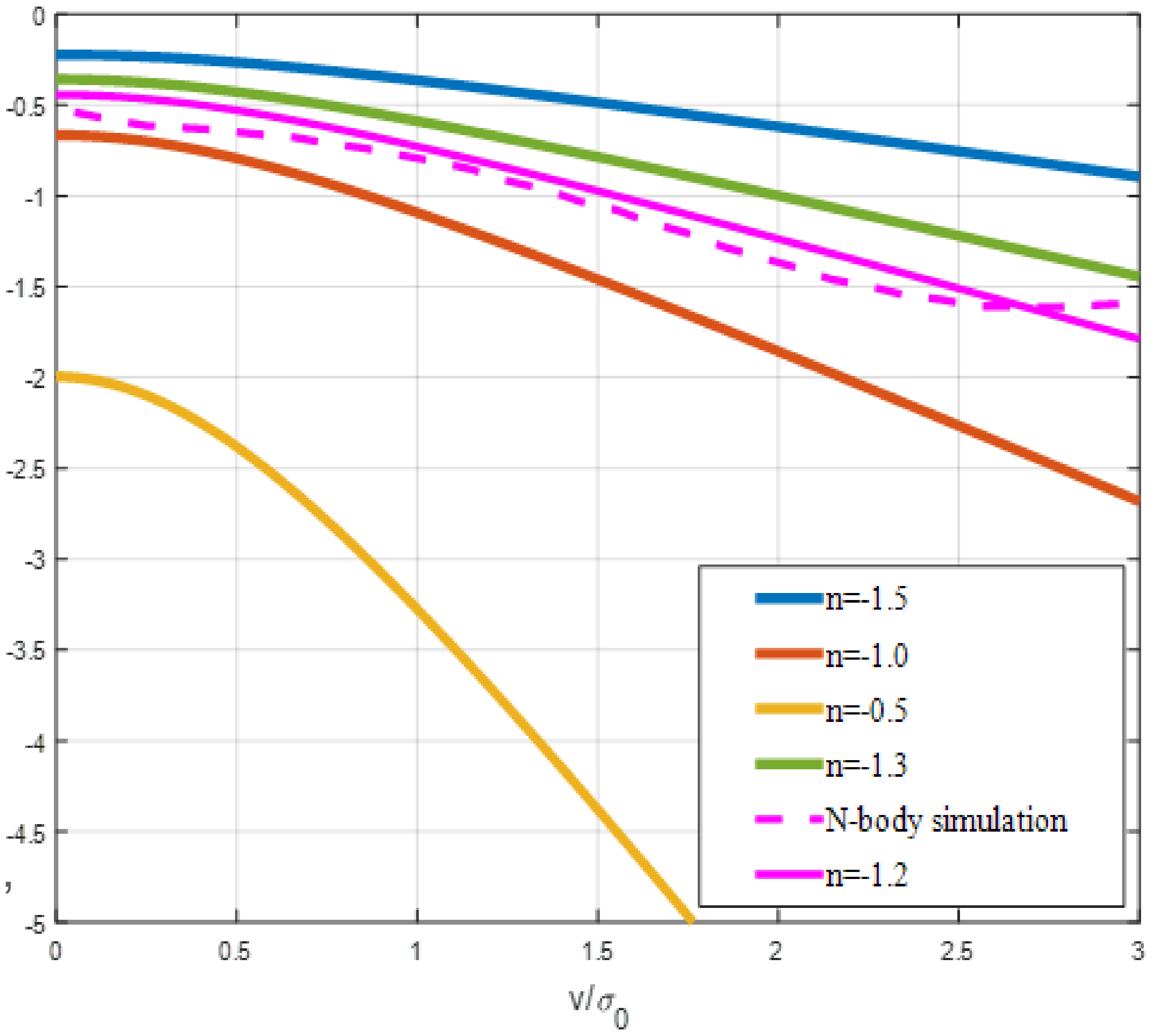
Inner halo, Newtonian behavior

Exponential wings for $|v| \gg v_0$

$$\varepsilon(v) \approx \frac{3}{2}\left(1 + \frac{2}{n}\right) v_0 v \propto v$$

Outer region of halo, non-Newtonian behavior

Deep-MOND?



Comparison with N-body simulation for particle energy ε(v)

MOND theory and acceleration fluctuation in DMF

- Empirical Tully-Fisher relation:

Flat rotation speed $v_f \propto M_b^{1/4}$ ⬅ observed baryonic mass

- MOND (Milgrom 1983) is an empirical model to reproduce flat rotation curve without dark matter.

$a_0 \approx 1.2\times 10^{-10}\, m/s^2$ Critical MOND acceleration

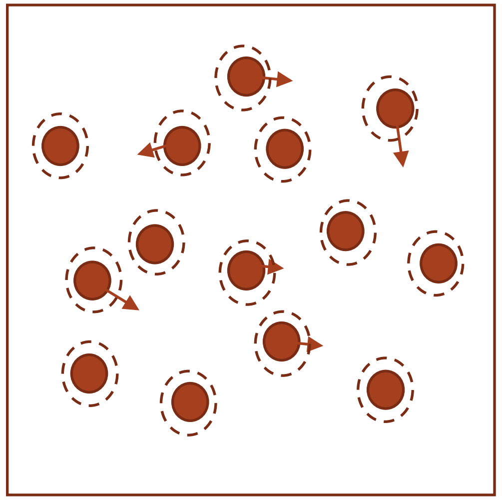
$F = ma$ $\quad a \gg a_0$ Newtonian

$F = m a^2/a_0 \propto a^2$ $\quad a \ll a_0$ Deep MOND

$$\frac{GMm}{r^2} = m\frac{\left(v_f^2/r\right)^2}{a_0} \Rightarrow v_f = \left(GMa_0\right)^{1/4}$$

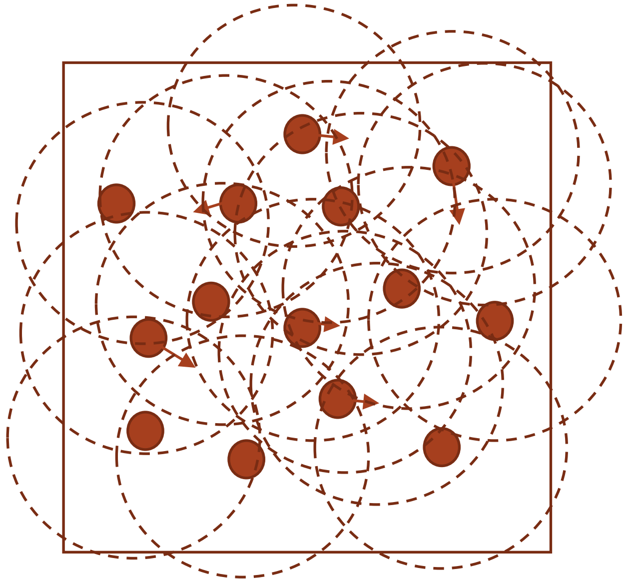
- What is the origin of MOND acceleration?
- What is the origin of deep "MOND"?
- Could MOND be an intrinsic property of dark matter flow in CDM cosmology?

- In kinetic theory of gases, molecules undergo random elastic collisions with a short-range of interaction. Only velocity fluctuation, no fluctuation of acceleration.



Short range: molecule acceleration vanishes

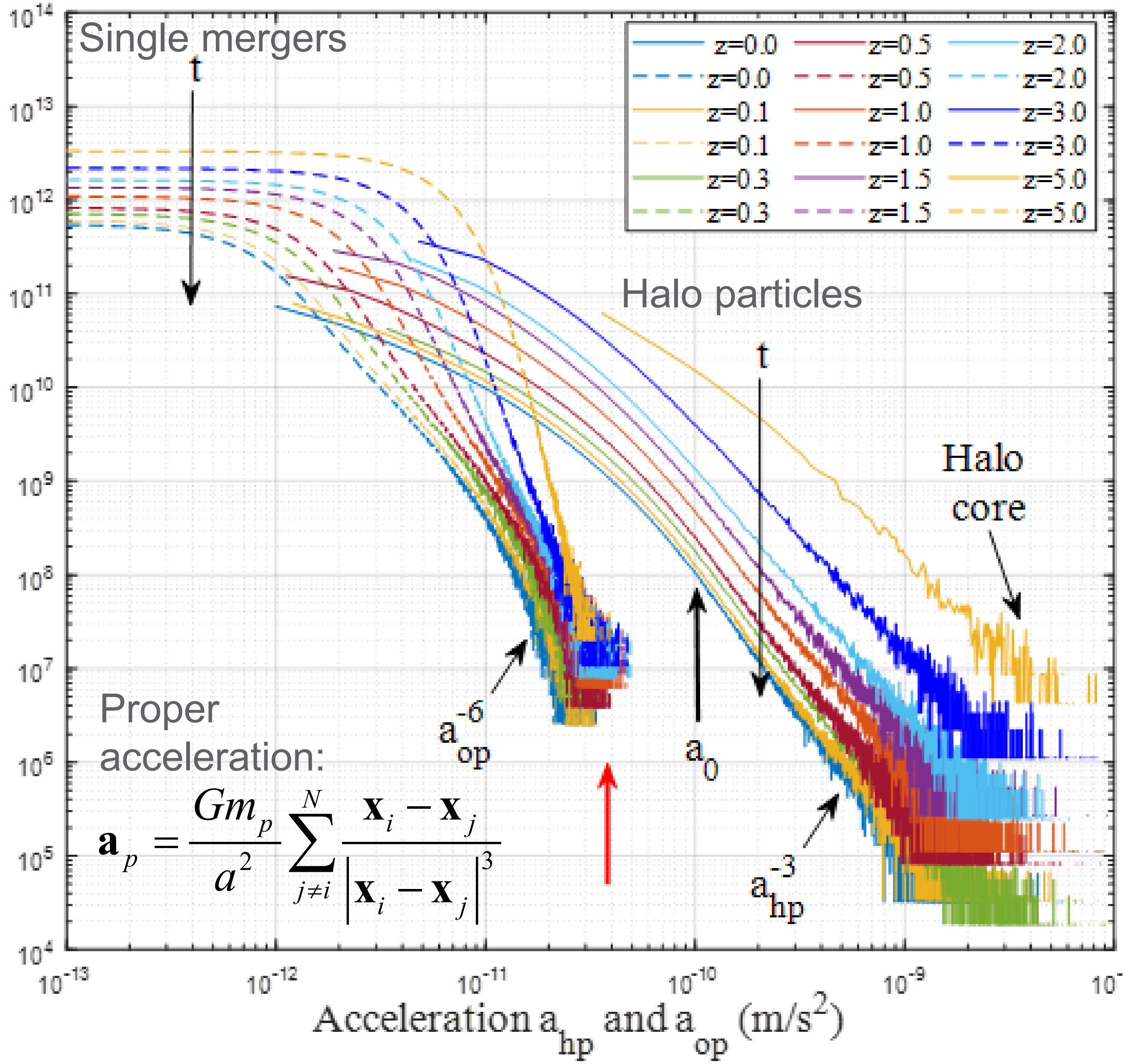
- The long-range gravity in dark matter flow leads to fluctuations in acceleration, in addition to the fluctuation in velocity.



Long range: nonvanishing and fluctuating acceleration

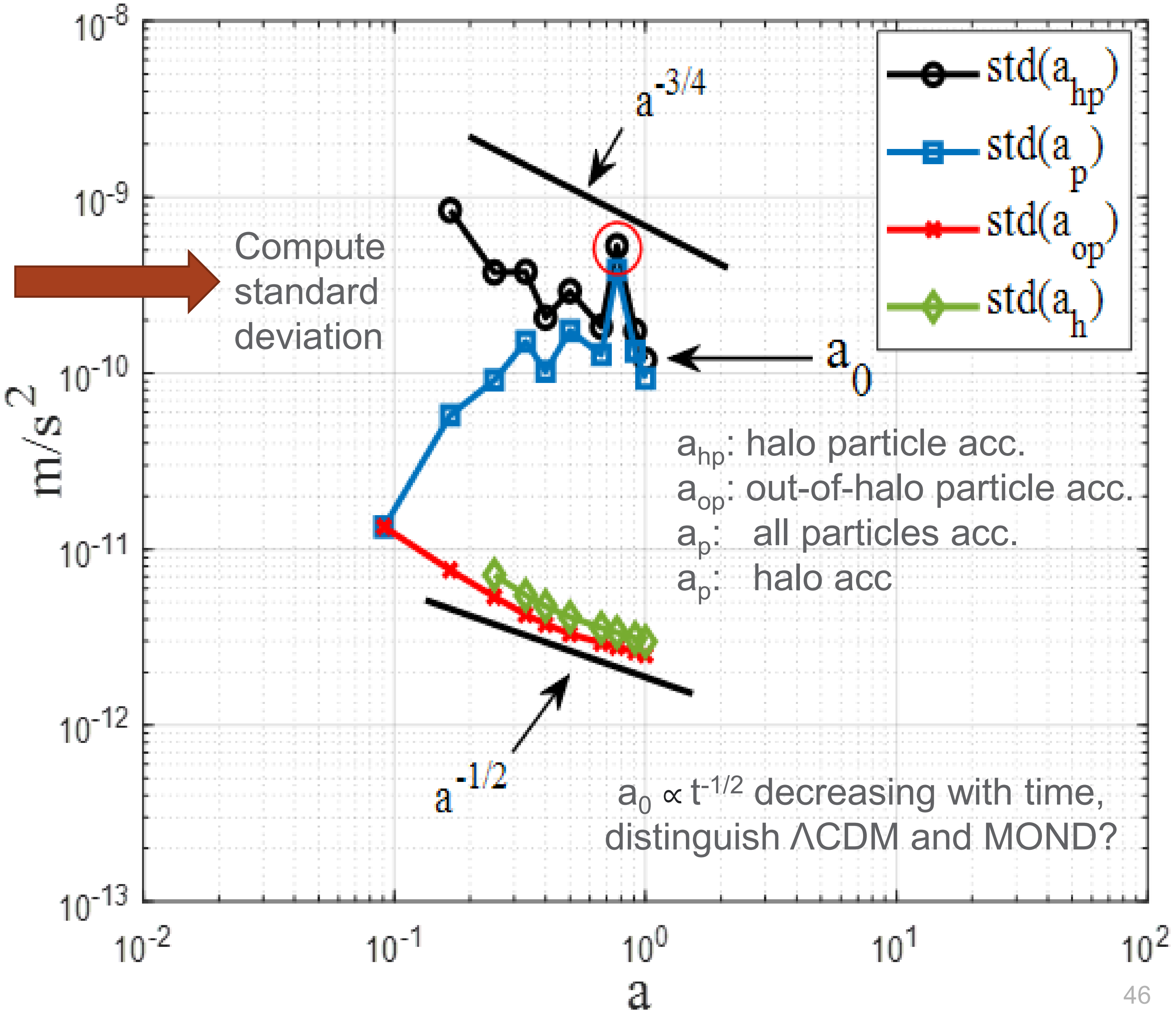
Acceleration distributions in dark matter

Acc fluctuation leads to distribution of acceleration



Time variation of acceleration fluctuation (DM only sim.)

Compute standard deviation



a_{hp}: halo particle acc.
a_{op}: out-of-halo particle acc.
a_p: all particles acc.
a_p: halo acc

$a_0 \propto t^{-1/2}$ decreasing with time, distinguish ΛCDM and MOND?

MOND acceleration a_0 from energy cascade

Assume a_0 is the typical acceleration scale of fluctuation,
u is the typical velocity scale of fluctuation, θ_{ur} is the angle of incidence.

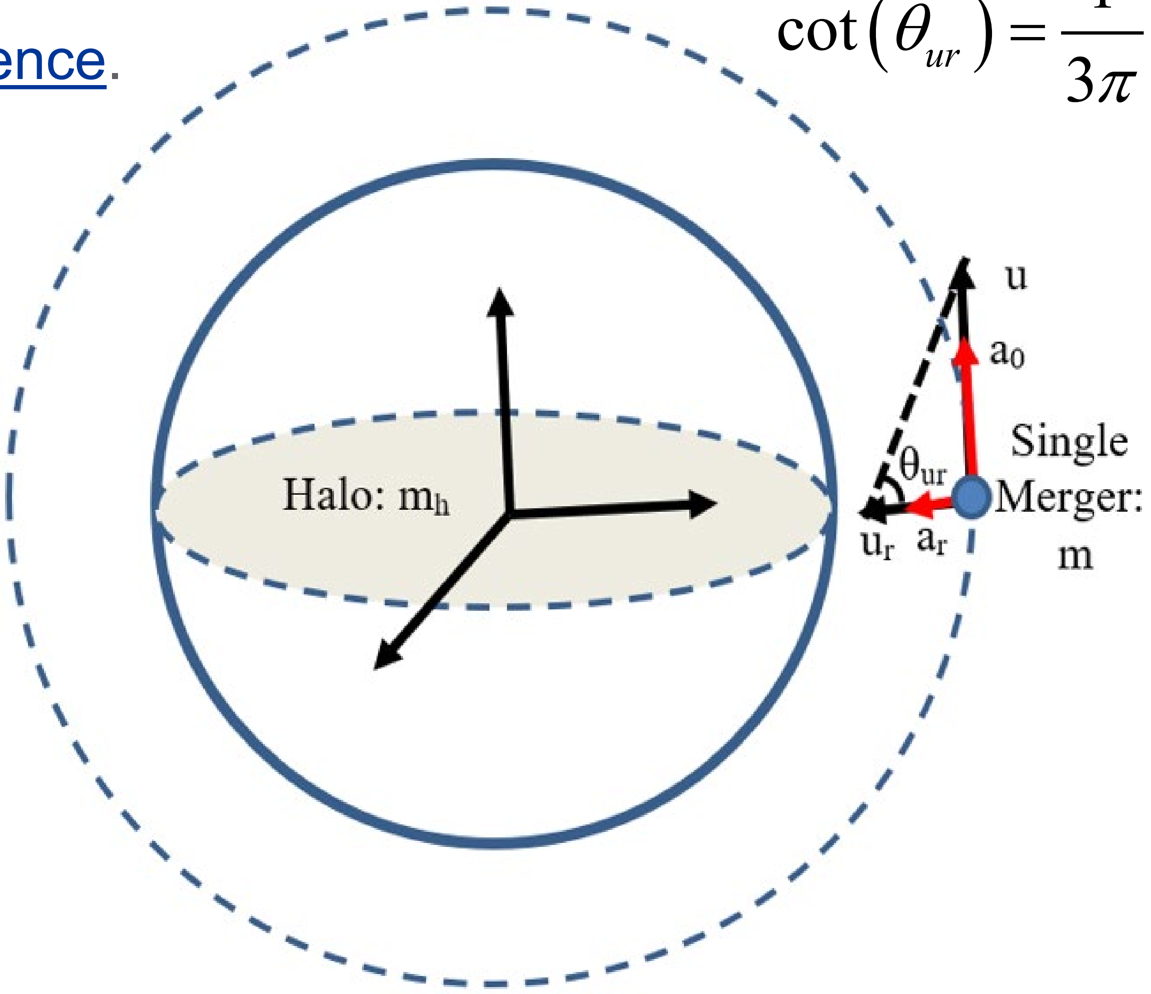
$$\cot(\theta_{ur}) = \frac{1}{3\pi}$$

The rate of energy cascade in terms of a_0, u and θ_{ur} :

$$\varepsilon_u = -a_r u_r = -a_0(a)\cot(\theta_{ur})u(a)\cot(\theta_{ur})$$

$$a_0(a) = -(3\pi)^2 \frac{\varepsilon_u}{u} = \frac{81}{4}\pi^2 H_0 \frac{u_0^2}{u} \propto a^{-3/4} \propto t^{-1/2}$$

Confirmed by simulations,
arXiv:2206.04333
arXiv:1712.01654
what about observations?



The rate of energy cascade:

$$\varepsilon_u \approx -\frac{3}{2}\frac{u^2}{t} = -\frac{3}{2}\frac{u_0^2}{t_0} = -\frac{9}{4}H_0 u_0^2 = -4.6\times 10^{-7}\frac{m^2}{s^3}$$

$$a_0(a=1) \approx 200 H_0 u_0 \approx 1.2\times 10^{-10}\, m/s^2$$

In Earth's atmosphere:

$$\varepsilon_u \approx 10^{-3}\, m^2/s^3$$

Potential connection with dark energy??

$$\rho_{vac} = \frac{\Lambda c^2}{8\pi G} = \frac{3\pi}{2G}\left(\frac{(3\pi)^2\varepsilon_u}{u_0}\right)^2 = \frac{3\pi}{2}\frac{a_0^2 H_0}{GH} \propto \frac{a_0^2}{H}$$

Milgrom coincidence

$$a_0(z=0) \approx c\frac{(\Lambda/3)^{1/2}}{2\pi}$$

- Ideal gas pressure P (N/m^2) ∝ temperature T ∝ velocity fluctuation
- DE density (N/m^2) ∝ a_0^2 ∝ acceleration fluctuation (implies an entropic origin?)

Redshift dependence of acceleration fluctuation a_0

How to compute the angle of incidence?

$$m_h = \frac{4}{3}\pi r_h^3 \Delta_c \bar{\rho} \Rightarrow v_{cir} = \frac{G m_h}{r_h} = H r_h \sqrt{\frac{\Delta_c}{2}} = 3\pi u_r$$

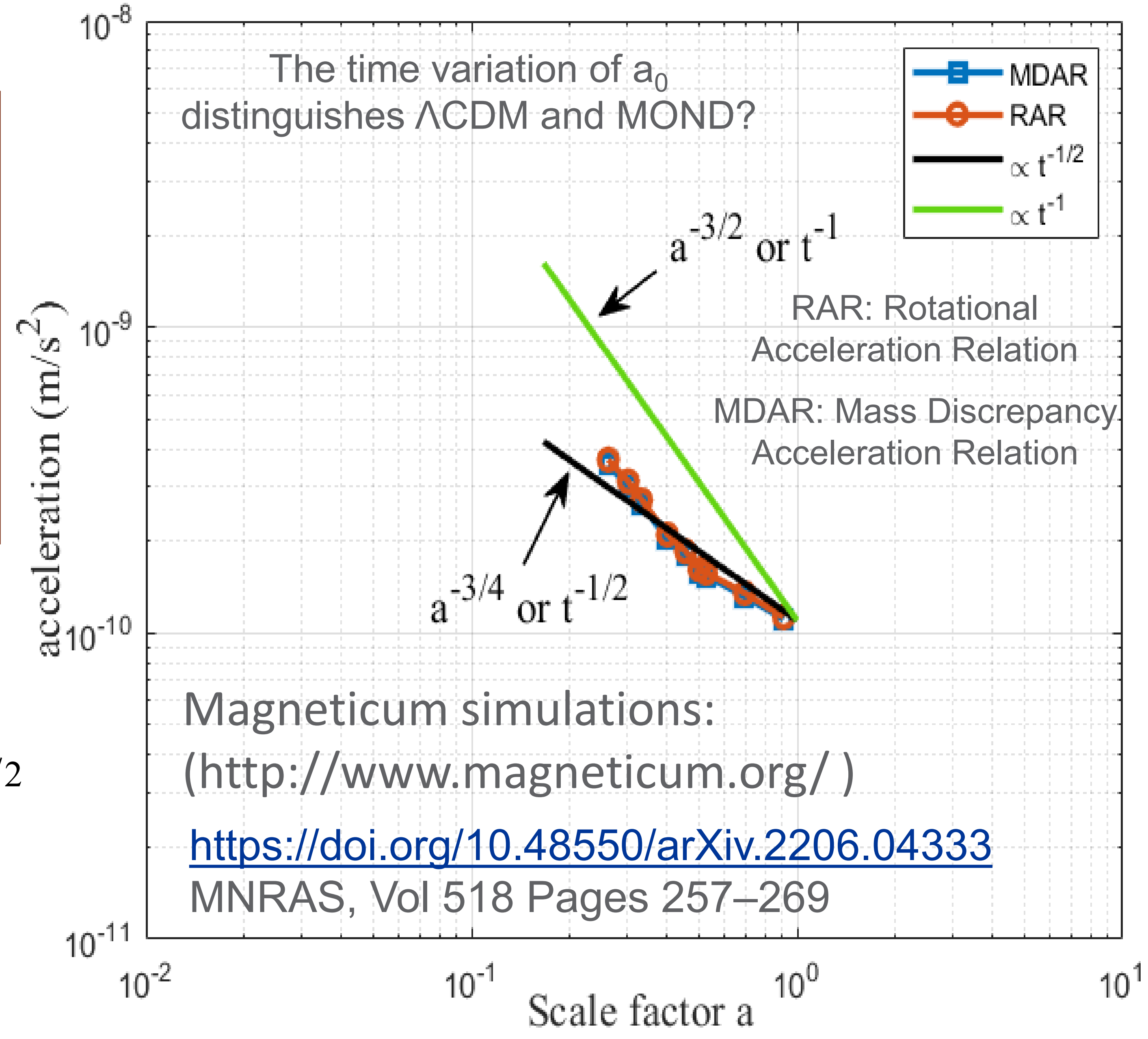
Critical density ratio: $\Delta_c = 2/(\beta_{s2})^2 = 18\pi^2$

$$\cot(\theta_{ur}) = \frac{u_r}{v_{cir}} = \beta_{s2} = \frac{1}{3\pi}$$

Finally, our Model predicts:

$$a_0(a) = -\frac{\Delta_c}{2}\cdot\frac{\varepsilon_u}{u} = -(3\pi)^2\frac{\varepsilon_u}{u} \propto a^{-3/4} \propto t^{-1/2}$$

Agree with hydrodynamic simulations



Magneticum simulations: (http://www.magneticum.org/)

https://doi.org/10.48550/arXiv.2206.04333

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The origin of deep MOND behavior?

- Fluctuation of acceleration introduces a scale of acceleration a_0
- Deep MOND for baryonic particles with acceleration $a_p \ll a_0$
- Consider baryonic mass in a one-dimensional dark matter fluid with a velocity fluctuation v_0 and acceleration fluctuation a_0 (Similar to Brownian motion)

$$\frac{1}{2}\frac{dv_p^2}{dt} = v_p \frac{dv_p}{dt} = a_p v_p = a_0 v_0 = -\varepsilon_u$$

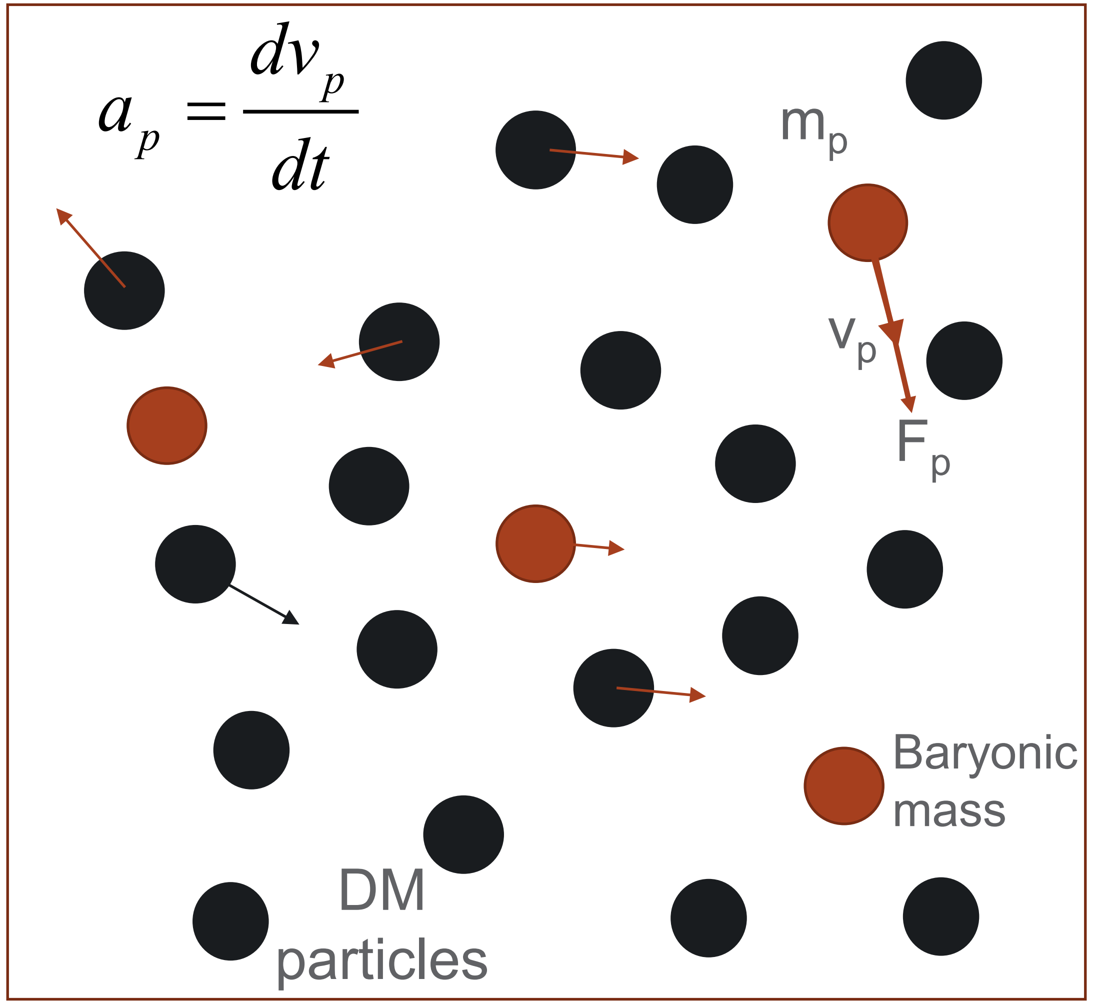
Constant rate of Energy cascade

$$\varepsilon_K(v) = v_0 v_p$$

Maximum entropy distribution: particle kinetic energy ε_k is proportional to velocity when $a_p \ll a_0$ (deep-MOND)

Power (Joule/second) of baryonic mass:

$$F_p v_p = m_p \frac{d\varepsilon_K}{dt} \quad \Rightarrow \quad F_p = m_p \frac{v_0}{v_p} a_p = m_p \frac{a_p^2}{a_0} \propto a_p^2$$



Baryonic mass immersed in DM fluid subject to external force F_p (two miscible phases)