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Cascade Theory for Turbulence, Dark Matter and SMBH Evolution

Dec 2022

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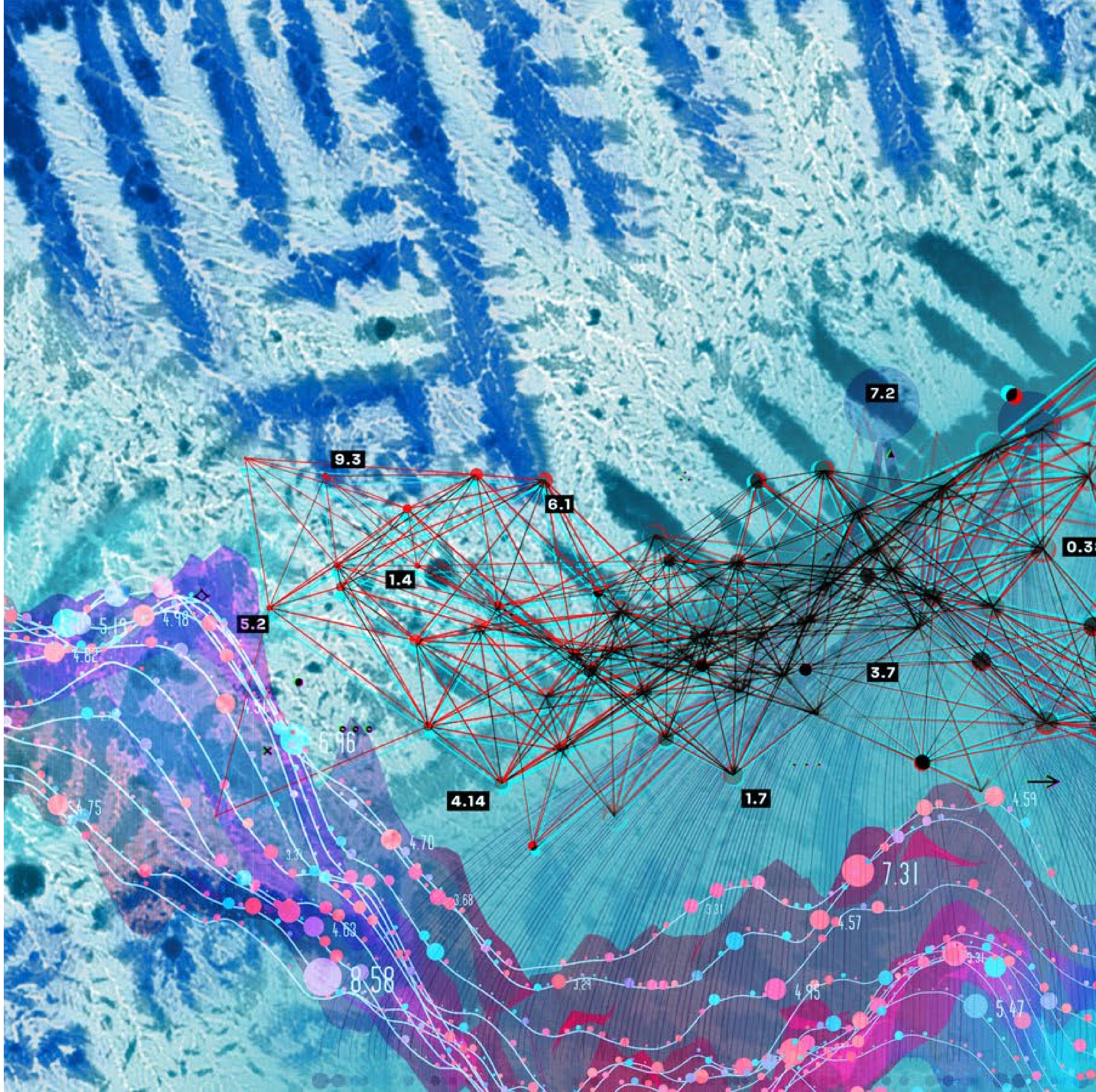

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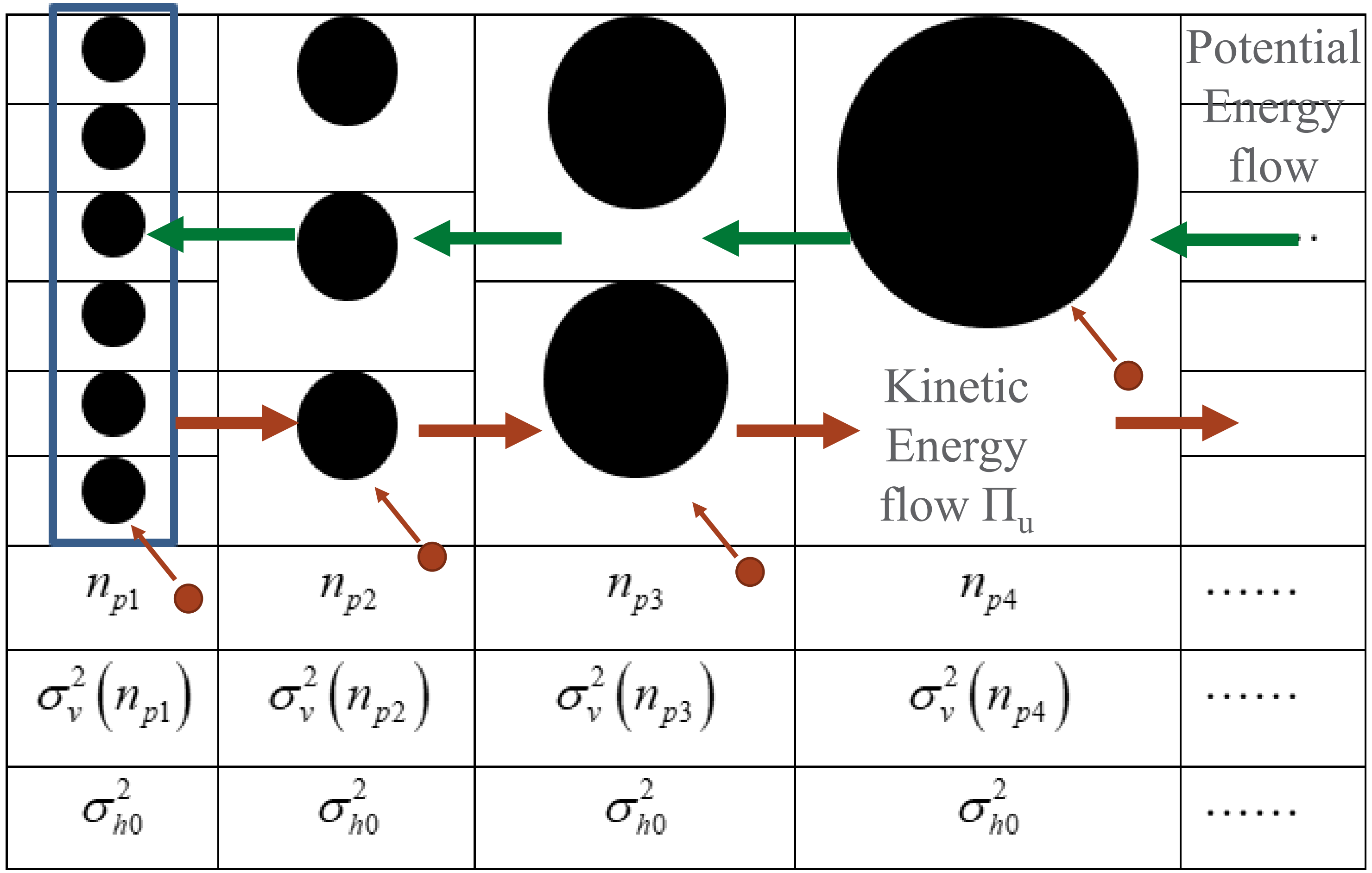
Outline

- Introduction
- Turbulence vs. the flow of dark matter: similarities and differences?
- Inverse mass cascade in dark matter flow
  - Random walk of halos in mass space and halo mass function
  - Random walk of dark matter in real space and halo density profile
- Energy cascade in dark matter flow
  - Universal scaling laws from N-body simulations and rotation curves
  - Dark matter properties from energy cascade
  - Uncertainty principle for energy cascade?
  - Extending to self-interacting dark matter
- Velocity/density correlation/moment functions
- Maximum entropy distributions for dark matter
- Energy cascade for the origin of MOND acceleration
- Energy cascade for the baryonic-to-halo mass relation
- Energy cascade for SMBH-bulge coevolution

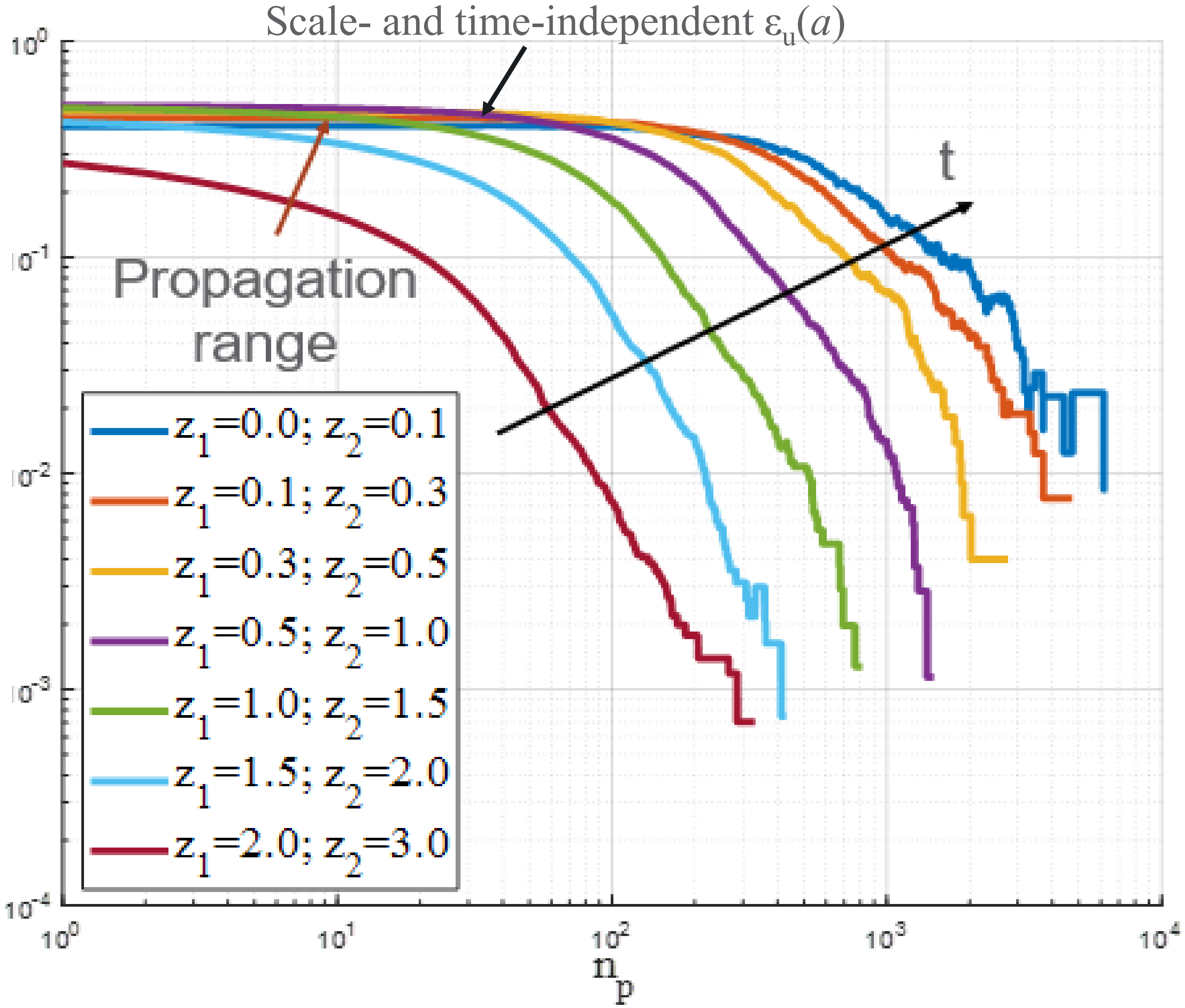
Relevant datasets are available at:

"A comparative study of dark matter flow & hydrodynamic turbulence and its applications"
http://dx.doi.org/10.5281/zenodo.6569901

Energy cascade in dark matter flow



- Identify all halos of different sizes
- Group halos according to the halo size n_p
- Kinetic energy flows from small to large mass scale through the merging with "single merger" (inverse cascade)
- Potential energy flows from large to small scales (direct cascade)



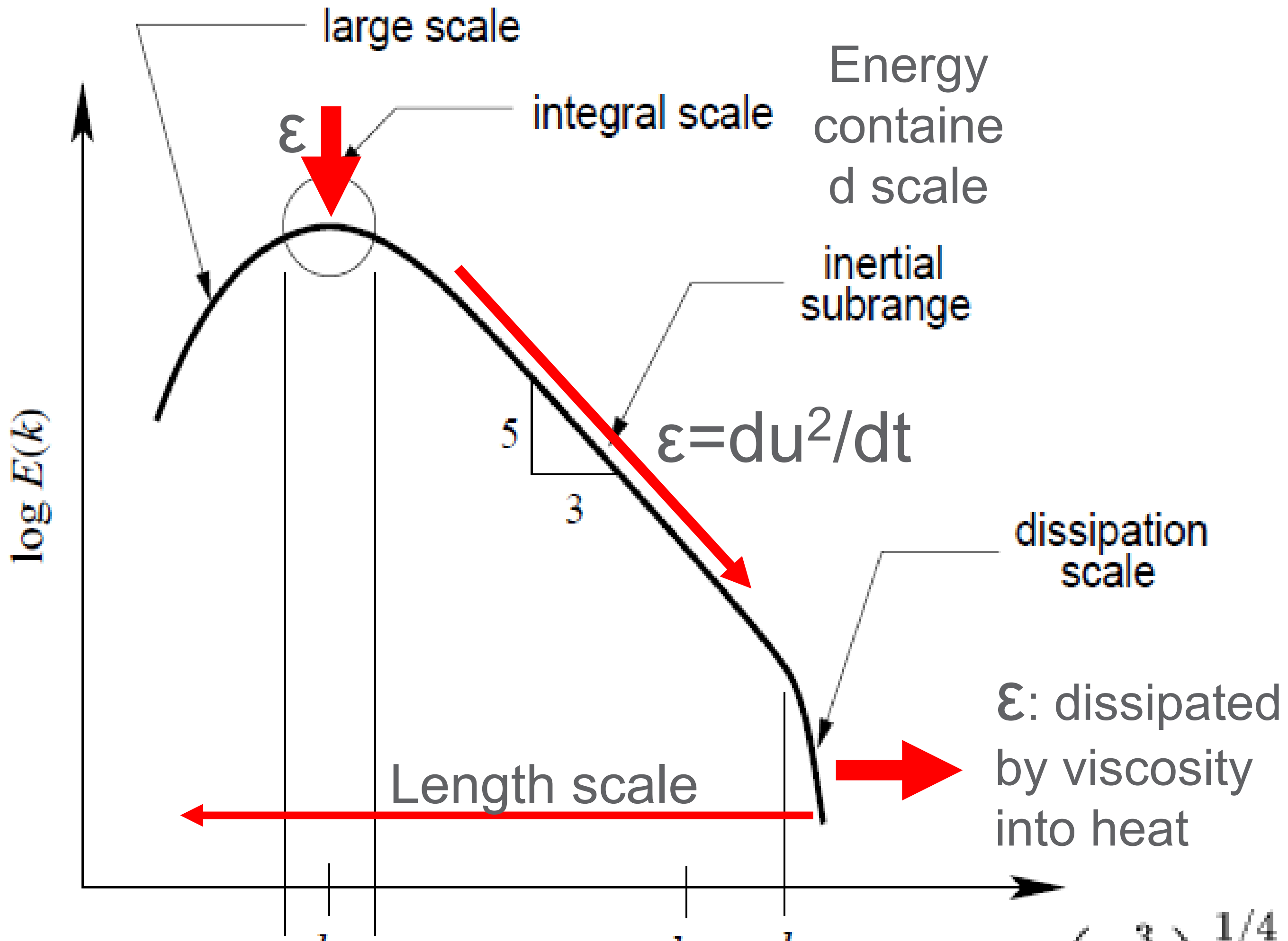
Rate of kinetic energy flux function $\pi_u(m_h,a)$

$$\Pi_u(m_h,a) = -\frac{\partial}{\partial t}\left[M_h(a)\int_{m_h}^{\infty} f_M(m,m_h^*)\sigma^2(m,a)dm\right]$$

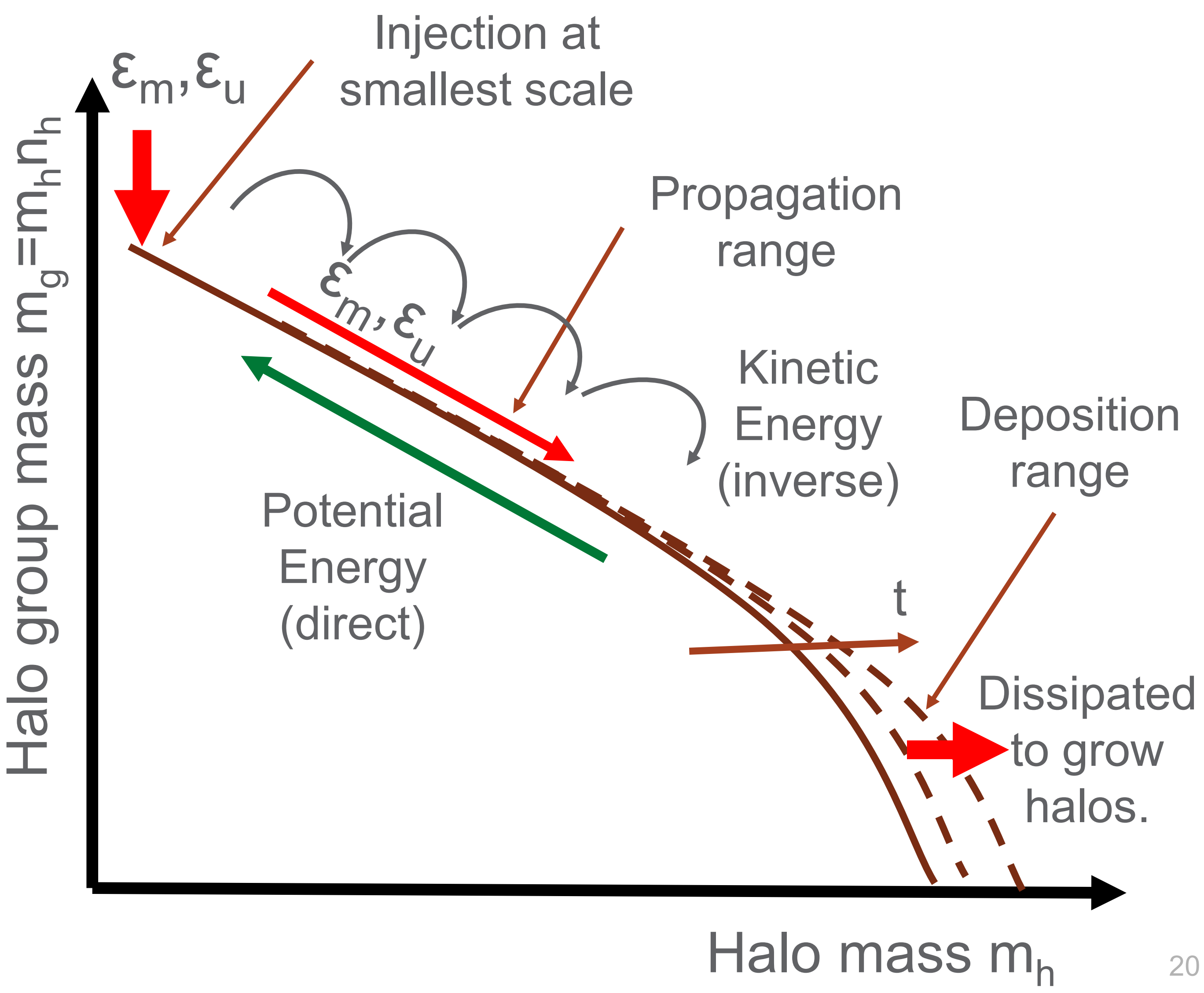
Energy cascade in turbulence and dark matter

Big whirls have little whirls, That feed on their velocity;
And little whirls have lesser whirls, And so on to viscosity.

Little halos have big halos, That feed on their mass;
And big halos have greater halos, And so on to growth.



$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$



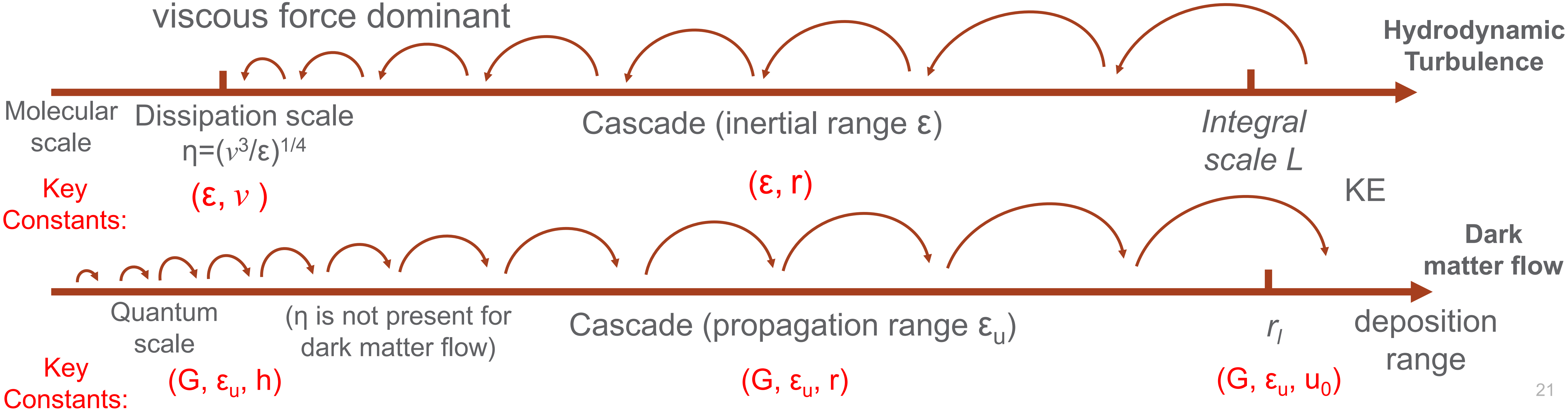
Energy cascade in turbulence and dark matter

Turbulence:

- Freely decaying (rate: ε)
- Direct energy cascade
- Vortex of different scales
- Integral scale: energy injection
- Inertial range:
  inertial >> viscous force
- Dissipation range:
  viscous force dominant

Dark matter flow:

- Freely growing (rate: ε_u): Virial theorem
- Inverse energy cascade
- Halos of different scales
- Collisionless, no dissipation range!
- The smallest length scale is not limited by viscosity.



Constant rate of energy cascade from N-body sim.

$$\frac{\partial E_y}{\partial t} + H\left(2K_p + P_y\right) = 0$$

Cosmic energy Equation (Irvine 1961)

$$K_p = -\varepsilon_{\mathbf{u}} t$$

Power-law for Peculiar kinetic energy K_p

$$P_y = \frac{7}{5}\varepsilon_{\mathbf{u}} t$$

Power-law for potential energy P_y

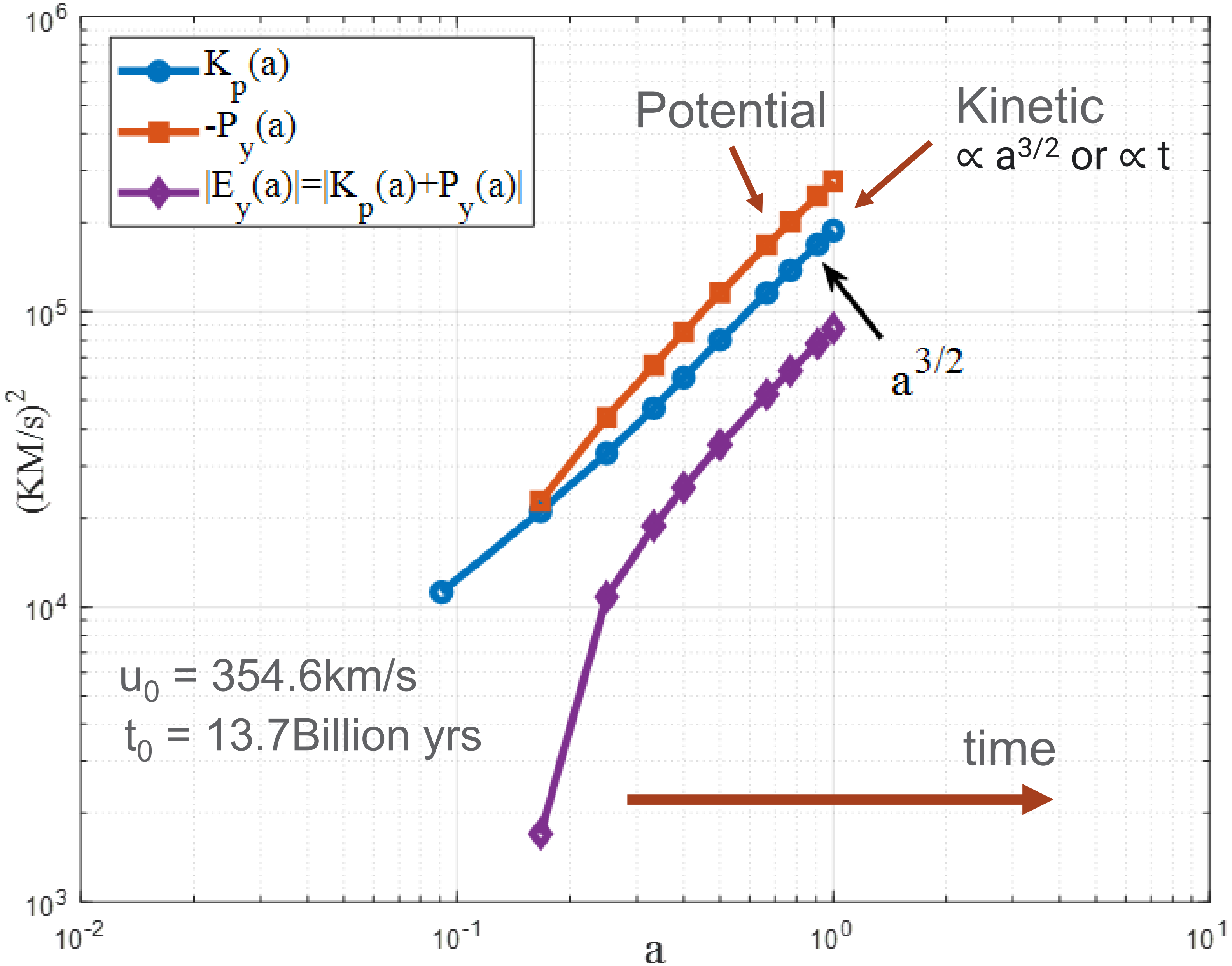
This rate ε_u is both time and scale independent, a fundamental constant!

From N-body simulation: (negative for inverse)

$$\varepsilon_u = -\frac{K_p}{t} = -\frac{3}{2}\frac{u_0^2}{t_0} \approx -4.6\times10^{-7}\frac{m^2}{s^3} < 0$$

In Earth's atmosphere: $\varepsilon \approx 10^{-3}\, m^2/s^3$

In Galaxy bulge: $\varepsilon_b \approx 10^{-4}\, m^2/s^3$



The time variation of specific kinetic and potential energies from *N*-body simulation.

Pair conservation equation for validation

Pair conservation equation (Peebles 1980) relates the pairwise velocity with density correlation ξ:

$$\frac{\langle \Delta u_L \rangle}{Har} = -\frac{\left(1+\bar{\xi}(r,a)\right)}{3\left(1+\xi(r,a)\right)} \frac{\partial \ln\left(1+\bar{\xi}(r,a)\right)}{\partial \ln a}$$

For large scale in linear regime, average correlation

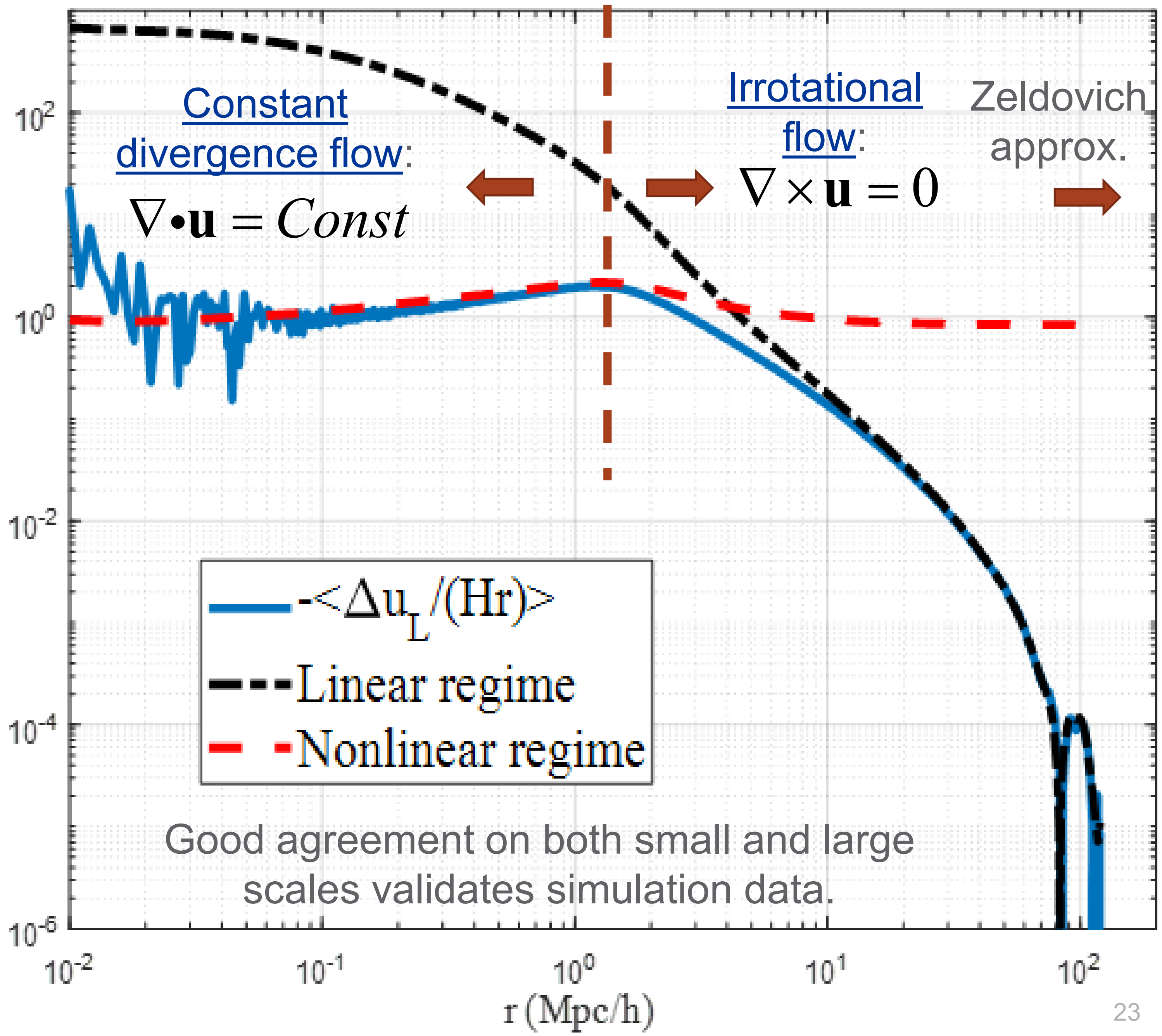
$\bar{\xi} \ll 1$ and $\partial \ln \bar{\xi} / \partial \ln a = 2$

$$\frac{\langle \Delta u_L \rangle}{Har} = -\frac{2\bar{\xi}(r,a)\left(1+\bar{\xi}(r,a)\right)}{3\left(1+\xi(r,a)\right)} \approx -\frac{2}{3}\bar{\xi}(r,a)$$

For small scale in non-linear regime (red dash),

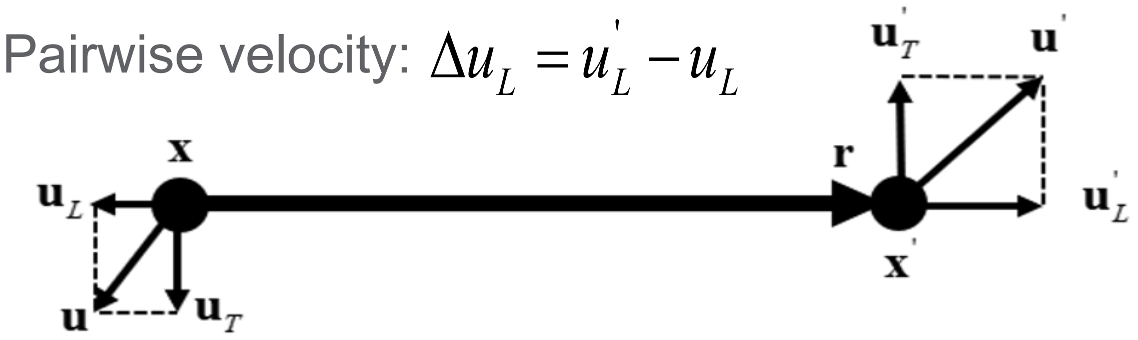
$\xi(r,a) \propto a^{\alpha} r^{\gamma}$ and $\partial \ln \bar{\xi} / \partial \ln a = \alpha$

Stable clustering hypothesis $\frac{\langle \Delta u_L \rangle}{Har} = -1$ → $\alpha = \gamma + 3$



2/3 law for kinetic energy confirmed by N-body sim.

Pairwise velocity: $\Delta u_L = u_L^{'} - u_L$



$$S_2^{lp}(r,a) = \left\langle (\Delta u_L)^2 \right\rangle = \left\langle (u_L^{'} - u_L)^2 \right\rangle$$

Pairwise velocity dispersion (represents the kinetic energy on scale r):

$$S_2^{lp}(r) - 2u^2 = S_{2r}^{lp} = v_r^2 \propto (\varepsilon_u r)^{2/3}$$

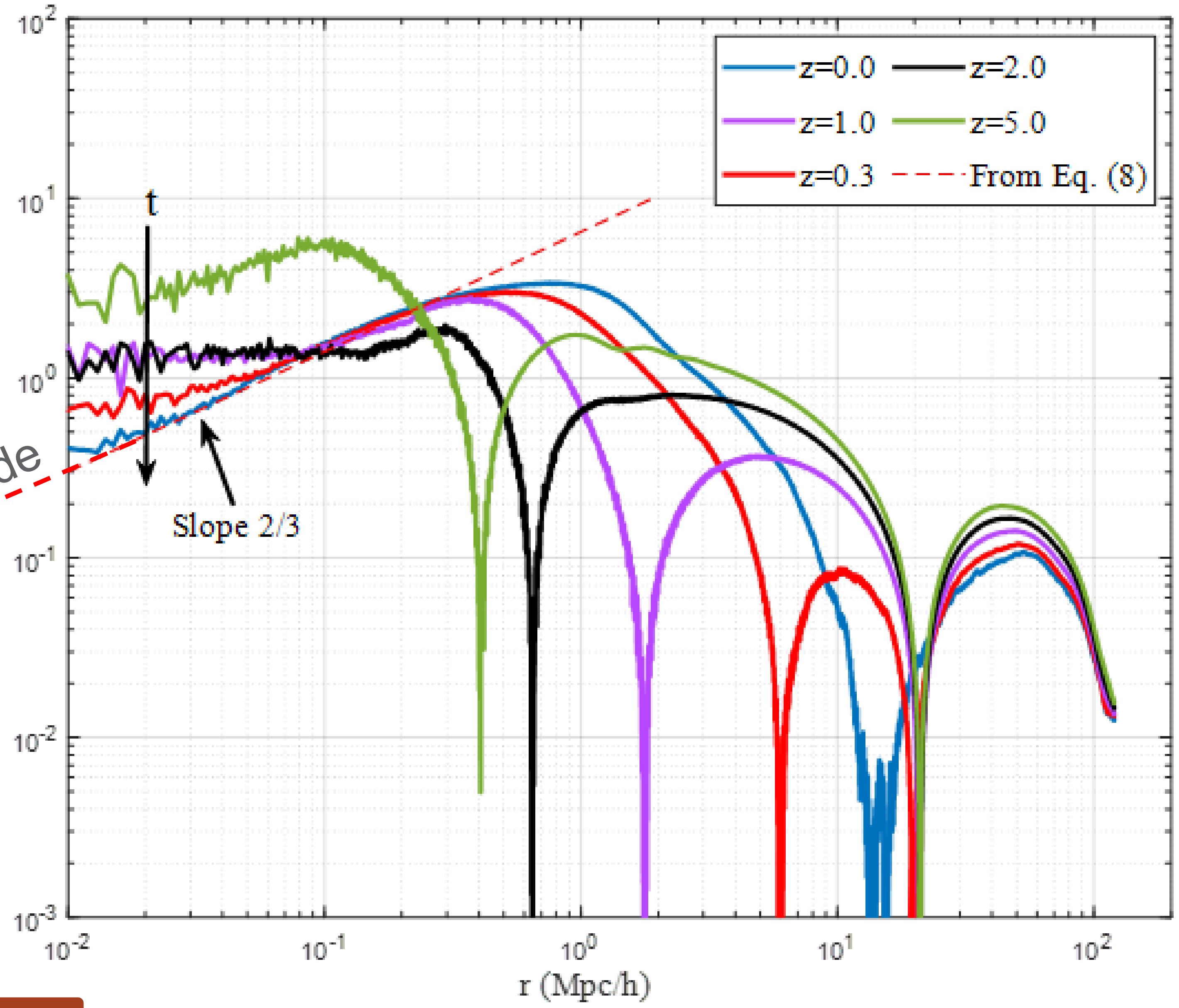
Kinetic Energy

$$(-\varepsilon_u) \propto \frac{v_r^2}{r} v_r = \frac{v_r^2}{r/v_r} = \frac{v_r^3}{r}$$

Acceleration Turnaround time

Cascade

Due to collisionless: Extend all the way to the smallest scale for dark matter properties



Variation of normalized reduced pairwise dispersion and two-thirds law

On scale r, kinetic energy follows a 2/3 law !

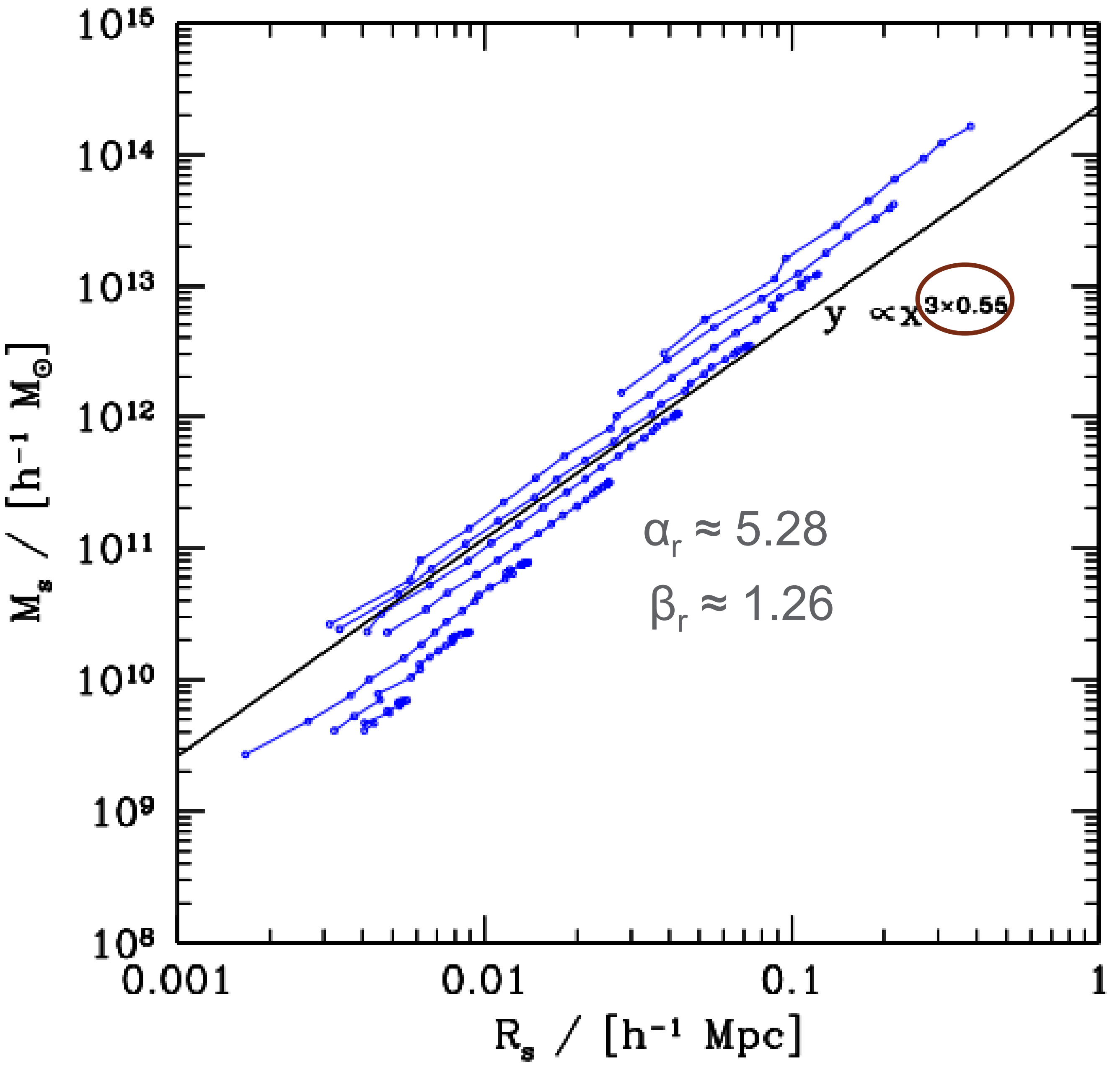
5/3 law for halo mass confirmed by N-body sim.

In propagation range, all relevant quantities are determined by G, ε_u, and scale *r*. This predicts:

Mass:	$m_r = \alpha_r \varepsilon_u^{2/3} G^{-1} r^{5/3}$	5/3 law
Density:	$\rho_r = \beta_r \varepsilon_u^{2/3} G^{-1} r^{-4/3}$	-4/3 law
Kinetic energy:	$v_r^2 = \left(\gamma_s \varepsilon_u\right)^{2/3} r^{2/3}$	2/3 law
Time:	$t_r \propto \varepsilon_u^{-1/3} r^{2/3}$	

Halo mass m_r enclosed in scale *r* can be obtained from N-body simulations

5/3 law confirmed by N-body simulations



Variation of halo core mass m_r with scale radius r_s follows a 5/3 law (Zhao et al. 2009)

-4/3 law for halo density confirmed by rotation curves

In propagation range, all relevant quantities are determined by G, ε_u, and scale *r*. This predicts:

Mass:	$m_r = \alpha_r \varepsilon_u^{2/3} G^{-1} r^{5/3}$	5/3 law
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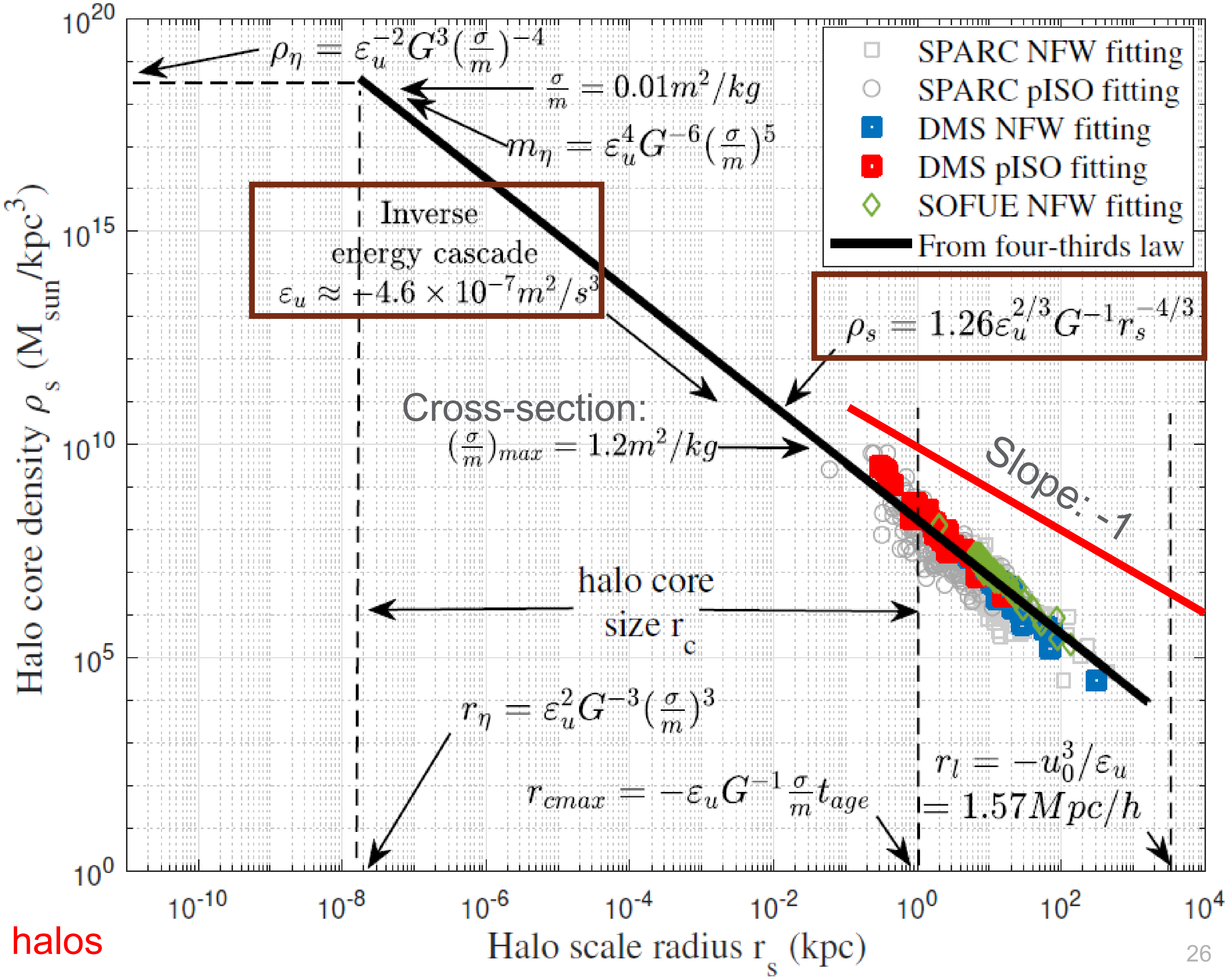
Time:	$t_r \propto \varepsilon_u^{-1/3} r^{2/3}$	

Halo core density ρ_s and scale r_s radius can be obtained from galaxy rotation curves



-4/3 law confirmed by rotation curves

Cuspy density for fully virialized collisionless DM halos

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Quick Recap II

In dark matter flow (DMF):

- Inverse cascade of kinetic energy from small to large scales (constant rate: ε_u m^2/s^3)
- Direct cascade of potential energy from large to small scales
- Two cascade connected by virial theorem

On any scale *r*, energy cascade predicts scaling laws on small scale:
(confirmed by N-body simulations and galaxy rotation curves)

Mass:	$m_r = \alpha_r \varepsilon_u^{2/3} G^{-1} r^{5/3}$	5/3 law
Density:	$\rho_r = \beta_r \varepsilon_u^{2/3} G^{-1} r^{-4/3}$	-4/3 law
Kinetic energy:	$v_r^2 = \left(\gamma_s \varepsilon_u\right)^{2/3} r^{2/3}$	2/3 law
Time:	$t_r \propto \varepsilon_u^{-1/3} r^{2/3}$	

Extend to the smallest scale for collisionless DM

Two hypothesis:

- Dark matter is fully collisionless
- Gravity is the only interaction

On the smallest scale:

$$m_X v_X \cdot l_X/2 = \hbar$$ Uncertainty principle

$$v_X^2 = G m_X / l_X$$ Virial theorem

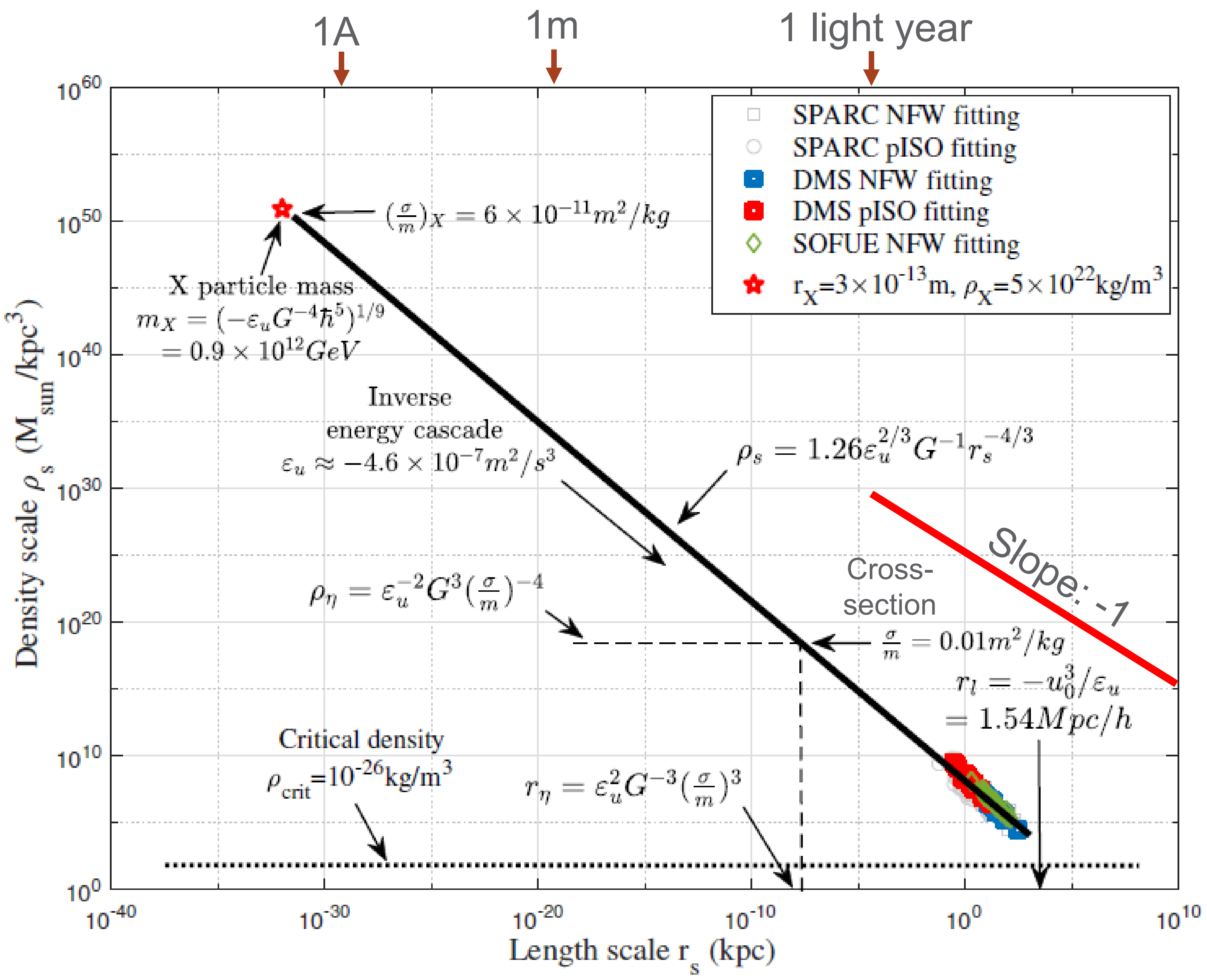
$$(-\varepsilon_u) = v_X^3 / l_X$$ Constant energy cascade

Energy cascade in DMF predicts:

Mass scale: $m_X \propto \left(-\varepsilon_u \hbar^5 / G^4\right)^{\frac{1}{9}} \approx 10^{12}\, GeV$

Length scale: $l_X \propto \left(-G\hbar/\varepsilon_u\right)^{\frac{1}{3}} \approx 10^{-13}\, m$

Time scale: $t_X \propto \left(G^2\hbar^2/\varepsilon_u^5\right)^{\frac{1}{9}} \approx 10^{-7}\, s$



Dark matter particle mass, size, and properties

Density scale: $\rho = m_X / l_X^3 \approx 5.33\times10^{22}\ kg/m^3$ ⟷ Nuclear density: $10^{17}\ kg/m^3$

Power scale (Joule/s): $\mu_X = m_X a_X \cdot v_X = -m_X \varepsilon_u = 7.44\times10^{-22}\ kg\cdot m^2/s^3 = 0.0046\, eV/s$

Energy scale: $\mu_X t_X / 4 = \hbar / t_X = \frac{1}{2} m_X v_X^2 = \boxed{0.87\times10^{-9}\, eV}$ ⟷ Rydberg energy of 13.6 eV for the ionization energy of the hydrogen atom

Particle lifetime: $\tau_X = \frac{m_X c^2}{\mu_X} = \boxed{-\frac{c^2}{\varepsilon_u}} = 6.2\times10^{15}\ yr$

If $\tau_X > 13.7\times10^9\ yr$ ➡ $\varepsilon_u < 0.21 m^2/s^3$

If instantons are responsible for the decay [1]:

$$\tau_X = \frac{\hbar e^{1/\alpha_X}}{m_X c^2} = 6.2\times10^{15}\ yr \Rightarrow \alpha_X \approx \frac{1}{136.85}$$

Dynamic viscosity: $\eta = -\varepsilon_u / G \approx 6900 Pa\cdot s$ Peanut Butter?

Pressure scale: Number density

$$P_X = \frac{m_X a_X}{l_X^2} = \frac{8\hbar^2}{m_X}\rho_{nX}^{5/3} = 1.84\times10^{10}\ Pa$$

analogue of the degeneracy pressure of Fermi gas

Cross section: $l_X^2 v_X = 4\times10^{-32}\ m^3 s^{-1}$

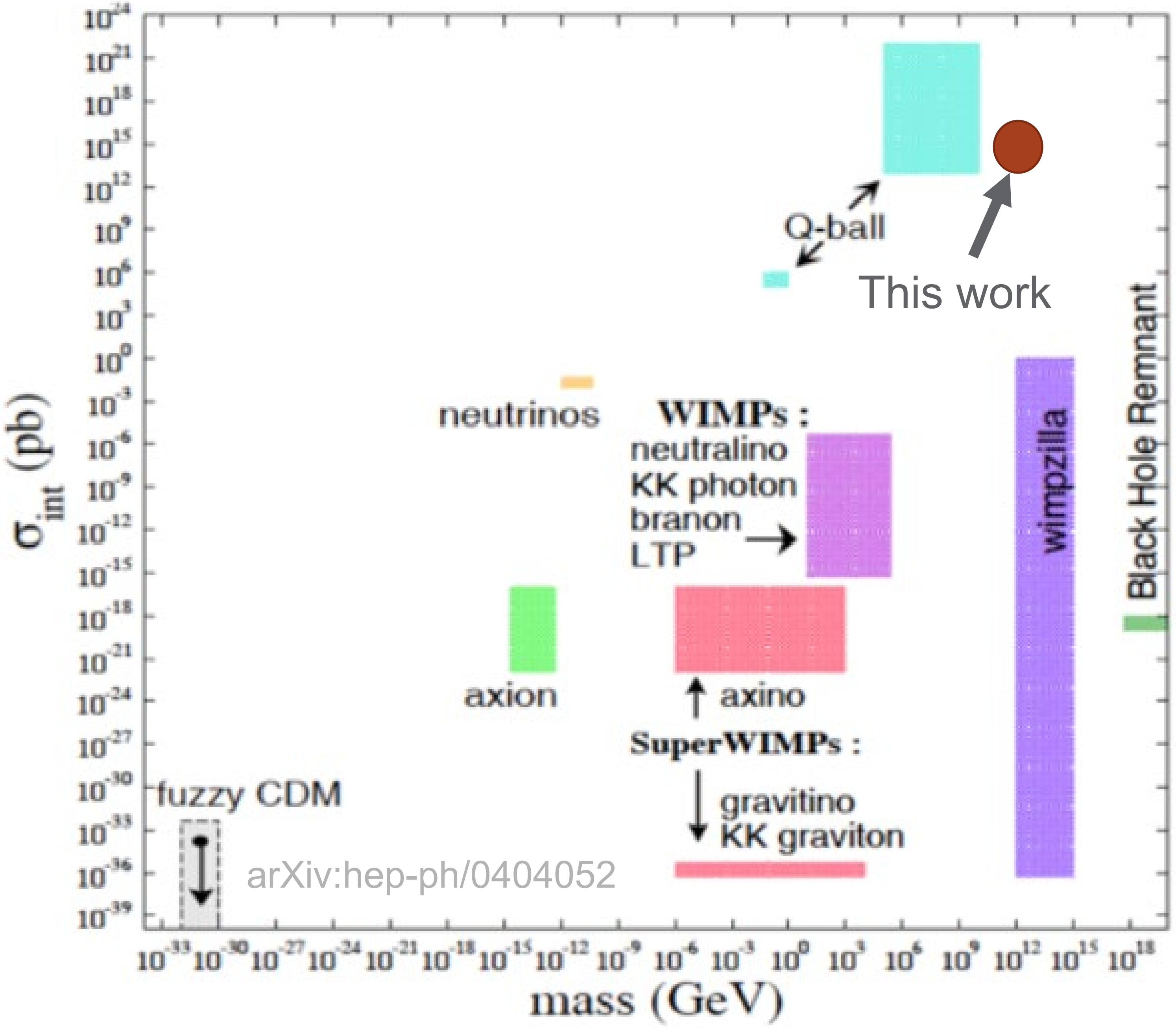
⇕

WIMP miracle: $\langle \sigma v \rangle = 3\times10^{-32}\ m^3 s^{-1}$

Kinematic viscosity for momentum transfer (collisionless): $\upsilon = \eta/\rho \approx 1.3\times10^{-19}\ m^2/s$

[1] Anchordoqui, L.A., et al.,Astroparticle Physics, 2021. 132.

Where is our prediction?



From this prediction:

- Much heavier than WIMP
- Much heavier than axion
- Comparable to Wimpzilla

Two hypothesis:

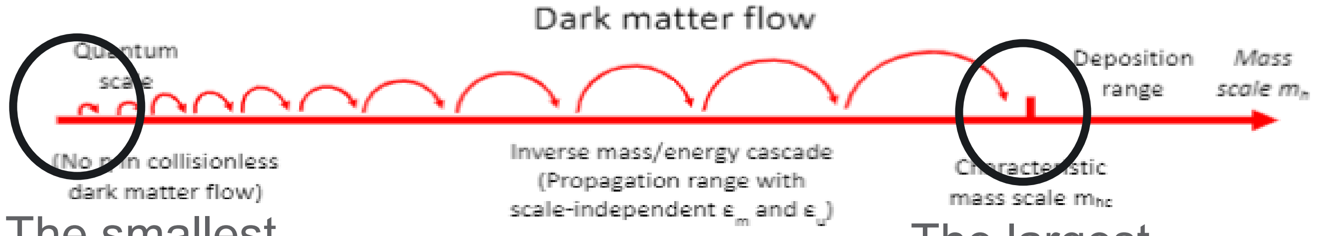
- DM is fully collisionless
- Gravity is the only interaction

If cannot detect DM at mass of 10^{12}Gev, then

- DM is self-interacting?
- Involve unknown forces?
- How to be consistent with cascade theory?

- Potential flaws in this argument?
- Any impacts on the detection methods?

Critical scales in collisionless dark matter flow



The smallest scale

The largest scale

On the smallest scale, three fundamental constants:

Gravitational constant	$G = 6.67\times10^{-11}\, m^3/\left(kg\cdot s^2\right)$
Rate of energy cascade	$\varepsilon_u = -4.6\times10^{-7}\, m^2/s^3$
Planck constant	$\hbar = 1.05\times10^{-34}\, kg\cdot m^2/s$

Simple dimensional analysis predicts:

Mass scale: $m_X \propto \left(-\varepsilon_u \hbar^5/G^4\right)^{\frac{1}{9}} \approx 10^{12}\, GeV$

Length scale: $l_X \propto \left(-G\hbar/\varepsilon_u\right)^{\frac{1}{3}} \approx 10^{-13}\, m$

Time scale: $t_X \propto \left(G^2\hbar^2/\varepsilon_u^5\right)^{\frac{1}{9}} \approx 10^{-6}\, s$

Three fundamental constants on large scale:

Gravitational constant	$G = 6.67\times10^{-11}\, m^3/\left(kg\cdot s^2\right)$
Rate of energy cascade	$\varepsilon_u = -4.6\times10^{-7}\, m^2/s^3$
Velocity dispersion	$u_0 \equiv u\left(a=1\right) = 354.61\, km/s$

Simple dimensional analysis predicts:

Mass scale: $m_L \propto -u_0^5/\left(G\varepsilon_u\right) \approx 9.14\times10^{13}\, M_\odot$

Length scale: $l_L \propto -u_0^3/\varepsilon_u \approx 3.14\, Mpc$

Time scale: $t_L \propto u_0^2/\varepsilon_u \approx 8.7\times10^{9}\, yr$

Critical scales for self-interacting dark matter

On the smallest length scale:

$$\rho_r(\sigma/m)v_r t_r = 1$$ Elastic scatter

$$v_s^2 = Gm_r(r_s)/r_s$$ Virial theorem

$$-\varepsilon_u = v_s^3/\gamma_s r_s$$ Constant energy cascade

All relevant quantities determined by G, cross-section σ/m and ε_u :

Length or minimum halo core size: $$r_\eta = \varepsilon_u^2 G^{-3}(\sigma/m)^3$$

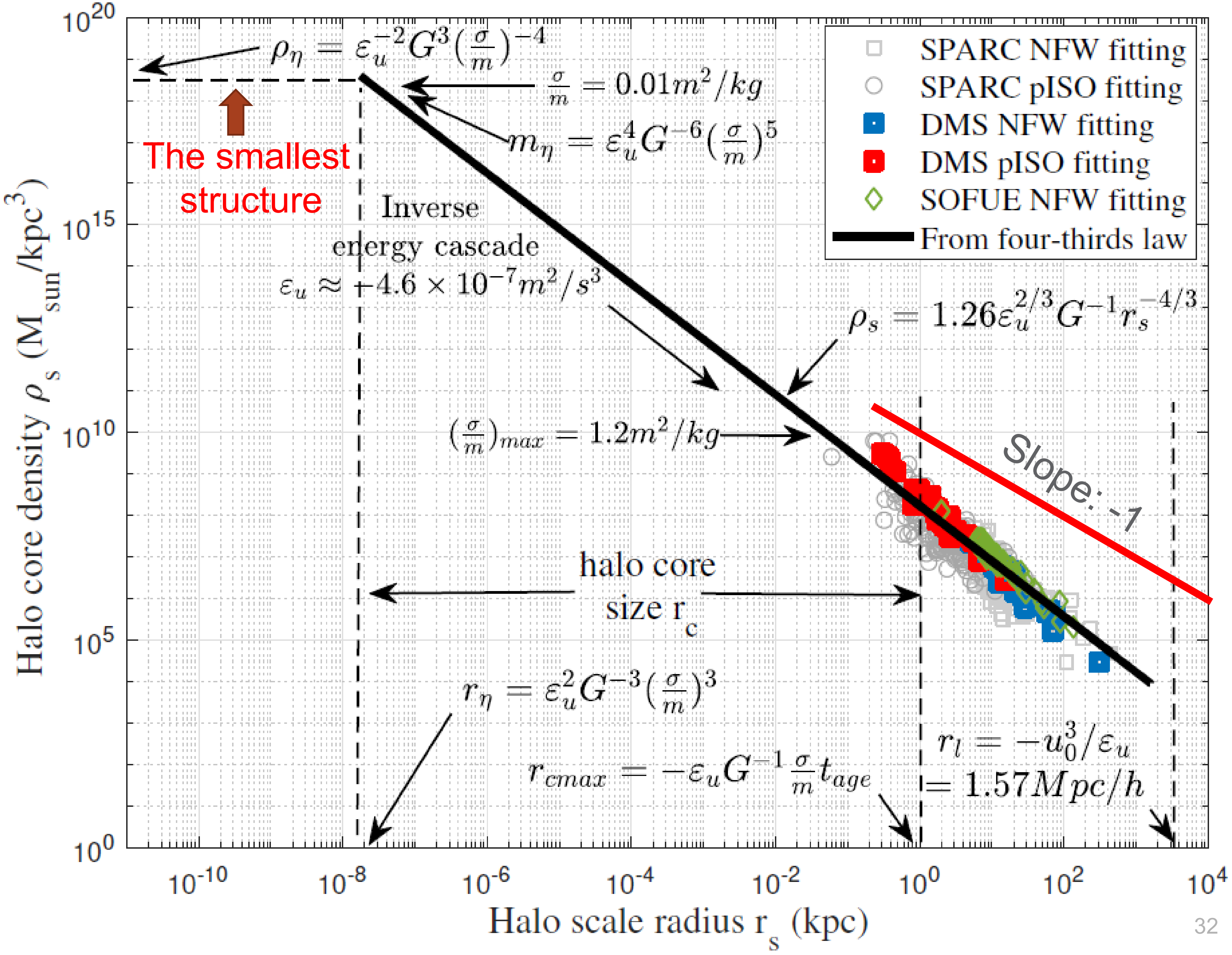
Mass scale: $$m_\eta = \varepsilon_u^4 G^{-6}(\sigma/m)^5$$

Density scale: $$\rho_\eta = \varepsilon_u^{-2} G^3(\sigma/m)^{-4}$$

Maximum halo core size r_{cmax} :

$$\rho_r \frac{\sigma}{m} v_r t_{age} = 1$$ t_{age}: age of Universe;

$$\frac{r_{c\max}}{(\sigma/m)} = -\varepsilon_u G^{-1} t_{age} \approx 10kpc\frac{g}{cm^2}$$



The origin of energy cascade: Uncertainty principle?

Position (**x**), Velocity (**v** = d**x**/dt), Acceleration (**a** = d**v**/dt)

For fully collisionless dark matter:

1) A unique "symmetry" between **x** and **v** in phase space:

- At given **x** , particles can have multiple **v** (multi-stream)
- For given **v,** particles can be at different **x**
- NOT possible for non-relativistic baryons

2) Due to the long-rang gravitational interaction,

- Fluctuations (uncertainty) in **x**
- Fluctuations (uncertainty) in **v**
- Fluctuations (uncertainty) in **a**

3) Two pairs of conjugate variables:

- Position **x** and momentum **p**
- Momentum **p** and acceleration **a**

Wave function for position: $\psi(x)$

Wave function for momentum: $\varphi(p)$

Wave function for acceleration: $\mu(a)$

$$\mu_X = -m_X \varepsilon_u = 7.44 \times 10^{-22} kg \cdot m^2/s^3$$

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \varphi(p) e^{ipx/\hbar} dp$$

$$\varphi(p) = \frac{1}{\sqrt{2\pi\mu_X}} \int_{-\infty}^{\infty} \mu(a) e^{ipa/\mu_X} da$$

Uncertainty principles: $\sigma_x \sigma_p \geq \hbar/2$ $\quad \sigma_p \sigma_a \geq \mu_X/2$

Postulated uncertainty principle for ***a*** and ***p*** leads to the constant rate of energy cascade:

$$\varepsilon_u = \mu_X/m_X = a_X v_X$$

Quick Recap III

- If DM is fully collisionless:
  - Scaling laws extended to the smallest scale (quantum)
  - Dark matter mass, size, density, pressure, lifetime, cross-section, etc.
  - The origin of cascade: uncertainty principle between momentum and acceleration?
- If DM is self-interacting:
  - The smallest scale determined by G, cross-section σ/m and ε_u
  - Smallest structure size (dependent on σ/m)
  - Maximum core size (dependent on σ/m)
  - Observational constraint for σ/m?

- Suggestions on the current work?
- Suggestions on the future work?
- Suggestion on the potential collaboration?
  - Hydrodynamic simulations?
  - Self-interaction DM simulations?
  - Code, data processing?