Cascade Theory for Turbulence, Dark Matter and SMBH Evolution

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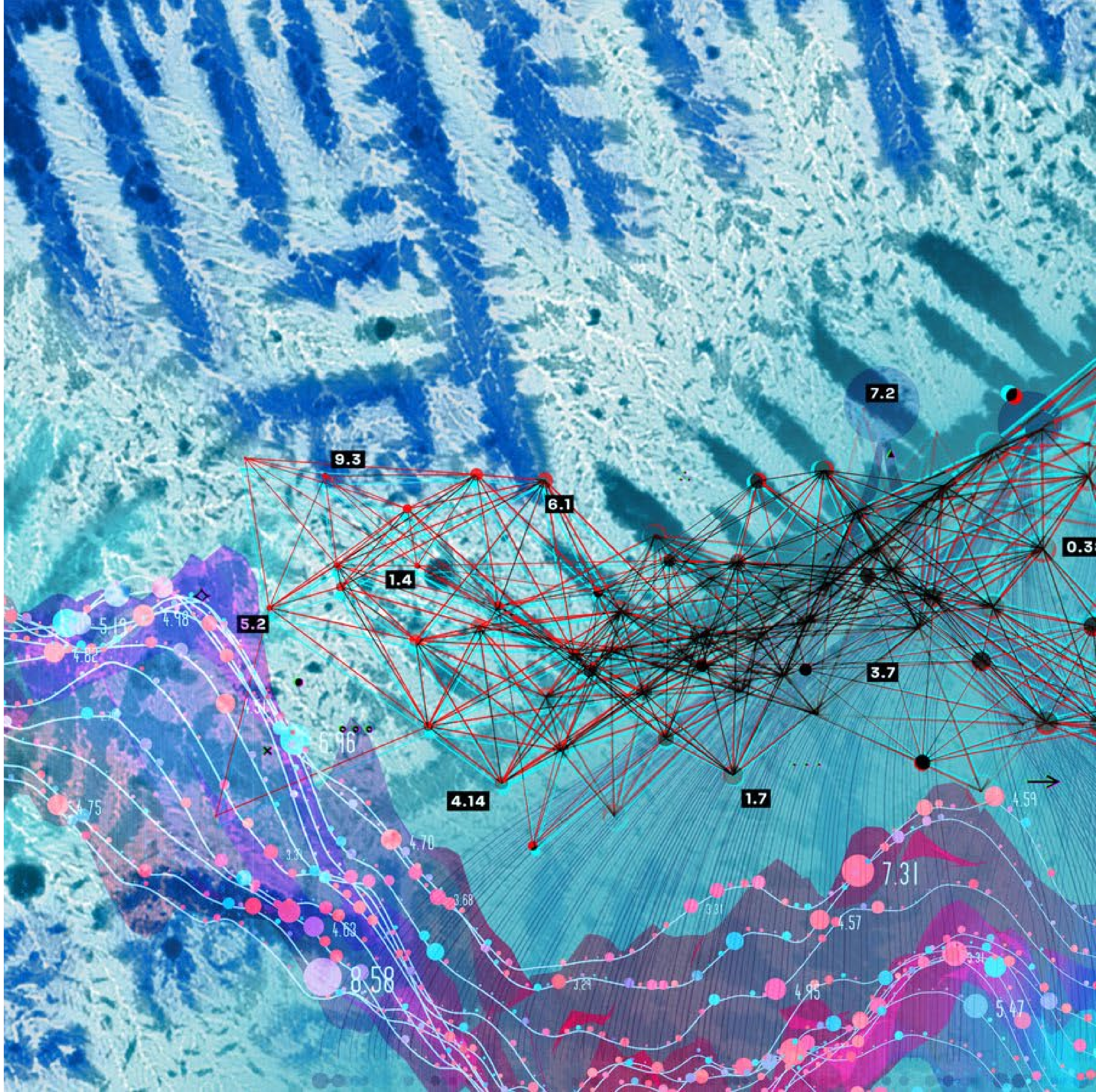


Outline

- Introduction
- Turbulence vs. the flow of dark matter: similarities and differences?
- Inverse mass cascade in dark matter flow
  - Random walk of halos in mass space and halo mass function
  - Random walk of dark matter in real space and halo density profile
- Energy cascade in dark matter flow
  - Universal scaling laws from N-body simulations and rotation curves
  - Dark matter properties from energy cascade
  - Uncertainty principle for energy cascade?
  - Extending to self-interacting dark matter
- Velocity/density correlation/moment functions
- Maximum entropy distributions for dark matter
- Energy cascade for the origin of MOND acceleration
- Energy cascade for the baryonic-to-halo mass relation
- Energy cascade for SMBH-bulge coevolution

Relevant datasets are available at:

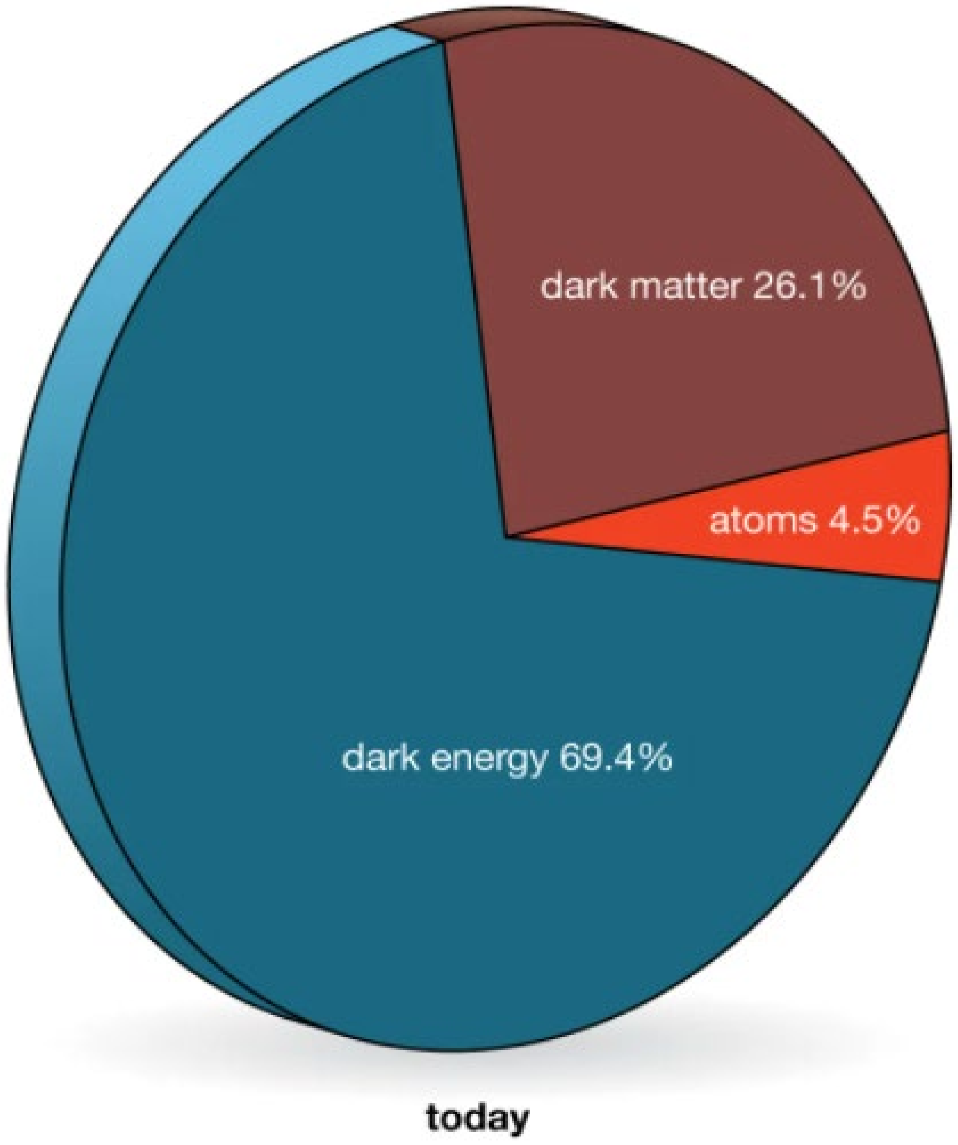
"A comparative study of dark matter flow & hydrodynamic turbulence and its applications"
http://dx.doi.org/10.5281/zenodo.6569901

Introduction

Content of universe

- Dark matter: 85% of the total matter.
- Dark matter flow (DMF): the widest presence in the universe.
- Hydrodynamic turbulence: the most familiar flow in our daily life.
- What are the similarities and differences?

During the pandemic, we find a time to think about and leverage this comparison for better understanding the nature of dark matter (DM) flow and DM properties.



What is dark matter?

No definite answer.

What it should not be?
- No electric charge
- No color charge (strong interactions)
- No strong self-interaction
- No fast decay: stable and long-lived
- Not any particles in standard model of particle physics

What it should be?
- Non-baryonic
- Cold (non-relativistic)
- Collisionless
- Dissipationless (optically dark)
- Sufficiently smooth with a fluid-like behavior

What is the nature of dark matter flow (DMF)? A special example of non-relativistic, self-gravitating, collisionless fluid dynamics (SG-CFD)

Might be a new opportunity for fluid dynamics contributing to the dark matter mystery, the biggest quest of contemporary astrophysics.

Brief timeline for dark matter research (~100 years)

Zwicky: Discovery of galaxy cluster velocity ~1000km/s

Rubin: Discovery of flat galaxy rotation curves

CMB fluctuations from COBE Confirms CDM prediction

ΛCDM as the standard cosmological model

1930s → **1960s** → **1970s** → **1980s** → **1992** → **1998** → **2000** → **2003-2008**

Discovery of the CMB

Cold Dark Matter (CDM) model proposed; MOND theory;

Evidence for Dark Energy and accelerating expansion: Type Ia supernova

WMAP and LSS data Confirm ΛCDM predictions

2030s ← **2022** ← **2013-2018**

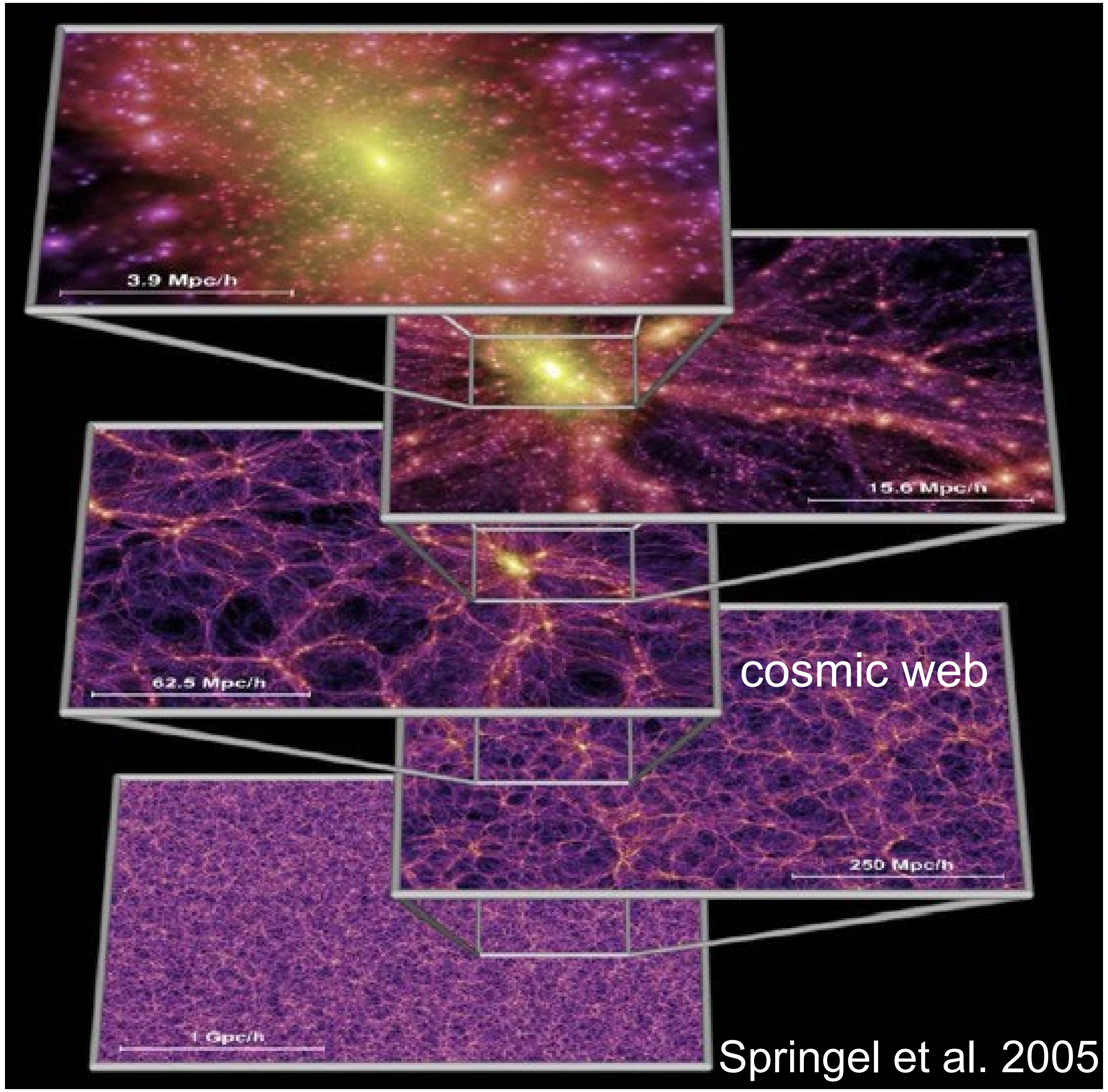
Discovery of dark matter particles??

James Webb Space Telescope

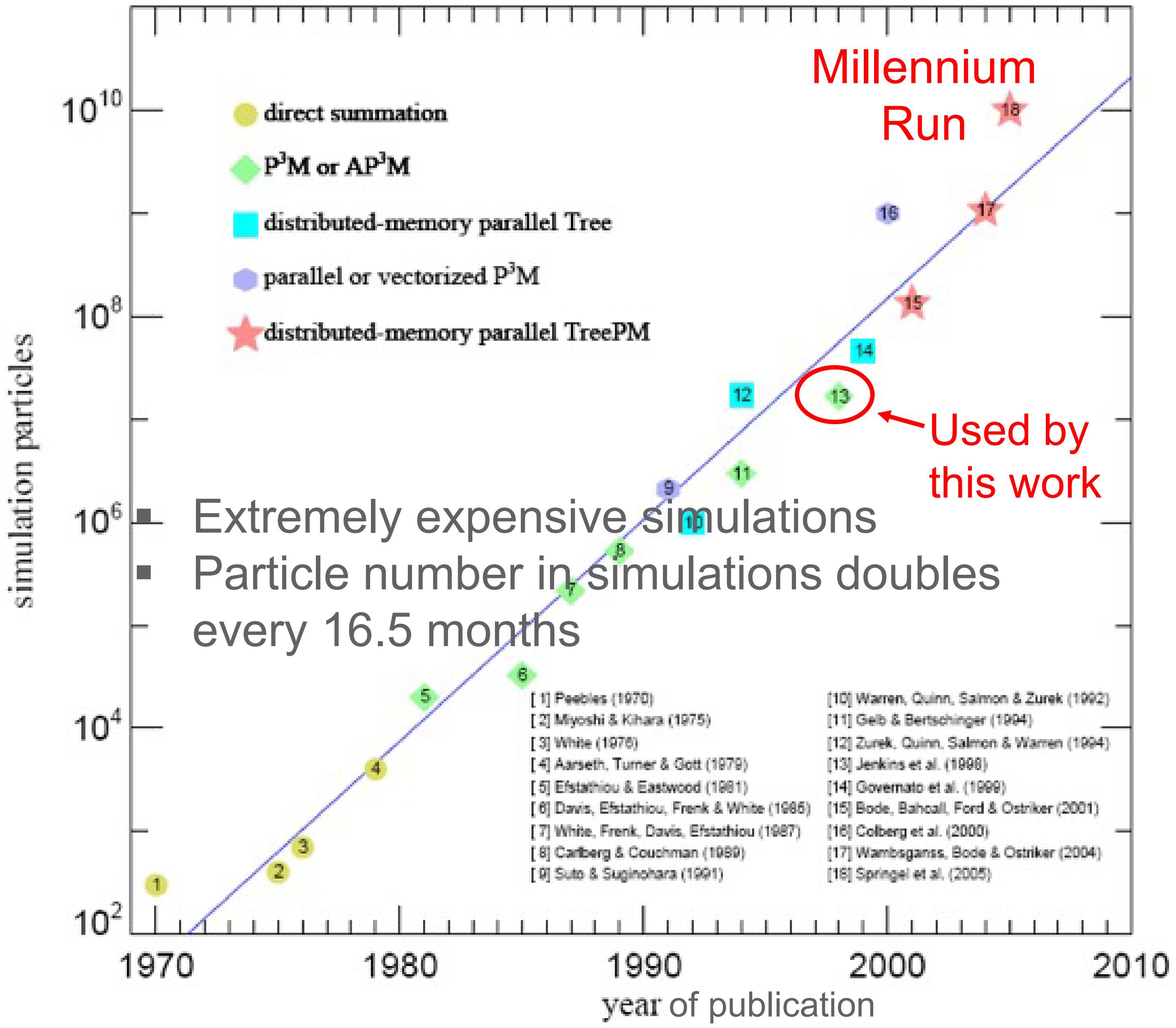
Planck data of CMB anisotropies Confirm ΛCDM predictions

COBE: COsmic Background Explorer (NASA)
WMAP: Wilkinson Microwave Anisotropy Probe (NASA)
LSS: Large Scale Structure (LSS) of the universe
CMB: Cosmic Microwave Background
ΛCDM: dark energy + cold dark matter (double dark)
Planck: European Space Agency (ESA)

Cosmological N-body simulations for the flow of DM



Springel et al. 2005



- Extremely expensive simulations
- Particle number in simulations doubles every 16.5 months

Dehnen and J. I. ReadThe European Physical Journal Plus, 126, 55

N-body simulations in this study

Run	Ω_0	Λ	h	Γ	σ_8	$L(Mpc/h)$	N_p	$m_p(M_\odot/h)$	$l_{soft}(Kpc/h)$
SCDM1	1.0	0.0	0.5	0.5	0.51	239.5	256^3	2.27×10^{11}	36

- N-body simulations carried out by the Virgo consortium. https://wwwmpa.mpa-garching.mpg.de/Virgo/data_download.html
- Standard CDM power spectrum (SCDM) with matter-dominant gravitational flow.
- Dark matter only simulations
- Similar analysis can be extended to other cosmological models and hydrodynamic simulations.
- All relevant datasets for this work are available at http://dx.doi.org/10.5281/zenodo.6569901

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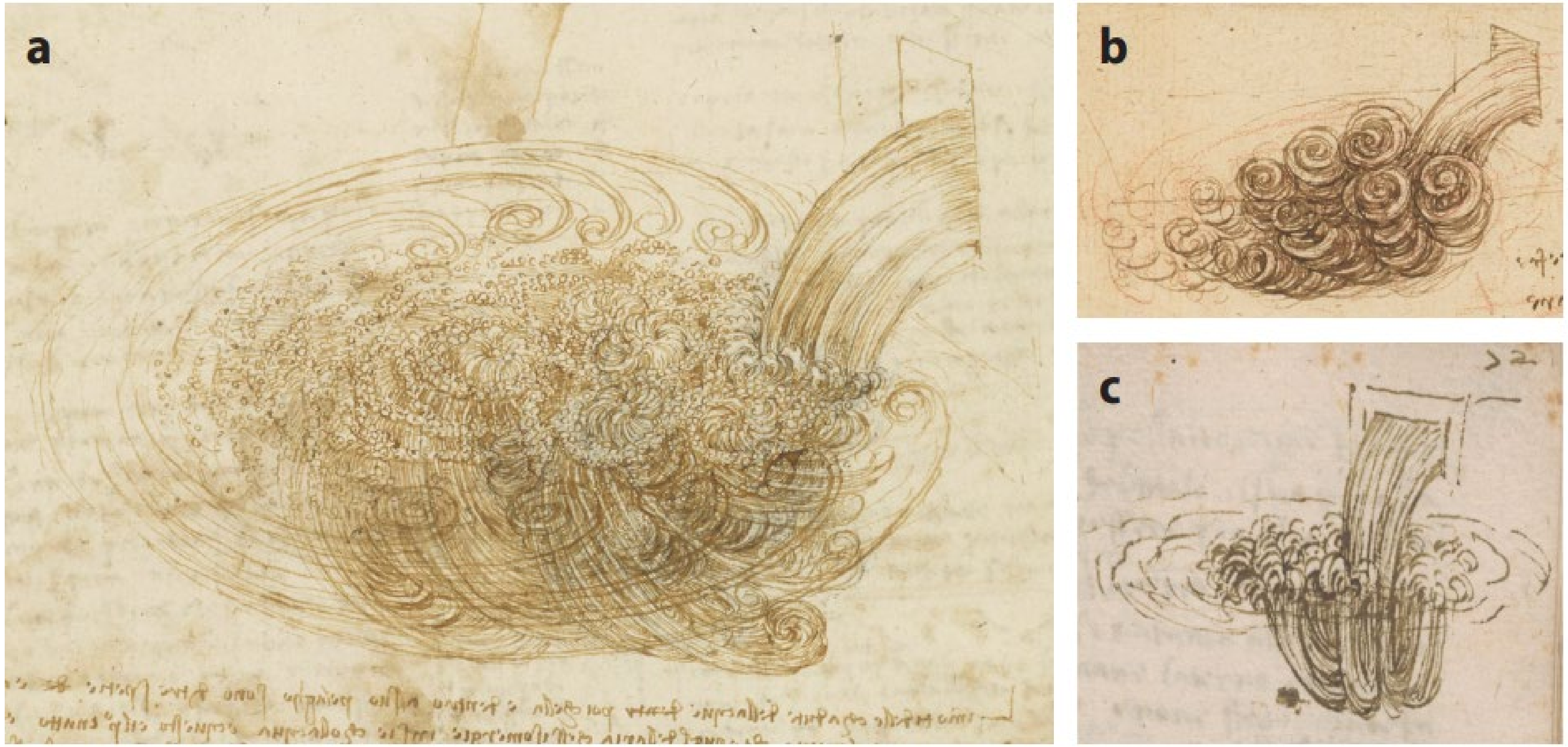
Brief timeline for turbulence research (~500 years)

- **1500s** — da Vinci: Earliest recognition of turbulence as a physical phenomenon
- **1877** — Boussinesq's concept of "eddy viscosity" replacing molecular viscosity
- **1894** — Reynolds number for transition from laminar to turbulence; Reynolds's decomposition (RANS) $v = \langle v \rangle + v'$ Reynolds stress;
- **1925** — Prandtl "mixing-length theory" for eddy viscosity Richardson "energy cascade"
- **1930s** — Taylor's statistical methods involving correlations power spectra; Karman and Howarth
- **1940s** — Kolmogorov: the "K41 theory" -5/3 law
- **1963** — Lorenz propose possible links between "deterministic chaos" and turbulence
- **1967** — Kraichnan's inverse cascade in 2-dimensional turbulence

RANS: Reynolds-averaged Navier-Stokes Equation;

What can we learn from turbulence?

da Vinci's sketch of turbulence (~1500 AD)



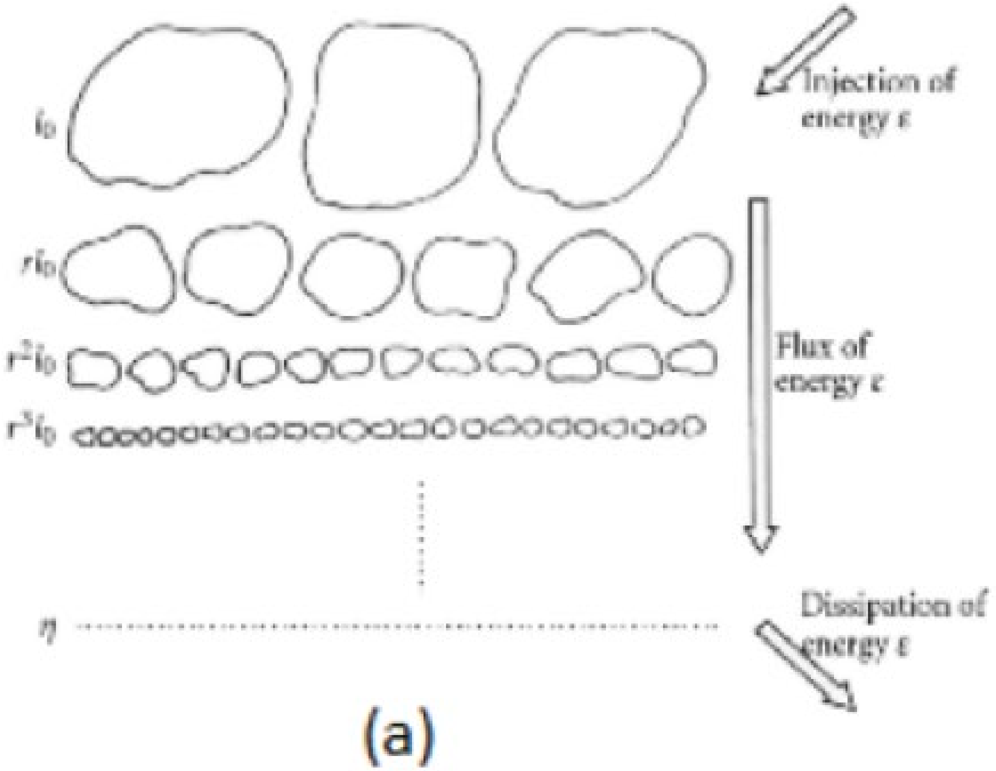
da Vinci sketch of turbulence: plunging water jet

- "turbolenza": the origin of modern word "turbulence"
- The pattern of flow with vortexes in fluid
- The random chaotic nature

". . . the smallest eddies are almost numberless, and large things are rotated only by large eddies and not by small ones, and small things are turned by small eddies and large."

Richardson's direct cascade (1922)

"Big whorls have little whorls, That feed on their velocity;
And little whorls have lesser whorls, And so on to viscosity."



Cascade: energy is injected on large scale, propagating across different scales, and dissipated on the smallest scale.

(a) : Cascade of energy, (b) : Lewis Richardson

Key attributes:

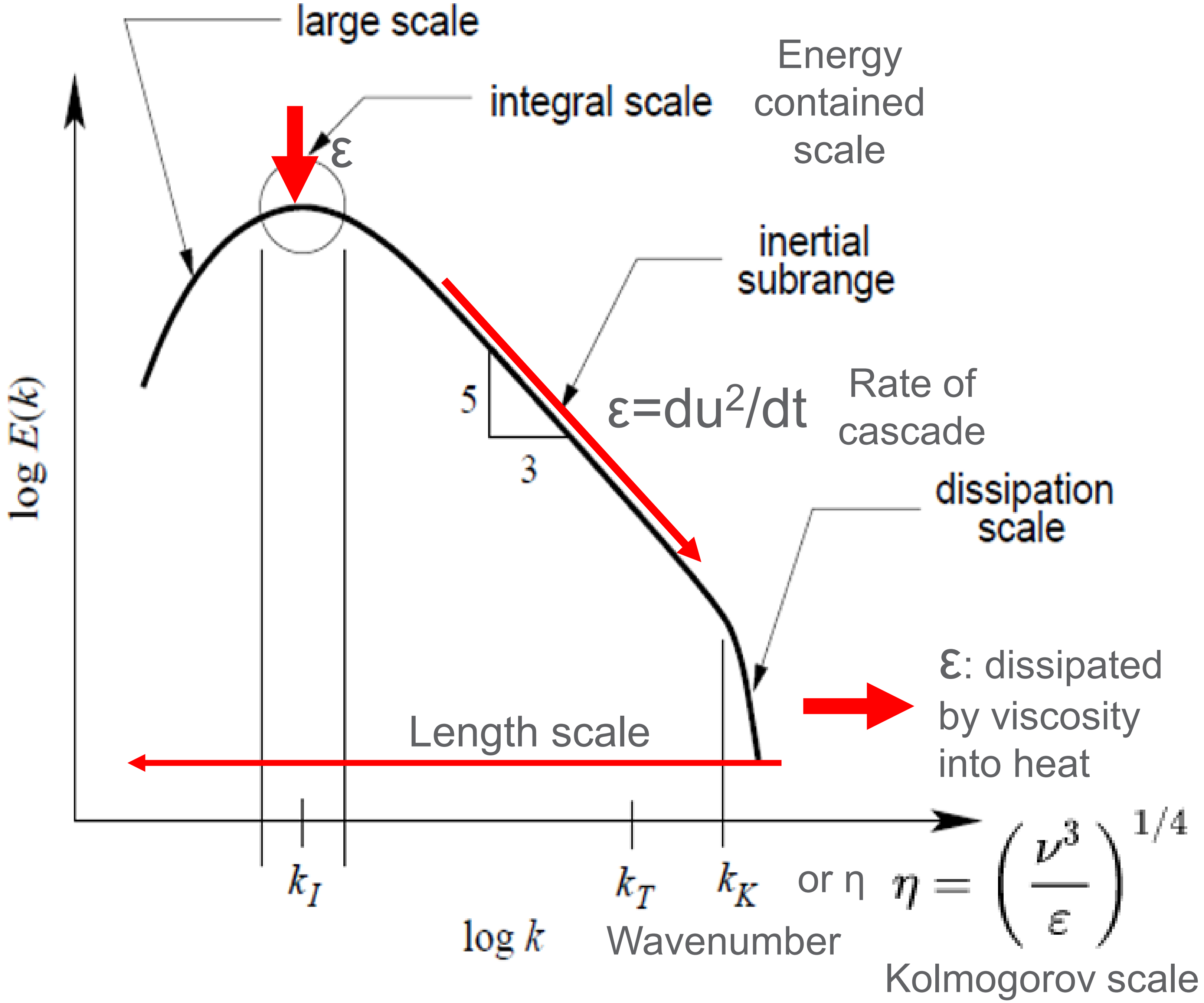
- Disorganized, chaotic, random;
- Nonrepeatability (sensitivity to initial conditions);

- Multiscale: large range of length and time scales;
- Dissipation mediated by viscosity;

- Three dimensionality;
- Time dependence;
- Rotationality (incompressible);
- Intermittency in space and time;

[1] "Weather Prediction by Numerical Process", Richardson, L.F. 1922

Direct energy cascade in turbulence (1940s)



- Freely decaying vs. forced stationary
- Integral scale: energy injection
- Inertial range: inertial >> viscous force
- Dissipation range: viscous dominant
- Dissipation scale: determined by kinematic viscosity (m^2/s) and rate of cascade (m^2/s^3)

Is there cascade in dark matter flow?
If yes, how does it initiate, propagate, and die?

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Hydrodynamic turbulence vs. dark matter flow

Key attributes of hydrodynamic turbulence:

- Chaotic, random;
- Nonrepeatability (sensitivity to initial conditions);
- Multiscale in length and time scales; Non-equilibrium;
- Intermittency in space and time;
- Dissipative and collisional
- Short-range interaction
- Velocity fluctuation
- Vortex as fundamental building block
- Maximum entropy distribution (Gaussian)
- Incompressible on all scales $\nabla \cdot \mathbf{v} = 0$
  - Divergence-free
  - Constant density
- Energy cascade from large to small length scales
- Vortex stretching responsible for energy cascade
  - Volume conserving
  - Shape changing
  - Uniform density
- Reynolds decomposition
- Reynolds stress for energy transfer between mean flow and random motion (turbulence)
- Closure problem, eddy viscosity, etc…
- Statistical theory: correlation/structure functions
- Scaling laws in inertial range

Key attributes of dark matter flow:

- Chaotic, random;
- Nonrepeatability;
- Multiscale in mass/length/time scales; Non-equilibrium;
- Intermittency in space and time;
- Dissipationless and collisionless
- Long-range gravity
- Velocity & acceleration fluctuation
- Halos as fundamental building block
- Maximum entropy distribution?? (X dist.)
- Flow behavior is scale-dependent (peculiar velocity)
  - Small scale: constant divergence $\nabla \cdot \mathbf{v} = \theta$
  - Large scale: irrotational (curl-free) $\nabla \times \mathbf{v} = 0$
- Mass/energy cascade from small to large mass scales
- Role of halos for energy cascade??
  - Halos are growing, rotating, with nonuniform density
  - Is halo shape changing important?
  - Mass cascade facilitates energy cascade?
- Velocity/acceleration decomposition?
- What facilitates the energy transfer between mean flow and random motion in dark matter??
- Self-closed model (analogue of NS) ?? Closure problem?
- Statistical theory: Kinematic and dynamic relations?
- Scaling laws in dark matter flow?

Common features

Critical MOND acceleration a_0?

This talk

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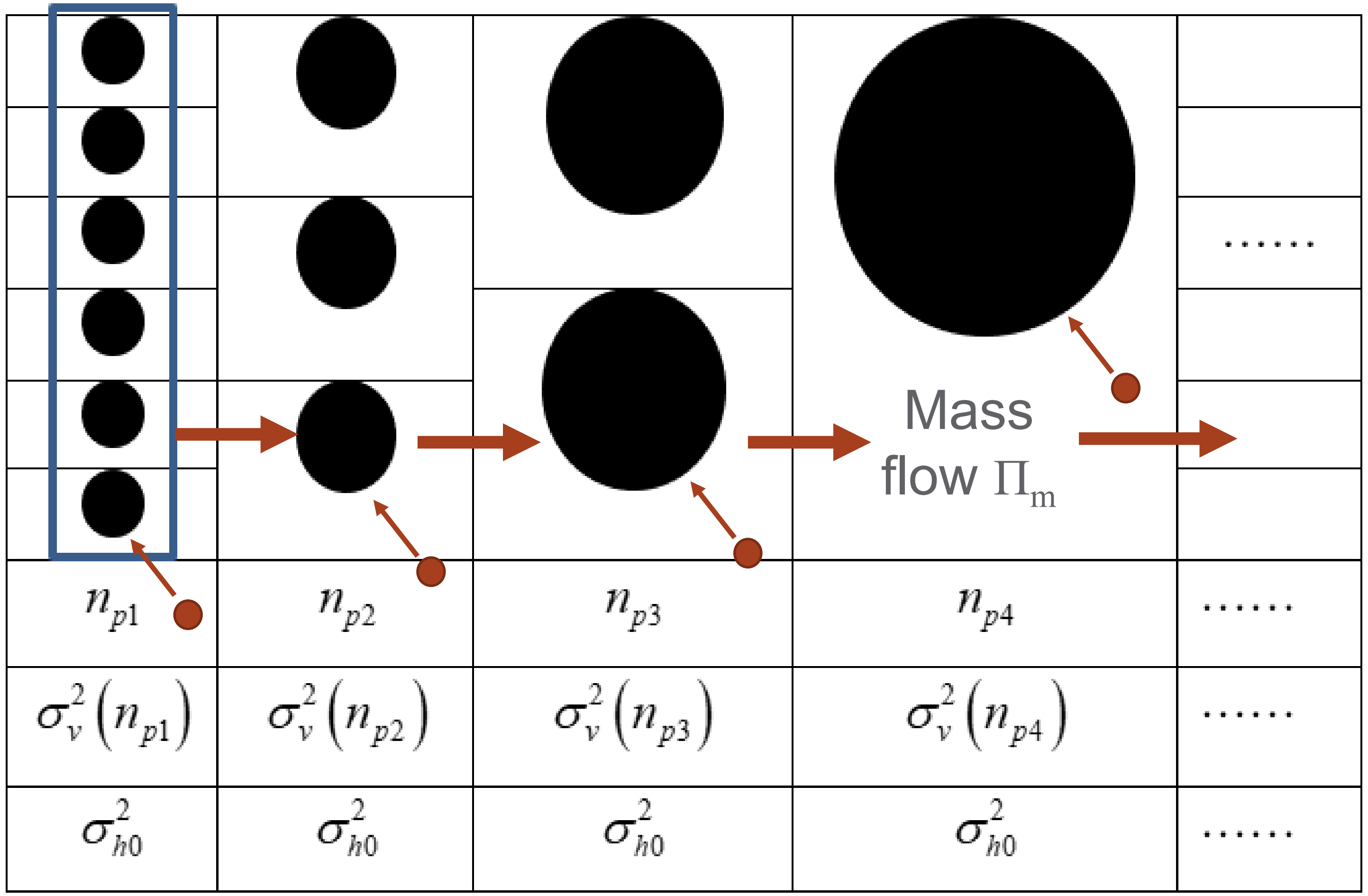
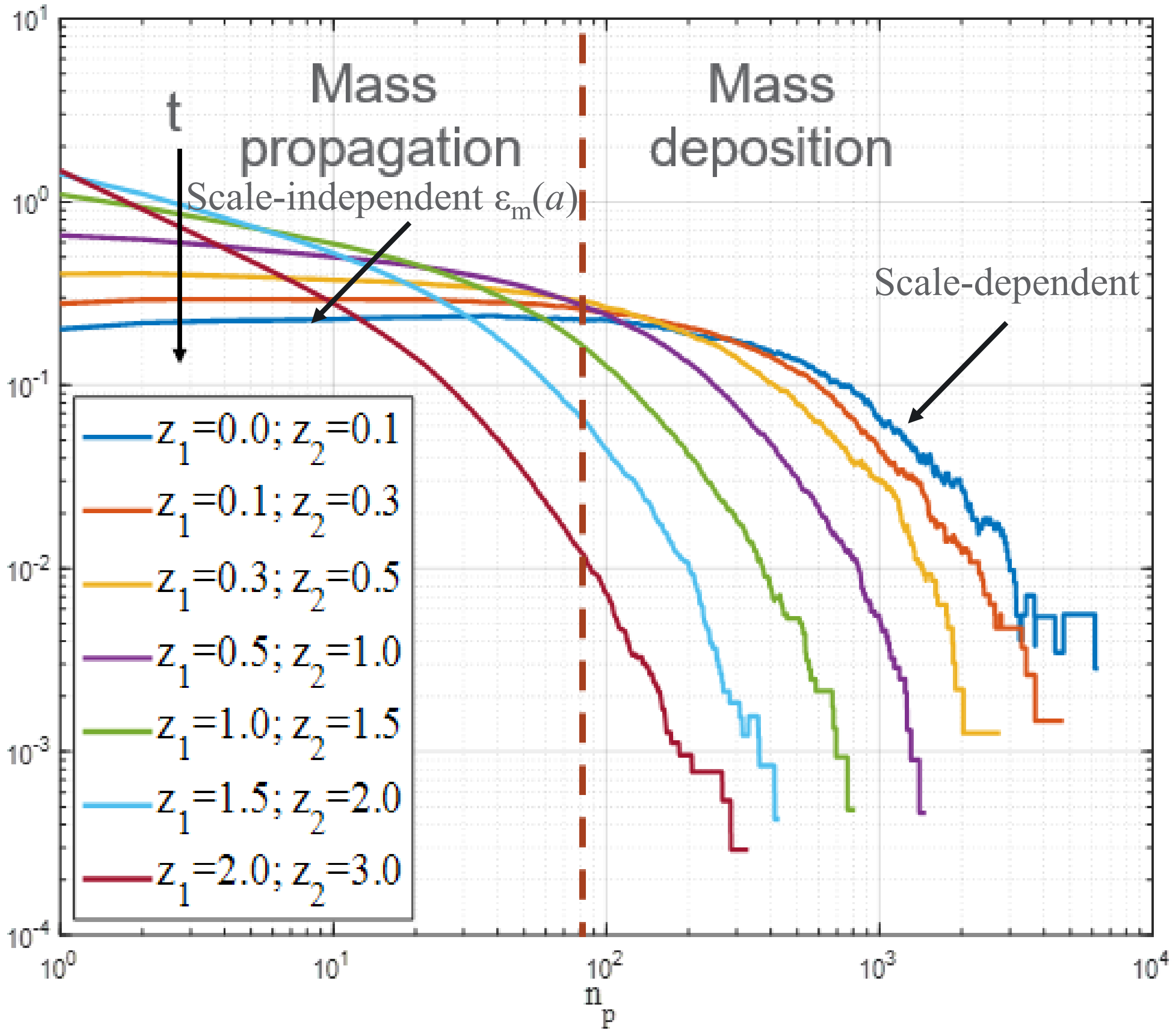
Inverse mass cascade in dark matter flow





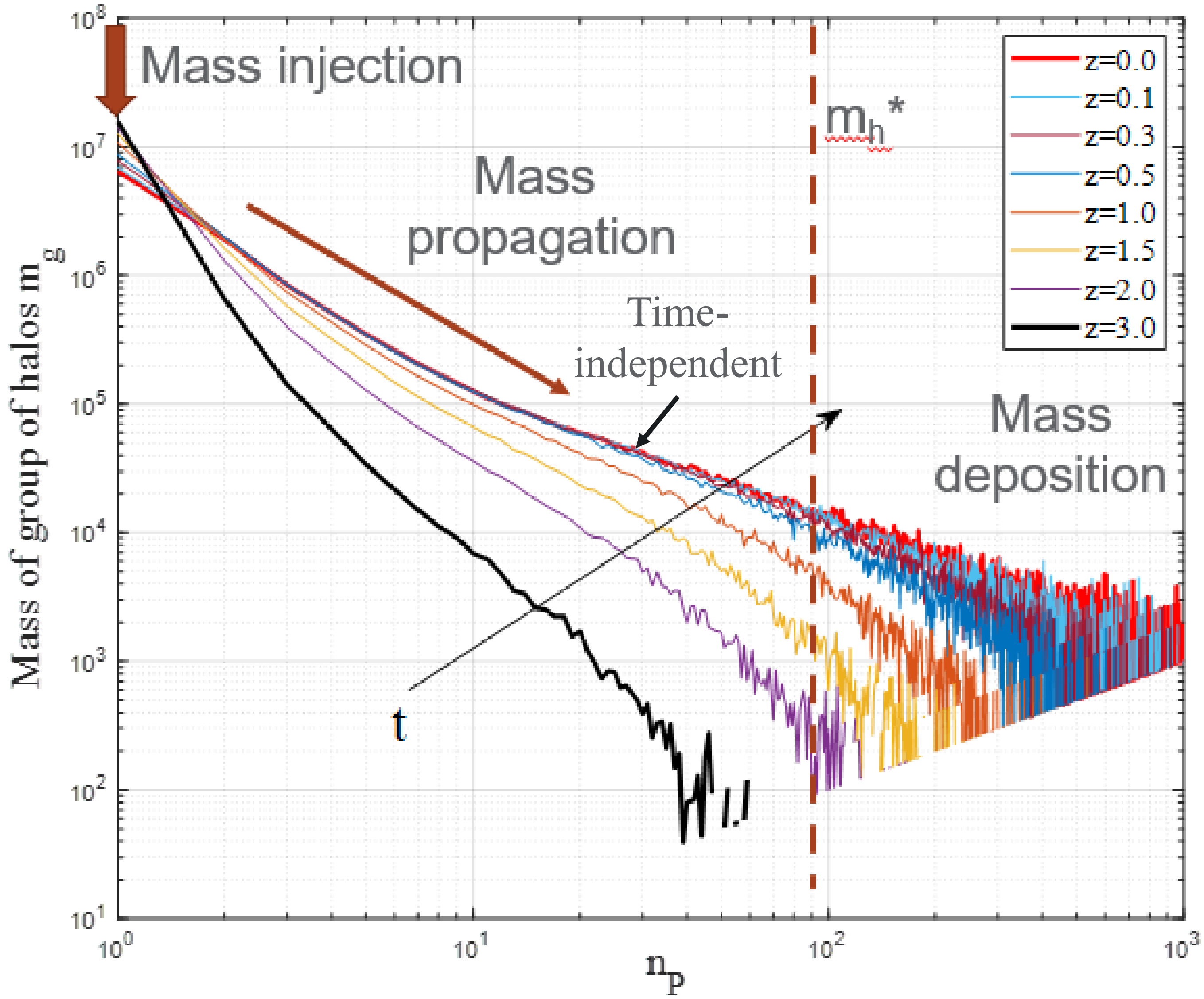
Mass cascade rate $\Pi_m(m_h, a)$ (normalized by Nm_p/t_0)

- Identify all halos of different sizes
- Group halos according to the halo size n_p
- Mass flow across halo groups from small to large mass scale (inverse) through the merging with "single merger"
- Cascade leads to random-walk of halos in mass space

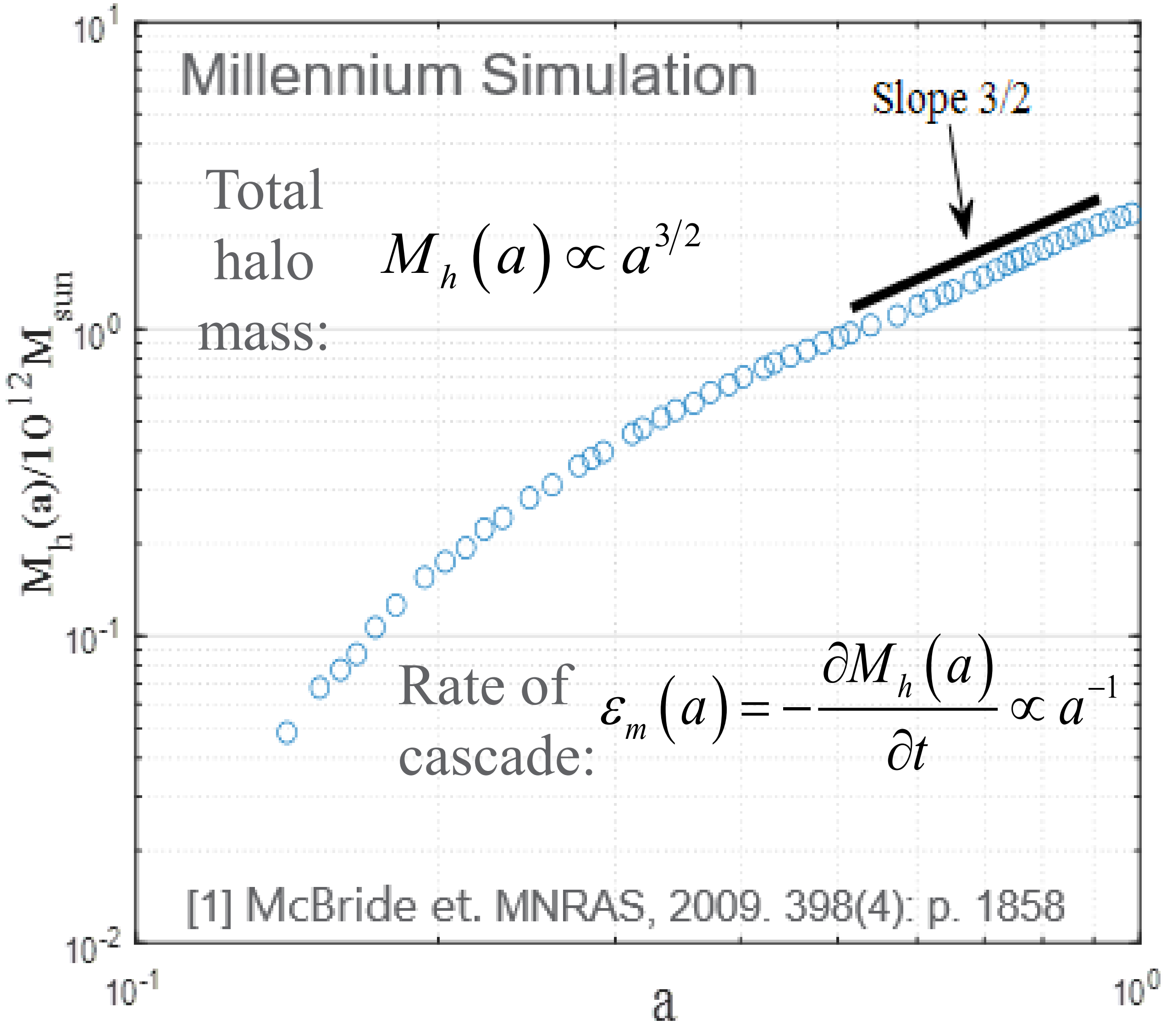
$$\Pi_m(m_h, a) = -\frac{\partial}{\partial t}\left[M_h(a)\int_{m_h}^{\infty} f_M(m, m_h^*)\,dm\right] \Rightarrow \frac{\partial \Pi_m}{\partial m_h} = \frac{1}{m_p}\frac{\partial m_g}{\partial t}$$

Total halo mass Mass function m_g: Group mass

Halo group mass and time variation of total halo mass

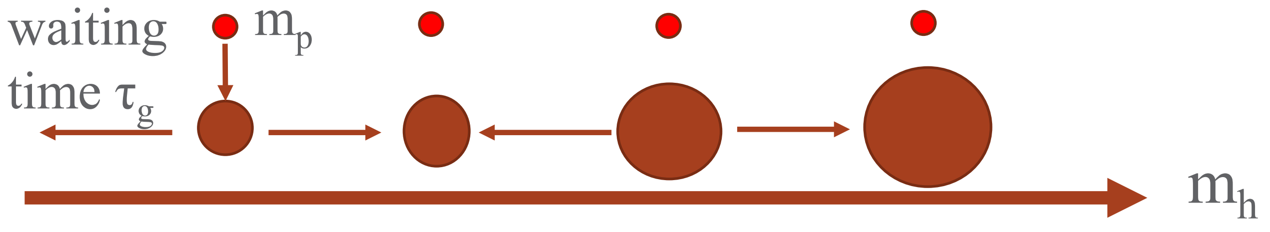


Halo group mass $m_g(m_h, a)$
(time-independent in mass propagation range)



The halo mass for type II halos (the dominant type for large halos, Fig. 2 in ref. [1]) exhibits a power law scaling

Random walk of halos and halo mass function



Merging frequency for halo group: $$f_h(m_h, a) \propto \text{surface area} \propto n_h m_h^{\lambda}$$

λ~2/3: Exponent for halo surface area.

Characteristic merging time for halo group: $$\tau_h(m_h, a) = 1/f_h$$

Characteristic merging time (lifetime) for a given halo: waiting time to merge

\# of halos in group

$$\tau_g(m_h, a) = n_h \tau_h$$

$$\boxed{\tau_g \propto m_h^{-\lambda}}$$ Position-dependent

The exponential distribution of waiting time to merge: $$P(\tau_{gr}) = \frac{1}{\tau_g} \exp\left(-\frac{\tau_{gr}}{\tau_g}\right)$$

1D Random walk equation in mass space (similar to diffusion):

$$\frac{\partial m_h(t)}{\partial t} = \frac{m_p \xi(t)}{\tau_g(m_h)} = \sqrt{2D_p(m_h)}\varsigma(t)$$

kg²/s Noise s$^{-1/2}$

Position-dependent diffusivity: $$\boxed{D_p(m_h) \propto m_h^{2\lambda}}$$

Fokker-Planck equation for distribution function:

$$\frac{\partial P_h}{\partial t} = \frac{\partial}{\partial m_h}\left[\sqrt{D_p}\frac{\partial}{\partial m_h}\left(\sqrt{D_p} P_h\right)\right] = D_{p0}\frac{\partial}{\partial m_h}\left[m_h^{\lambda}\frac{\partial}{\partial m_h}\left(m_h^{\lambda} P_h\right)\right]$$

Solving Fokker-Planck Eq. leads to Halo mass function:

$$f_M(m_h, a) = \frac{(1-\lambda)}{\sqrt{\pi\eta_0}}\left(\frac{m_h^*}{m_h}\right)^{\lambda}\frac{1}{m_h^*}\exp\left[-\frac{1}{4\eta_0}\left(\frac{m_h}{m_h^*}\right)^{2-2\lambda}\right]$$

Reduce to Press-Schechter (PS) if λ=2/3 !

(single λ here, how about double λ?)

Double-λ mass function from mass cascade

λ: halo geometry parameter; naturally, we can have two different λ in two different ranges.
λ_1 for mass propagation range (small halos);
λ_2 for mass deposition range (large halos);

$$f_M(m_h,a)=\frac{(1-\lambda)}{\sqrt{\pi\eta_0}}\left(\frac{m_h^*}{m_h}\right)^{\lambda_1}\frac{1}{m_h^*}\exp\left[-\frac{1}{4\eta_0}\left(\frac{m_h}{m_h^*}\right)^{2-2\lambda_2}\right]$$

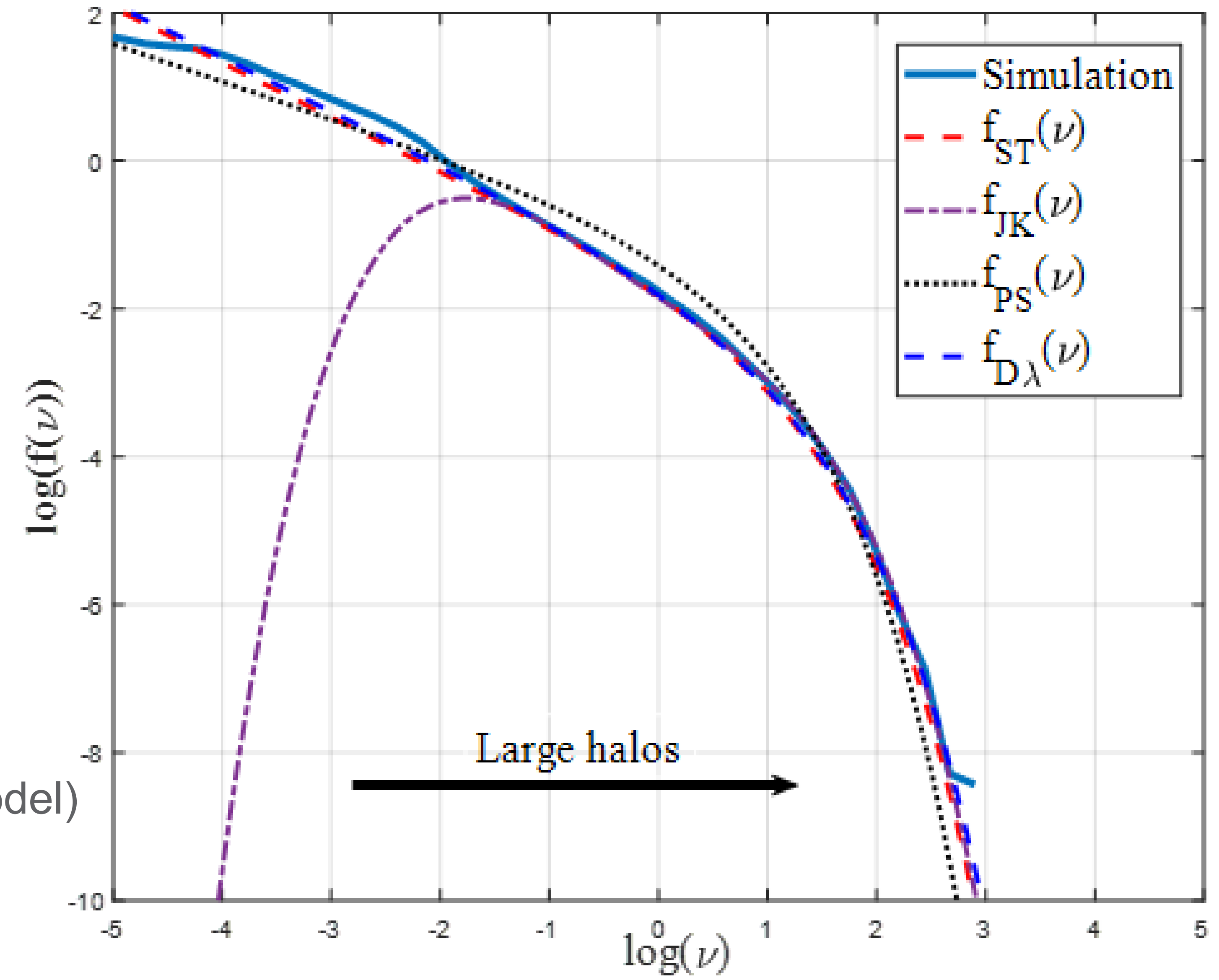
Two-parameter double-λ mass function:

$$f_{D\lambda}(\nu)=\frac{\left(2\sqrt{\eta_0}\right)^{-q}}{\Gamma(q/2)}\nu^{q/2-1}\exp\left(-\frac{\nu}{4\eta_0}\right)\qquad q=\frac{1-\lambda_1}{1-\lambda_2}$$

- PS mass function (require δc and spherical collapse model)
- ST model (modified PS) from ellipsoid collapse
- JK mass function by data fitting
- From simulation $q\approx 0.6$ → $\lambda_1\approx 4/5 \quad \lambda_2\approx 2/3$

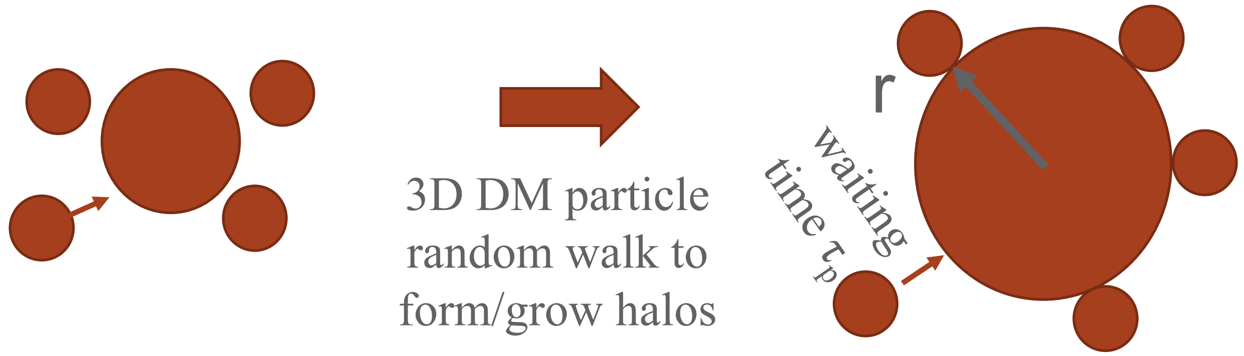
$$\boxed{f_{PS}(\nu)=\frac{1}{\sqrt{2\pi}}\nu^{-1/2}\exp\left(-\frac{\nu}{2}\right)}$$

$$\boxed{f_{ST}(\nu)=\frac{(1+1/(q\nu)^p)\sqrt{2q}}{\Gamma(1/2)+2^{-p}\Gamma(1/2-p)}\frac{1}{2\sqrt{\nu}}e^{-q\nu/2}}$$



Comparison between different mass functions and N-body simulation

Random walk of DM and double-γ halo densify profile



Waiting time dependent on halo size r (position-dependent):

$$\boxed{\tau_p(r) \propto m_r(r)^{-\lambda} \propto r^{-\gamma}}$$

The larger halo, the shorter waiting time

3D Random walk equation:

$$\frac{d\mathbf{X}_t}{dt} = \sqrt{2D_P(\mathbf{X}_t)}\,\xi(t).$$

$$\boxed{D_P(\mathbf{X}_t) = D_0(t) r^{2\gamma}}$$

Fokker-Planck equation for distribution function:

$$\frac{\partial P_r(\mathbf{X},t)}{\partial t} = D_0 \frac{\partial}{\partial X_i}\left[r^\gamma \frac{\partial}{\partial X_i}\left(r^\gamma P_r(\mathbf{X},t)\right)\right]$$

$$\boxed{\begin{aligned}\alpha &= 2-2\gamma_2\\ \beta &= \frac{2-2\gamma_2}{2-\gamma_1}\end{aligned}}$$

Double-γ halo density profile: $x = r/r_s(t)$

$$\rho_{D\gamma}\left(x \equiv \frac{r}{r_s(t)}\right) = \frac{\alpha\beta^{-(1/\alpha+1/\beta)}}{4\pi\Gamma(1/\alpha+1/\beta)} x^{\frac{\alpha}{\beta}-2} \exp\left(-\frac{x^\alpha}{\beta}\right)$$

Reduce to Einasto if α=2β !

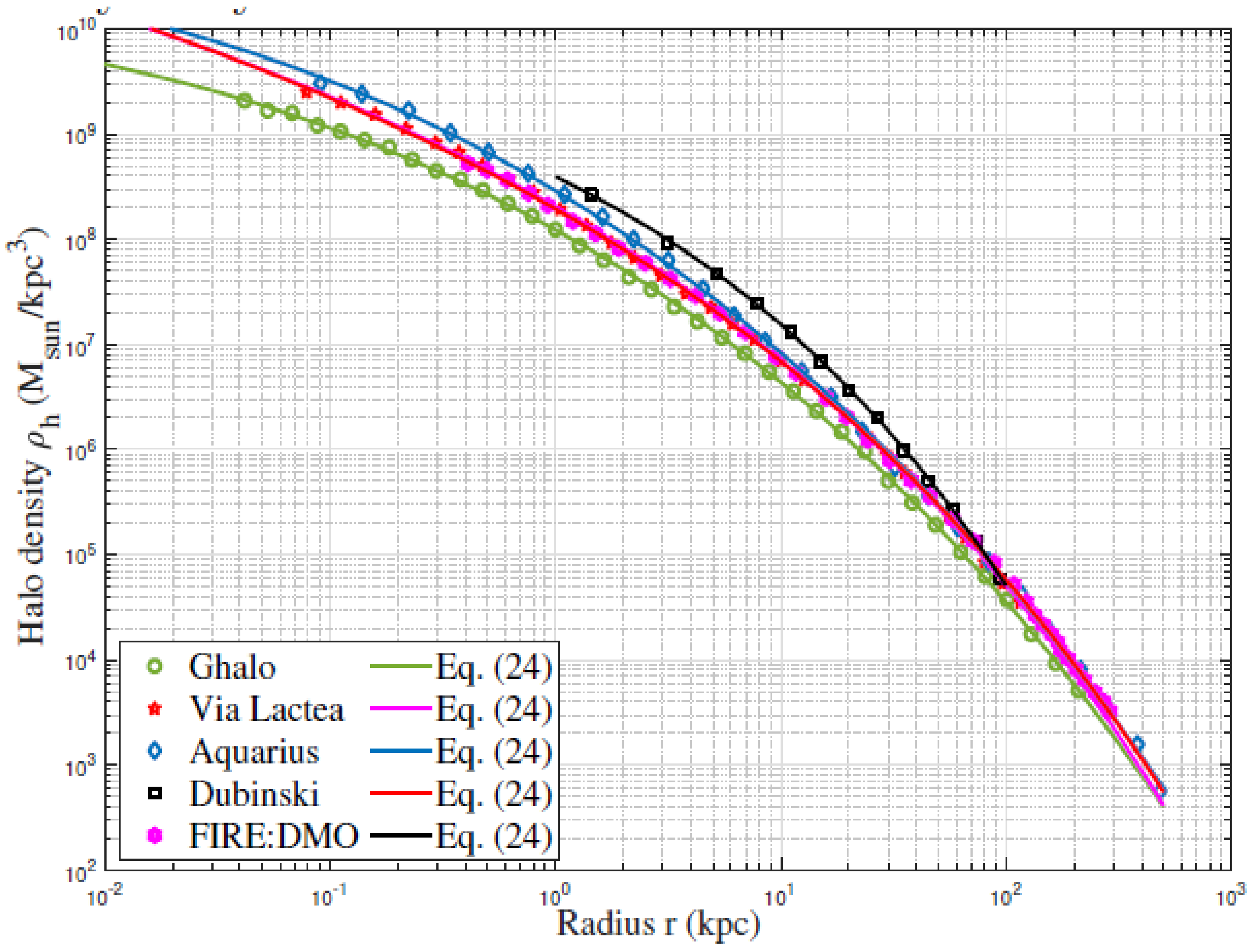


FIG. 2. Halo density profiles for simulated halos: 1) Ghalo [51]; 2) Via Lactea [52]; 3) Aquarius [53]; 4) Dubinski [54]; 5) FIRE:DMO [30]. The double-γ density model (Eq. (24)) was also used to fit all simulated halos for the entire range of r.

Quick Recap I

In (incompressible) hydrodynamic turbulence:

- Energy cascade is well established
  - Direct energy cascade from large to small scales (3D)
  - Inverse energy cascade from small to large scales (2D)
- No mass cascade involved

In dark matter flow:

- Inverse mass cascade from small to large scales (rate: ε_m kg/s)
- Mass cascade leads to the random walk of halos in mass space
- Random walk of halos in mass space leads to halo mass function (just like diffusion)

- Random walk of DM particles leads to halo density profile
- Halo density profile and mass function share the same origin.
- Halo density and mass function share similar functional form

- Both random walks involve a position-dependent waiting time (or diffusivity)
- No critical density ratio δc or spherical/ellipsoidal collapse model required

Halo mass function and density profile

$$f_{D\lambda}(\nu) = \frac{\left(2\sqrt{\eta_0}\right)^{-q}}{\Gamma(q/2)} \nu^{q/2-1} \exp\left(-\frac{\nu}{4\eta_0}\right)$$

$$\rho_{D\gamma}(x) = \frac{\alpha\beta^{-(1/\alpha+1/\beta)}}{4\pi\Gamma(1/\alpha+1/\beta)} x^{\frac{\alpha}{\beta}-2} \exp\left(-\frac{x^\alpha}{\beta}\right)$$