SUMS OF THREE FIBONACCI NUMBERS CLOSE TO A POWER OF 2

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Abstract

In this paper, we find all the sums of three Fibonacci numbers which are close to a power of 2. This paper continues and extends previous work of Hasanalizade.

1. Introduction

The Fibonacci sequence $\{F_n\}_{n\geq 0}$ is the binary recurrence sequence given by

$$F_{n+2} = F_{n+1} + F_n \quad \text{for } n \geq 0$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$. The Fibonacci numbers are famous for possessing wonderful and amazing properties. The problem of determining all integer solutions to Diophantine equations with Fibonacci numbers has piqued the curiosity of mathematicians and there is a wide literature on this subject. For instance, Bugeaud et al. [4] studied the Fibonacci numbers, that is, perfect powers and found that 1, 2 and 8 are the only powers of 2 in the Fibonacci sequence. Later, Bravo and Luca [3] solved the Diophantine equation $F_n + F_m = 2^a$ in non-negative integers n, m and a with $n \geq m$. Following that, Bravo and Bravo [1] found all solutions of the Diophantine equation $F_n + F_m + F_l = 2^a$ in non-negative integers n, m, l and a with $n \geq m \geq l$. Subsequently, Chim and Ziegler [6] determined all solutions of the Diophantine equations

$$F_{n_1} + F_{n_2} = 2^{a_1} + 2^{a_2} + 2^{a_3}$$

and

$$F_{m_1} + F_{m_2} + F_{m_3} = 2^{t_1} + 2^{t_2}$$

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in non-negative integers $n_1, n_2, m_1, m_2, m_3, a_1, a_2, a_3, t_1, t_2$ with $n_1 \geq n_2, m_1 \geq m_2 \geq m_3, a_1 \geq a_2 \geq a_3$ and $t_1 \geq t_2$.

An integer n is said to be close to a positive integer m if it satisfies

$$|n - m| < \sqrt{m}.$$

By using the above concept Chern and Cui [5] found all the Fibonacci numbers which are close to a power of 2. More precisely they found all solutions of the inequality

$$|F_n - 2^m| < 2^{m/2}.$$

Later, Bravo et al. [2] extended the previous work of [5] by taking into consideration the k-generalized Fibonacci sequence $\{F_n^{(k)}\}$ and solved the Diophantine equation

$$|F_n^{(k)} - 2^m| < 2^{m/2}$$

in non-negative integers n, k, m with $k \geq 2$ and $n \geq 1$.

Recently, Hasanlizade [8] extended the previous work of [5] by considering the sum of Fibonacci numbers and studied the sum of two Fibonacci numbers close to a power of 2. In particular, he solved

$$|F_n + F_m - 2^a| < 2^{a/2},$$

in positive integers n, m and a with $n \geq m$.

Motivated by the above literature, we extend the work of [5] and [8], and search for the sum of three Fibonacci numbers which are close to a power of 2. More specifically, we study the Diophantine inequality

$$|F_n + F_m + F_l - 2^a| < 2^{a/2}, \tag{1}$$

in positive integers n, m, l and a with $n \geq m \geq l$.

In particular, our main result concerning (1) is the following. A list of solutions of (1) is given in Table 1.

Theorem 1. *There are exactly 280 solutions* $(n, m, l, a) \in \mathbb{N}^4$ *to Diophantine Inequality* (1). *All the solutions satisfy* $n \leq 42$ *and* $a \leq 28$.

2. Auxiliary Results

In this section, we will review several well-known results which will be used later. The Binet formula for the Fibonacci sequence is

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad (n \geq 0), \tag{2}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{-1}{\alpha}$ are the roots of the characteristic equation $x^2-x-1 = 0$.

Lemma 1. *For every positive integer $n \geq 1$, we have*

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1}. \tag{3}$$

Lemma 2 ([10, Equation (5)])**.** *For every positive integer $n > 1$, we have*

$$0.38\alpha^n \leq F_n \leq 0.48\alpha^n.$$

To prove our main result, we use a few applications of a Baker-type lower bound for non-zero linear forms in logarithms of algebraic numbers. We state a result of Matveev [9] about the general lower bound for linear forms in logarithms. Applying a version of the Baker-Davenport reduction method we reduce the large bound. We first recall some basic notations from algebraic number theory.

Let η be an algebraic number of degree d with minimal primitive polynomial

$$f(X) := a_0X^d + a_1X^{d-1} + \cdots + a_d = a_0\prod_{i=1}^{d}(X-\eta^{(i)}) \in \mathbb{Z}[X],$$

where the a_i's are relatively prime integers, $a_0 > 0$, and the $\eta^{(i)}$'s are conjugates of η. Then

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d}\log\left(\max\{|\eta^{(i)}|, 1\}\right)\right) \tag{4}$$

is called the *logarithmic height* of η. In particular, if $\eta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\eta) = \log\max\{|p|, q\}$.

With these established notations, Matveev (see [9] or [4, Theorem 9.4]), proved the ensuing result.

Theorem 2. *Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$, $\gamma_1, \gamma_2, \ldots, \gamma_t$ be positive real numbers of $\mathbb{K}$, and $b_1, ..., b_t$ rational integers. Put*

$$B \geq max\{|b_1|, |b_2|, \ldots, |b_t|\},$$

and

$$\Lambda := \gamma_1^{b_1} \ldots \gamma_t^{b_t} - 1.$$

Let $A_1, ..., A_t$ be real numbers such that

$$A_i \geq max\{Dh(\gamma_i), |\log\gamma_i|, 0.16\}, \quad i = 1, \ldots, t.$$

Then, assuming that $\Lambda \neq 0$, we have

$$|\Lambda| > exp\left(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1+\log D)(1+logB)A_1\cdots A_t\right).$$

The following criterion of Legendre, a well-known result from the theory of Diophantine approximation, is used to reduce the upper bounds on variables that are too large.

Lemma 3. *Let τ be an irrational number, $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$ be all the convergents of the continued fraction of τ, and M be a positive integer. Let N be a non-negative integer such that $q_N > M$. Then putting $a(M) := \max\{a_i : i = 0, 1, 2, \ldots, N\}$, the inequality*

$$\left| \tau - \frac{r}{s} \right| > \frac{1}{(a(M)+2)s^2},$$

holds for all pairs (r, s) of positive integers with $0 < s < M$.

Another result which will play an important role in our proof is due to Dujella and Pethö [7, Lemma 5 (a)].

Lemma 4. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := ||\mu q|| - M||\gamma q||$, where $|| \cdot ||$ denotes the distance from the nearest integer. If $\epsilon > 0$, then there exists no solution to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w},$$

in positive integers u, v and w with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

Lemma 5. *For any non-zero real number x, we have the following:*

(a) $0 < x < |e^x - 1|$.

(b) If $x < 0$ and $|e^x - 1| < 1/2$, then $|x| < 2|e^x - 1|$.

3. Proof of Theorem 1

3.1. Upper Bound for n

First of all, observe that (n, m, l, a) is a solution of (1) for $m = l = 1$ and $m = l = 2$. So from now on, we assume that $m \geq 2$ and $l \geq 2$. Since $F_n + F_{n-1} = F_{n+1}$, we can assume that $n > m + 1$ and $n > l + 1$. In particular, $n - m \geq 2$ and $n - l \geq 2$.

Proposition 1. *There are exactly 280 solutions $(n, m, l, a) \in \mathbb{N}^4$ to (1) with $n \leq 550$. All solutions satisfy $n \leq 42$ and $a \leq 28$. The list of solutions is given in Table 1.*

Proof. The solutions were found by a brute force search with *Mathematica*. □

$\lvert F_i + F_1 + F_1 - 2\rvert < \sqrt{2}, i = 1, 2$	$\lvert F_{10} + F_4 + F_i - 2^6\rvert < 2^3, i = 1, 2, 3, 4$
$\lvert F_2 + F_2 + F_i - 2\rvert < \sqrt{2}, i = 1, 2$	$\lvert F_{10} + F_5 + F_i - 2^6\rvert < 2^3, i = 1, 2, 3, 4, 5$
$\lvert F_i + F_1 + F_2 - 2^2\rvert < 2, i = 1, 2$	$\lvert F_{10} + F_6 + F_i - 2^6\rvert < 2^3, i = 1, 2, \cdots, 6$
$\lvert F_2 + F_2 + F_i - 2^2\rvert < 2, i = 1, 2$	$\lvert F_{10} + F_7 + F_i - 2^6\rvert < 2^3, i = 1, 2, 3, 4$
$\lvert F_3 + F_1 + F_1 - 2^2\rvert < 2$	$\lvert F_{10} + F_9 + F_9 - 2^7\rvert < 2^{7/2}$
$\lvert F_3 + F_2 + F_i - 2^2\rvert < 2, i = 1, 2$	$\lvert F_{10} + F_{10} + F_i - 2^7\rvert < 2^{7/2}, i = 6, 7, 8$
$\lvert F_3 + F_3 + F_i - 2^2\rvert < 2, i = 1, 2$	$\lvert F_{11} + F_8 + F_i - 2^7\rvert < 2^{7/2}, i = 6, 7, 8$
$\lvert F_3 + F_3 + F_3 - 2^3\rvert < 2^{3/2}$	$\lvert F_{11} + F_9 + F_i - 2^7\rvert < 2^{7/2}, i = 1, 2, \ldots, 7$
$\lvert F_4 + F_1 + F_1 - 2^2\rvert < 2$	$\lvert F_{11} + F_{11} + F_{11} - 2^8\rvert < 2^4$
$\lvert F_4 + F_2 + F_i - 2^2\rvert < 2, i = 1, 2$	$\lvert F_{12} + F_{10} + F_{10} - 2^8\rvert < 2^4$
$\lvert F_4 + F_3 + F_i - 2^3\rvert < 2^{3/2}, i = 1, 2, 3$	$\lvert F_{12} + F_{11} + F_i - 2^8\rvert < 2^4, i = 6, 7, 8, 9$
$\lvert F_4 + F_4 + F_i - 2^3\rvert < 2^{3/2}, i = 1, 2, 3, 4$	$\lvert F_{13} + F_5 + F_i - 2^8\rvert < 2^4, i = 4, 5$
$\lvert F_5 + F_1 + F_1 - 2^3\rvert < 2^{3/2}$	$\lvert F_{13} + F_6 + F_i - 2^8\rvert < 2^4, i = 1, 2, \cdots, 6$
$\lvert F_5 + F_2 + F_i - 2^3\rvert < 2^{3/2}, i = 1, 2$	$\lvert F_{13} + F_7 + F_i - 2^8\rvert < 2^4, i = 1, 2, \cdots, 7$
$\lvert F_5 + F_3 + F_i - 2^3\rvert < 2^{3/2}, i = 1, 2, 3$	$\lvert F_{13} + F_8 + F_i - 2^8\rvert < 2^4, i = 1, 2, \ldots, 7$
$\lvert F_5 + F_4 + F_i - 2^3\rvert < 2^{3/2}, i = 1, 2, 3$	$\lvert F_{13} + F_9 + F_i - 2^8\rvert < 2^4, i = 1, 2, 3, 4$
$\lvert F_5 + F_5 + F_i - 2^4\rvert < 2^2, i = 4, 5$	$\lvert F_{13} + F_{12} + F_{12} - 2^9\rvert < 2^{9/2}$
$\lvert F_6 + F_1 + F_1 - 2^3\rvert < 2^{3/2}$	$\lvert F_{13} + F_{13} + F_i - 2^9\rvert < 2^{9/2}, i = 9, 10$
$\lvert F_6 + F_2 + F_i - 2^3\rvert < 2^{3/2}, i = 1, 2$	$\lvert F_{14} + F_{11} + F_i - 2^9\rvert < 2^{9/2}, i = 9, 10$
$\lvert F_6 + F_4 + F_i - 2^4\rvert < 2^2, i = 3, 4$	$\lvert F_{14} + F_{12} + F_i - 2^9\rvert < 2^{9/2}, i = 1, 2, \cdots, 7$
$\lvert F_6 + F_5 + F_i - 2^4\rvert < 2^2, i = 1, 2, 3, 4, 5$	$\lvert F_{15} + F_{14} + F_i - 2^{10}\rvert < 2^5, i = 6, 7, 8, 9, 10$
$\lvert F_6 + F_6 + F_i - 2^4\rvert < 2^2, i = 1, 2, 3, 4$	$\lvert F_{16} + F_4 + F_4 - 2^{10}\rvert < 2^5$
$\lvert F_7 + F_1 + F_1 - 2^4\rvert < 2^2$	$\lvert F_{16} + F_5 + F_i - 2^{10}\rvert < 2^5, i = 1, 2, 3, 4, 5$
$\lvert F_7 + F_2 + F_i - 2^4\rvert < 2^2, i = 1, 2$	$\lvert F_{16} + F_6 + F_i - 2^{10}\rvert < 2^5, i = 1, 2, \cdots, 6$
$\lvert F_7 + F_3 + F_i - 2^4\rvert < 2^2, i = 1, 2, 3$	$\lvert F_{16} + F_7 + F_i - 2^{10}\rvert < 2^5, i = 1, 2, \cdots, 7$
$\lvert F_7 + F_4 + F_i - 2^4\rvert < 2^2, i = 1, 2, 3, 4$	$\lvert F_{16} + F_8 + F_i - 2^{10}\rvert < 2^5, i = 1, 2, \ldots, 8$
$\lvert F_7 + F_5 + F_i - 2^4\rvert < 2^2, i = 1, 2$	$\lvert F_{16} + F_9 + F_i - 2^{10}\rvert < 2^5, i = 1, 2, \ldots, 9$
$\lvert F_7 + F_6 + F_6 - 2^5\rvert < 2^{5/2}$	$\lvert F_{16} + F_{10} + F_i - 2^{10}\rvert < 2^5, i = 1, 2, \cdots, 7$
$\lvert F_7 + F_7 + F_i - 2^5\rvert < 2^{5/2}, i = 1, 2, 3, 4, 5, 6$	$\lvert F_{16} + F_{16} + F_i - 2^{10}\rvert < 2^5, i = 9, 10, 11$
$\lvert F_8 + F_4 + F_4 - 2^5\rvert < 2^{5/2}$	$\lvert F_{17} + F_{13} + F_{13} - 2^{11}\rvert < 2^{11/2}$
$\lvert F_8 + F_5 + F_i - 2^5\rvert < 2^{5/2}, i = 1, 2, 3, 4, 5$	$\lvert F_{17} + F_{14} + F_i - 2^{11}\rvert < 2^{11/2}, i = 9, 10, 11$
$\lvert F_8 + F_6 + F_i - 2^5\rvert < 2^{5/2}, i = 1, 2, 3, 4, 5, 6$	$\lvert F_{20} + F_{16} + F_{14} - 2^{13}\rvert < 2^{13/2}$
$\lvert F_8 + F_7 + F_i - 2^5\rvert < 2^{5/2}, i = 1, 2, 3, 4$	$\lvert F_{21} + F_{21} + F_{21} - 2^{15}\rvert < 2^{15/2}$
$\lvert F_8 + F_8 + F_8 - 2^6\rvert < 2^3$	$\lvert F_{22} + F_{21} + F_{19} - 2^{15}\rvert < 2^{15/2}$
$\lvert F_9 + F_1 + F_1 - 2^5\rvert < 2^{5/2}$	$\lvert F_{23} + F_{18} + F_{17} - 2^{15}\rvert < 2^{15/2}$
$\lvert F_9 + F_2 + F_i - 2^5\rvert < 2^{5/2}, i = 1, 2$	$\lvert F_{23} + F_{19} + F_i - 2^{15}\rvert < 2^{15/2}, i = 1, 2, \ldots, 11$
$\lvert F_9 + F_3 + F_i - 2^5\rvert < 2^{5/2}, i = 1, 2$	$\lvert F_{24} + F_{22} + F_{17} - 2^{16}\rvert < 2^8$
$\lvert F_9 + F_7 + F_7 - 2^6\rvert < 2^3$	$\lvert F_{26} + F_{20} + F_{18} - 2^{17}\rvert < 2^{17/2}$
$\lvert F_9 + F_8 + F_i - 2^6\rvert < 2^3, i = 3, 4, 5, 6, 7$	$\lvert F_{29} + F_{20} + F_{18} - 2^{19}\rvert < 2^{19/2}$
$\lvert F_9 + F_9 + F_i - 2^6\rvert < 2^3, i = 1, 2, 3, 4$	$\lvert F_{41} + F_{40} + F_{29} - 2^{28}\rvert < 2^{14}$
$\lvert F_{10} + F_1 + F_1 - 2^6\rvert < 2^3$	$\lvert F_{42} + F_{28} + F_{27} - 2^{28}\rvert < 2^{14}$
$\lvert F_{10} + F_2 + F_i - 2^6\rvert < 2^3, i = 1, 2$	$\lvert F_{42} + F_{29} + F_i - 2^{28}\rvert < 2^{14}, i = 1, 2, \ldots, 22$
$\lvert F_{10} + F_3 + F_i - 2^6\rvert < 2^3, i = 1, 2, 3$	

Table 1: Solutions of Inequality (1)

Due to Proposition 1, for the rest of paper, we assume that $n > 550$. Using Lemma 2 for $n > m > l > 1$, we have

$$0.38\alpha^n < F_n < F_n + F_m + F_l < 0.48\alpha^n + 0.48\alpha^{n-1} + 0.48\alpha^{n-2} < 0.97\alpha^n. \qquad (5)$$

Let us now get a relationship between n and a. Without affecting generality, we may now assume $a \geq 2$. Combining (1) with the upper bound of (5), we get that

$$2^{a-1} \leq 2^a - 2^{\frac{a}{2}} < F_n + F_m + F_l < 0.97\alpha^n < \alpha^n. \qquad (6)$$

When we combine (1) with the lower bound of (5), we get

$$0.38\alpha^n < F_n + F_m + F_l < 2^a + 2^{\frac{a}{2}} < 2^{a+1}.$$

Thus

$$n\frac{\log \alpha}{\log 2} + \frac{\log 0.38}{\log 2} - 1 < a < n\frac{\log \alpha}{\log 2} + 1, \qquad (7)$$

where $\frac{\log \alpha}{\log 2} = 0.6942\ldots$. Hence, we have $a < n$. Using (2) in (1), we get

$$\frac{\alpha^n + \alpha^m + \alpha^l}{\sqrt{5}} - F_n + F_m + F_l = \frac{\beta^n + \beta^m + \beta^l}{\sqrt{5}}. \qquad (8)$$

Taking the absolute values on both sides in (8), we obtain

$$\left|\frac{\alpha^n(1 + \alpha^{m-n} + \alpha^{l-n})}{\sqrt{5}} - 2^a\right| < \frac{|\beta|^n + |\beta|^m + |\beta|^l}{\sqrt{5}} + 2^{\frac{a}{2}} < \frac{3}{5} + 2^{\frac{a}{2}} < 2^{\frac{a}{2}+1},$$

for all $n \geq 4$ and $m, l \geq 2$. Dividing both sides of the above relation by 2^a, we get

$$\left|\frac{\alpha^n(1 + \alpha^{m-n} + \alpha^{l-n})}{2^a\sqrt{5}} - 1\right| < 2^{-\frac{a}{2}+1}. \qquad (9)$$

For the left-hand side, we apply Theorem 2 with the following data. Set

$$\Lambda_1 := 2^{-a}\alpha^n\frac{(1 + \alpha^{m-n} + \alpha^{l-n})}{\sqrt{5}} - 1.$$

Note that $\Lambda_1 \neq 0$. If $\Lambda_1 = 0$, we would get the relation

$$2^a\sqrt{5} = \alpha^n + \alpha^m + \alpha^l. \qquad (10)$$

Conjugating the above relation in $\mathbb{Q}(\sqrt{5})$, we get

$$-2^a\sqrt{5} = \beta^n + \beta^m + \beta^l. \qquad (11)$$

Equations (10) and (11) lead to

$$\alpha^n < \alpha^n + \alpha^m + \alpha^l = |\beta^n + \beta^m + \beta^l| < |\beta^n| + |\beta^m| + |\beta^l| < 1,$$

which is impossible for positive integers n. Hence $\Lambda_1 \neq 0$. We take $t := 3$,

$$\gamma_1 := 2, \quad \gamma_2 := \alpha, \quad \gamma_3 := \frac{(1+\alpha^{m-n}+\alpha^{l-n})}{\sqrt{5}},$$

and

$$b_1 := -a, \quad b_2 := n, \quad b_3 := 1.$$

Note that $\mathbb{K} := \mathbb{Q}(\sqrt{5})$ contains $\gamma_1, \gamma_2, \gamma_3$ and has $D := [\mathbb{K} : \mathbb{Q}] = 2$. Since $a < n$, we can take $B := \max\{|b_1|, |b_2|, |b_3|\} = n$. Since $h(\gamma_1) = \log 2$ and $h(\gamma_2) = \frac{\log \alpha}{2}$, it follows that we can take $A_1 := 1.4 > Dh(\gamma_1)$ and $A_2 := 0.5 > Dh(\gamma_2)$. To estimate $h(\gamma_3)$, we begin by observing that

$$\gamma_3 = \frac{(1+\alpha^{m-n}+\alpha^{l-n})}{\sqrt{5}} < \frac{3}{\sqrt{5}} \quad \text{and} \quad \gamma_3^{-1} = \frac{\sqrt{5}}{(1+\alpha^{m-n}+\alpha^{l-n})} < \sqrt{5},$$

so it follows that $|\log \gamma_3| < 1$. Using the properties of logarithmic height, we get

$$\begin{aligned} h(\gamma_3) &\leq \log\sqrt{5} + |m-n|\left(\frac{\log\alpha}{2}\right) + |l-n|\left(\frac{\log\alpha}{2}\right) + 2\log 2 \\ &= \log(4\sqrt{5}) + (n-m)\left(\frac{\log\alpha}{2}\right) + (n-l)\left(\frac{\log\alpha}{2}\right). \end{aligned}$$

Thus, we can take $A_3 := 5 + (2n-m-l)\log\alpha > \max\{2h(\gamma_3), |\log\gamma_3|, 0.16\}$. Then by using Theorem 2, we have

$$\begin{aligned} \log|\Lambda_1| &> -1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1+\log 2) \times (1+\log n) \times 1.4 \times 0.5 \times \\ &\qquad (5 + (2n-m-l)\log\alpha) \\ &> -1.4 \times 10^{12} \times \log n \times (5 + (2n-m-l)\log\alpha), \end{aligned}$$

where $1 + \log n < 2\log n$ holds for all $n \geq 3$. By comparing the above inequality with (9), we get

$$\left(\frac{a}{2} - 1\right)\log 2 < 1.4 \times 10^{12} \times \log n \times (5 + (2n-m-l)\log\alpha). \tag{12}$$

Now consider the second linear form in logarithms by rewriting (8) differently. Using (2), we get that

$$\frac{\alpha^n}{\sqrt{5}} - (F_n + F_m + F_l) = \frac{\beta^n}{\sqrt{5}} - F_m + F_l.$$

Again, combining the above relation with (1), we get that

$$\left|\frac{\alpha^n}{\sqrt{5}} - 2^a\right| < 2^{\frac{a}{2}} + \frac{|\beta|^n}{\sqrt{5}} + F_m + F_l, \tag{13}$$

where $|\beta|^n < 1/2$ for all $n \geq 2$. Dividing both sides of Inequality (13) by $\frac{\alpha^n}{\sqrt{5}}$ and taking into account that $\alpha > \sqrt{2}$ and $n > m > l$, we obtain

$$\begin{aligned} |1 - 2^a\alpha^{-n}\sqrt{5}| &< \frac{2^{\frac{a}{2}}\sqrt{5}}{\alpha^n} + \frac{\sqrt{5}}{2\alpha^n} + \frac{\sqrt{5}}{\alpha^{n-m}} + \frac{\sqrt{5}}{\alpha^{n-l}} \\ &< 2\sqrt{5}\max\{\alpha^{m-n}, \alpha^{l-n}, \alpha^{a-n}\}. \end{aligned} \tag{14}$$

For the left-hand side, we apply Theorem 2 with the following data. Set

$$\Lambda_2 := 2^a\alpha^{-n}\sqrt{5} - 1.$$

Note that $\Lambda_2 \neq 0$. If $\Lambda_2 = 0$, then we have that $2^a = \frac{\alpha^n}{\sqrt{5}}$. So $\alpha^{2n} \in \mathbb{Z}$, which is not possible. Therefore $\Lambda_2 \neq 0$. We take $t := 3$,

$$\gamma_1 := 2, \quad \gamma_2 := \alpha, \quad \gamma_3 := \sqrt{5},$$

and

$$b_1 := a, \quad b_2 := -n, \quad b_3 := 1.$$

Note that $\mathbb{K} := \mathbb{Q}(\sqrt{5})$ contains $\gamma_1, \gamma_2, \gamma_3$ and has $D := 2$. Since $a < n$, we deduce that $B := \max\{|b_1|, |b_2|, |b_3|\} = n$. The logarithmic heights for γ_1, γ_2 and γ_3 are calculated as follows:

$$h(\gamma_1) = \log 2, \quad h(\gamma_2) = \frac{\log\alpha}{2} \quad \text{and} \quad h(\gamma_3) = \log\sqrt{5}.$$

Thus, we can take

$$A_1 := 1.4, \quad A_2 := 0.5 \quad \text{and} \quad A_3 := 1.7.$$

As before, applying Theorem 2, we have

$$\begin{aligned} \log|\Lambda_2| &> -1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1 + \log 2) \times (1 + \log n) \times 1.4 \times 0.5 \times 1.7 \\ &> -2.31 \times 10^{12} \times \log n. \end{aligned}$$

Comparing the above inequality with (14) implies that

$$\min\{(n-m)\log\alpha, (n-l)\log\alpha, (n-a)\log\alpha\} < 2.4 \times 10^{12}\log n.$$

Now the argument splits into three cases.

`Case 1.` $\min\{(n-m)\log\alpha, (n-l)\log\alpha, (n-a)\log\alpha\} = (n-m)\log\alpha$.
In this case, we have

$$(n-m)\log\alpha < 2.4 \times 10^{12}\log n.$$

Case 2. $\min\{(n-m)\log\alpha, (n-l)\log\alpha, (n-a)\log\alpha\} = (n-l)\log\alpha$.
In this case, we have

$$(n-l)\log\alpha < 2.4\times 10^{12}\log n.$$

Now by using Case 1 and Case 2 in (12), we have

$$\begin{aligned}\left(\frac{a}{2}-1\right)\log 2 &< 1.4\times 10^{12}\times\log n\times(5+(2n-m-l)\log\alpha)\\ &< 1.4\times 10^{12}\times\log n\times(5+4.8\times 10^{12}\log n)\\ &< 6.86\times 10^{24}\log^2 n.\end{aligned}$$

Combining the above with the lower bound of (7) and by a calculation in *Mathematica*, we get

$$a < 9\times 10^{28} \quad \text{and} \quad n < 1.87\times 10^{29}.$$

Case 3. $\min\{(n-m)\log\alpha, (n-l)\log\alpha, (n-a)\log\alpha\} = (n-a)\log\alpha$.
In this case, we have

$$(n-a)\log\alpha < 2.4\times 10^{12}\log n. \tag{15}$$

Note that the upper bound of (7) yields

$$n-a > n\left(1-\frac{\log\alpha}{\log 2}\right)-1. \tag{16}$$

From (15) and (16), we obtain

$$a < 3.84\times 10^{14} \quad \text{and} \quad n < 5.53\times 10^{14}.$$

Thus, in all the three cases, we have

$$a < 9\times 10^{28} \quad \text{and} \quad n < 1.87\times 10^{29}.$$

Now we need to reduce the bound of n.

3.2. Reducing the Bound on n

Let us assume that $n-m > 550$, $n-l > 550$ and $n-a > 550$. In order to apply Lemma 2, we put

$$z_1 := a\log 2 - n\log\alpha + \log\sqrt{5}.$$

Since we have assumed that $\min\{n-m, n-l, n-a\} > 550$, we have $|e^{z_1}-1| < 1/2$. Thus, by Lemma 2, we have that $|z_1| < 2|e^{z_1}-1|$. Replacing z_1 by its formula and by (14), we get that

$$|a\log 2 - n\log\alpha + \log\sqrt{5}| < \frac{4\sqrt{5}}{\alpha^{\min\{n-m,n-l,n-a\}}}.$$

Dividing both sides of the above inequality by $\log\alpha$, we can conclude

$$0 < \left| a\frac{\log 2}{\log\alpha} - n + \frac{\log\sqrt{5}}{\log\alpha}\right| < \frac{4\sqrt{5}}{\log\alpha}\cdot\alpha^{-w}, \tag{17}$$

where $w = \min\{n-m, n-l, n-a\}$. Putting now

$$\gamma := \frac{\log 2}{\log\alpha}, \quad \mu := \frac{\log\sqrt{5}}{\log\alpha}, \quad A := \frac{4\sqrt{5}}{\log\alpha} \quad \text{and} \quad B := \alpha,$$

the above Inequality (17) implies

$$0 < |a\gamma - n + \mu| < AB^{-w}.$$

It is clear that γ is an irrational number. We also put $M := 9\times 10^{28}$ which is an upper bound for a. We find that the convergent $\frac{p}{q} = \frac{p_{69}}{q_{69}}$ is such that $q = q_{69} > 6M$. By using this we have that $\epsilon = ||\mu q_{69}|| - M||\gamma q_{69}|| > 0$. Therefore

$$n - m < \frac{\log\left(\frac{4\sqrt{5}q}{\epsilon \ \log\alpha}\right)}{\log\alpha} < 157$$

or

$$n - l < \frac{\log\left(\frac{4\sqrt{5}q}{\epsilon \ \log\alpha}\right)}{\log\alpha} < 157$$

or

$$n - a < \frac{\log\left(\frac{4\sqrt{5}q}{\epsilon \ \log\alpha}\right)}{\log\alpha} < 157.$$

Thus, we have that either $n - m < 157$ or $n < 517$. The latter case contradicts our assumption that $n > 550$. Inserting the upper bound for $n - m$ into (12), we get that $a < 3.93\times 10^{15}$. Finally, we shall use Inequality (9) to improve the bound on n, where we put

$$z_2 := a\log 2 - n\log\alpha + \log\psi(n-m, n-l),$$

where $\psi(t,s) := \sqrt{5}(1+\alpha^{-t}+\alpha^{-s})^{-1}$. Therefore, (9) implies that

$$|1 - e^{z_2}| < \frac{2}{2^{a/2}}. \tag{18}$$

Note that $z_2 \neq 0$. Thus, we distinguish the following cases. If $z_2 > 0$, then

$$0 < z_2 < e^{z_2} - 1 \leq \frac{2}{2^{a/2}}.$$

Suppose $z_2 < 0$. Then, from (18) we have that $|1-e^{z_2}| < 1/2$ and therefore $e^{|z_2|} < 2$. Since $z_2 < 0$, we have

$$0 < |z_2| \leq e^{|z_2|} - 1 = e^{|z_2|}|e^{z_2} - 1| < \frac{4}{2^{a/2}}.$$

In any case, we have that

$$0 < |z_2| \leq \frac{4}{2^{a/2}}.$$

Replacing z_2 in the above inequality and dividing both sides by $\log\alpha$, we conclude that

$$0 < \left| a\frac{\log 2}{\log\alpha} - n + \frac{\log\psi(n-m, n-l)}{\log\alpha} \right| < \frac{4}{\log\alpha} \cdot 2^{-a/2}. \tag{19}$$

Here we take $M := 3.93 \times 10^{15}$ which is an upper bound for a. By virtue of Lemma 2 with

$$\gamma := \frac{\log 2}{\log\alpha}, \quad \mu := \frac{\log\psi(n-m, n-l)}{\log\alpha}, \quad A := \frac{4}{\log\alpha}, \quad B := \sqrt{2}$$

to Inequality (19) for all choices $n - m \in \{1, \dots, 157\}$ and $n - l \in \{1, \dots, 157\}$ except when

$$(n-m, n-l) \in \{(0,3), (1,1), (1,5), (3,0), (3,4), (4,3), (5,1), (7,8), (8,7)\}.$$

Furthermore, with the help of *Mathematica*, we find that if (n, m, l, a) is a possible solution of (1), excluding those cases mentioned previously, then $n \leq 550$. This is false because we have assumed $n > 550$. Finally, we deal with the special cases where

$$(n-m, n-l) \in \{(0,3), (1,1), (1,5), (3,0), (3,4), (4,3), (5,1), (7,8), (8,7)\}.$$

As can be seen, there are certain symmetric cases and there is no significant difference in their study. Therefore, we deal with the cases $(n-m, n-l) \in \{(1,1), (3,0), (4,3), (5,1), (8,7)\}$. For these special cases, we have that

$$\frac{\log\psi(t,s)}{\log\alpha} = \begin{cases} 0 & \text{if } (t,s) = (1,1); \\ 0 & \text{if } (t,s) = (3,0); \\ 1 & \text{if } (t,s) = (4,3); \\ 2 - \frac{\log 2}{\log\alpha} & \text{if } (t,s) = (5,1); \\ 3 - \frac{\log 2}{\log\alpha} & \text{if } (t,s) = (8,7). \end{cases}$$

Note that, when we apply Lemma 2 to the Inequality (19), the parameters γ and μ appearing in Lemma 2 are linearly dependent and the corresponding value of ϵ is always negative. When $n - m = 1$ and $n - l = 1$ from (19), we have

$$0 < |a\gamma - n| < \frac{4}{\log\alpha} \cdot 2^{-\frac{1}{2}\left(n\frac{\log\alpha}{\log 2} + \frac{\log 0.38}{\log 2} - 1\right)}. \tag{20}$$

We recall that $a < 3.93 \times 10^{15}$. Let $[a_0, a_1, a_2, a_3, a_4, \dots] = [1, 2, 3, 1, 2, \dots]$ be the continued fraction of γ. A quick search using *Mathematica* reveals that

$$q_{34} < 3.93 \times 10^{15} < q_{35}.$$

Furthermore, $a_M := \max\{a_i : i = 0, 1, \ldots, 35\} = a_{17} = 134$. So by Lemma 2 we have

$$|a\gamma - n| > \frac{1}{(a_M + 2)a}.$$

Comparing estimates (19) and (20) we get that $n < 112$. Similarly, one can get $n < 112$ in all other cases. This again contradicts our assumption that $n > 550$. Hence, the result is proved.

4. The Code

In this section, for the sake of completeness, we present the *Mathematica* code used to confirm Proposition 1:

```
Catch[Do[{n,m,l,a};
If[Abs[Fibonacci[n] + Fibonacci[m] + Fibonacci[l] - 2^(a)]
 < 2^(a/2) && n >= m >= l, Print[{n,m,l,a}]],{n,1,550},
{m,1,548},{l,1,548},{a,1,100}]]
```

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